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A CLASS OF UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS ASSOCIATED WITH THE CONVOLUTION STRUCTURE

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Abstract: Making use of convolution structure, we introduce a new class of univalent functions with positive coefficients. Among the results presented in this paper include the coefficient bounds, distortion inequalities, extreme points and integral mean inequalities for this generalized class of functions are studied.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k , \qquad (a_k \geq 0) \tag{1}$$

which are analytic and univalent in the open unite disc $U = \{ z : z \in C \text{ and} |z| < 1\}$.

Given two functions disc $f, g \in \mathcal{A}$, where f is given by (1) and g is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k , \qquad (b_k \geq 0),$$

The Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z), \qquad z \in U.$$

Salagean [14] introduced the following operator which is popularly known as the Salagean derivative operator :

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = Df(z) = zf^{'}(z)$$

and in general,

$$D^n f(z) = D\left(D^{n-1} f(z)\right)(n \in N_0 = N \cup \{0\}).$$

We easily find from (1) that

$$D^n f(z) = z + \textstyle\sum_{k=2}^{\infty} k^n a_k z^k \ (f \in \mathcal{A}; n \in N_0). \qquad (2)$$

We now introduce a new class $\mathcal{A}_g^*(n, \alpha, \beta)$ of function belonging to the class $\mathcal{A}$ which consist of functions $f(z)$ of the form (1) satisfying the following inequality:

$$\mathcal{Re}\left\{\frac{(1-\beta)D^{n+1}(f * g)(z) + \beta D^{n+2}(f * g)(z)}{(1-\beta)D^n(f * g)(z) + \beta D^{n+1}(f * g)(z)}\right\} < \alpha \qquad \left(0 \le \beta \le 1, 1 < \alpha \le \frac{4}{3}, n \in N_0\right) \qquad (3)$$

Remark1. If we put $g(z) = \frac{z}{1-z}$ and $n = 0, \beta = 0$ we get the class $V(\alpha)$ studied by Uralegaaddi etc [19], if we put$g(z) = \frac{z}{1-z}$ and $n = 0, \beta = 1$ we get the class $U(\alpha)$ studied by Uralegaaddi etc [19].

Remark2. If we put $g(z) = \frac{z}{1-z}$ and $\beta = 0$ we get the class studied by Dixit and Chandra [8].

Note that in [6] Dixit, Porwal and A. Dixit have also studied the univalent function with positive coefficient by the use of Salagean operator.

Several authors such as ([1-5,7,9],[11],[12],[15-18]) studied the classes of star like and convex functions with negative coefficients only. In the present paper analogues to these results an attempt has been made to study above mentioned classes of functions with positive coefficients associated with the convolution structure defined by Salagean operator. It is worthy to note that, a family of analytic univalent functions related to uniformly star like and uniformly convex functions with positive coefficient in the open unit disc U have been extensively studied by Porwal and Dixit [13].

In fact by taking Salagean operator and convolution structure, we have to made a unified study for the classes introduced by Uralegaddi etc.[19] several properties like coefficient bounds, distortion theorem, theorem involving extreme points and integral means inequalities for this generalized class of functions have been studied.

2. Main Result

The following theorem lays the foundation of our systematic study of the class $\mathcal{A}_g^*(n,\alpha,\beta)$ difined in the preceding section.

Theorem 2.1 A function $f(z)$ given by (1.2) is in $\mathcal{A}_g^*(n,\alpha,\beta)$ iff for $0 \le \beta \le 1, 1 < \alpha \le \frac{4}{3}, n \in N$

$$\sum_{k=2}^{\infty}[(k-\alpha)(1-\beta+\beta k)]k^n a_k b_k \ \le (\alpha-1). \tag{4}$$

Proof. Assume that $f \in \mathcal{A}_g^*(n,\alpha,\beta)$ then in view of (1.2) to (1.3), we have

$$\mathcal{R}e\left\{\frac{(1-\beta)D^{n+1}(f*g)(z)+\beta D^{n+2}(f*g)(z)}{(1-\beta)D^{n}(f*g)(z)+\beta D^{n+1}(f*g)(z)}\right\}$$
$$= \mathcal{R}e\left\{\frac{(1-\beta)z+\sum_{k=2}^{\infty}(1-\beta)k^{n+1}a_k b_k z^k+\beta\left(z+\sum_{k=2}^{\infty}k^{n+2}a_k b_k z^k\right)}{(1-\beta)z+\sum_{k=2}^{\infty}(1-\beta)k^{n}a_k b_k z^k+\beta\left(z+\sum_{k=2p+1}^{\infty}k^{n+1}a_k b_k z^k\right)}\right\}$$
$$= \mathcal{R}e\left\{\frac{z+\sum_{k=2}^{\infty}(1-\beta+k\beta)k^{n+1}a_k b_k z^k}{z+\sum_{k=2}^{\infty}(1-\beta+k\beta)k^{n}a_k b_k z^k}\right\} < \alpha$$

If we choose z to be real and let $r \to 1^-$

$$1+\sum_{k=2}^{\infty}(1-\beta+k\beta)k^{n+1}a_k b_k \le \ \alpha+\alpha\sum_{k=2}^{\infty}(1-\beta+k\beta)k^{n}a_k b_k$$
$$\sum_{k=2}^{\infty}(k-\alpha)(1-\beta+k\beta)k^{n}a_k b_k \le \ \alpha-1$$

Conversely assume that (2.1) holds for $f(z) \in \mathcal{A}$, let us define the function $F(z)$ by

$$F(z)=\frac{(1-\beta)D^{n+1}(f*g)(z)+\beta D^{n+2}(f*g)(z)}{(1-\beta)D^{n}(f*g)(z)+\beta D^{n+1}(f*g)(z)}.$$

It suffices to show that

$$\left|\frac{F(z)-1}{F(z)-(2\alpha-1)}\right| < 1\,. \qquad (z \in U)$$

We note that

$$\left|\frac{F(z)-1}{F(z)-(2\alpha-1)}\right| = \left|\frac{\frac{(1-\beta)D^{n+1}(f*g)(z)+\beta D^{n+2}(f*g)(z)}{(1-\beta)D^{n}(f*g)(z)+\beta D^{n+1}(f*g)(z)}-1}{\frac{(1-\beta)D^{n+1}(f*g)(z)+\beta D^{n+2}(f*g)(z)}{(1-\beta)D^{n}(f*g)(z)+\beta D^{n+1}(f*g)(z)}-(2\alpha-1)}\right|$$

$$= \left|\frac{(1-\beta)D^{n+1}(f*g)(z)+\beta D^{n+2}(f*g)(z)-\{(1-\beta)D^{n}(f*g)(z)+\beta D^{n+1}(f*g)(z)\}}{(2\alpha-1)\{(1-\beta)D^{n}(f*g)(z)+\beta D^{n+1}(f*g)(z)\}-\{(1-\beta)D^{n+1}(f*g)(z)+\beta D^{n+2}(f*g)(z)\}}\right|$$

$$= \left|\frac{\sum_{k=2}^{\infty}(1-\beta+k\beta)k^{n+1}a_k b_k z^k-\sum_{k=2}^{\infty}(1-\beta+k\beta)k^{n}a_k b_k z^k}{(2\alpha-2)z+(2\alpha-1)\sum_{k=2}^{\infty}(1-\beta+k\beta)k^{n}a_k b_k z^k-\sum_{k=2}^{\infty}(1-\beta+k\beta)k^{n+1}a_k b_k z^k}\right|$$

$$= \left|\frac{\sum_{k=2}^{\infty}(1-\beta+k\beta)(k-1)k^{n}a_k b_k z^k}{(2\alpha-2)z+\sum_{k=2}^{\infty}(1-\beta+k\beta)(2\alpha-1-k)k^{n}a_k b_k z^k}\right|$$

The last expression is bounded above by 1 if

$$\sum_{k=2}^{\infty}(1-\beta+k\beta)(k-1)k^{n}a_k b_k \le (2\alpha-2)+\sum_{k=2}^{\infty}(1-\beta+k\beta)(2\alpha-1-k)k^{n}a_k b_k$$

Which is equivalent to the condition (4).This completes the proof of the theorem.

Finally, we note that the assertion (4) of Theorem 2.1 is sharp, the external function being

$$f(z)=z+\frac{(\alpha-1)}{[(k-\alpha)(1-\beta+k\beta)]k^{n}b_k}z^k.$$

Corollary 2.2 Let the function $f(z)$defined by (2) belong to the class $\mathcal{A}_g^*(n,\alpha,\beta)$ then

$$a_k \le \frac{(\alpha-1)}{[(k-\alpha)(1-\beta+k\beta)]k^{n}b_k}, \qquad (k\ge 2).$$

Theorem 2.2 Let $1<\alpha_2\le\alpha_1\le\frac{4}{3},\ n\in N$ and $0\le\beta\le 1$ then

$$\mathcal{A}_g^*(n,\alpha_1,\beta)\supseteq\mathcal{A}_g^*(n,\alpha_2,\beta).$$

Proof : Let the function $f(z)$defined by (3) in the class $\mathcal{A}_g^*(n,\alpha_2,\beta)$. Then by the Theorem 2.1, we have

$$\sum_{k=2}^{\infty}[(k-\alpha_2)(1-\beta+k\beta)]k^{n}a_k b_k \ \le p^{n}(\alpha_2-1)$$

consequently

$$\sum_{k=2}^{\infty}[(k-\alpha_1)(1-\beta+k\beta)]k^{n}a_k b_k \le \sum_{k=2}^{\infty}[(k-\alpha_2)(1-\beta+k\beta)]k^{n}a_k b_k$$
$$\le p^{n}(\alpha_2-1)$$

This completes the proof of the Theorem 2.2 with the aid of the Theorem 2.1.

Theorem 2.3 For $0\le\beta\le 1, 1<\alpha_2\le\alpha_1\le\frac{4}{3},\ n\in N,$

$$\mathcal{A}_g^*(n+1,\alpha,\beta)\subseteq\mathcal{A}_g^*(n,\alpha,\beta).$$

Proof : Let the function $f(z)$ defined by (3) be in the class $\mathcal{A}_g^*(n+1,\alpha,\beta)$. Then by the Theorem 2.1, we have

$$\sum_{k=2}^{\infty}[(k-\alpha)(1-\beta+k\beta)]k^{n+1}a_k b_k \le (\alpha-1)$$

consequently,

$$\sum_{k=2}^{\infty}[(k-\alpha)(1-\beta+k\beta)]k^{n}a_k b_k \le \sum_{k=2}^{\infty}[(k-\alpha)(1-\beta+k\beta)]k^{n+1}a_k b_k \le (\alpha-1).$$

This completes the proof of Theorem 2.3 with the aid of Theorem 2.1.

3. Distortion Inequalities

In this section, we shall prove distortion theorem for the functions belonging to the class

Theorem 3.1 Let the function $f(z)$ of the form (2) be in the class $\mathcal{A}_g^*(n,\alpha,\beta)$ then $|z|=r<1$, we have

$$|f(z)| \le r + \frac{(\alpha-1)r^2}{[(2-\alpha)(1+\beta)]2^n b_2} \qquad (5)$$

and

$$|f(z)| \ge r - \frac{(\alpha-1)r^2}{[(2-\alpha)(1+\beta)]2^n b_2}. \qquad (6)$$

Proof. Since $(z) \in \mathcal{A}_g^*(n,\alpha,\beta)$, we apply Theorem 2.1.

$$(2-\alpha)(1+\beta)2^n b_2 \sum_{k=2}^{\infty} a_k \le \sum_{k=2}^{\infty}[(k-\alpha)(1-\beta+k\beta)]k^{n}a_k b_k \le \alpha-1.$$

Thus we have

$$\sum_{k=2}^{\infty} a_k \le \frac{(\alpha-1)}{[(2-\alpha)(1+\beta)]2^n b_2} \qquad (7)$$

From (1.2) and (3.3) we obtain

$$|f(z)| \le |z| + |z|^2 \sum_{k=2}^{\infty} a_k$$

$$\le r + \frac{(\alpha-1)r^2}{[(2-\alpha)(1+\beta)]2^n b_2}$$

and

$$|f(z)| \geq r - \frac{(\alpha-1)r^2}{[(2-\alpha)(1+\beta)]2^n b_2}.$$

This completes the proof of the Theorem 3.1.

Theorem 3.2 Let the function $f(z)$ of the form (2) be in the class $\mathcal{A}_g^*(n,\alpha,\beta)$ then $|z| = r < 1$, we have

$$|f'(z)| \leq 1 + \frac{(\alpha-1)r}{[(2-\alpha)(1+\beta)]2^{n-1} b_2} \tag{8}$$

and

$$|f'(z)| \geq 1 - \frac{(\alpha-1)r}{[(2-\alpha)(1+\beta)]2^{n-1} b_2}. \tag{9}$$

The equalities in (8) and (9) are attained for the function $f(z)$ given by

$$|f(z)| \leq z + \frac{(\alpha-1)z^2}{[(2-\alpha)(1+\beta)]2^n b_2}.$$

Proof. We have

$$|f'(z)| \leq 1 + \sum_{k=2}^{\infty} k a_k |z|^{k-1} \leq 1 + r\sum_{k=2}^{\infty} k a_k$$

Since $f(z) \in \mathcal{A}_g^*(n,\alpha,\beta)$, we have

$$(2-\alpha)(1+\beta)2^{n-1} b_2 \sum_{k=2}^{\infty} k a_k \leq \sum_{k=2}^{\infty} [(k-\alpha)(1-\beta+k\beta)] k^n a_k b_k \leq \alpha - 1$$

Thus we have

$$\sum_{k=2}^{\infty} k a_k \leq \frac{(\alpha-1)}{[(2-\alpha)(1+\beta)]2^{n-1} b_2}$$

hence

$$|f'(z)| \leq 1 + \frac{(\alpha-1)r}{[(2-\alpha)(1+\beta)]2^{n-1} b_2}.$$

4. Extreme Points

Theorem 4.1 Let $f(z) = z$ and

$$f_k(z) = z + \frac{(\alpha - 1)}{[(k-\alpha)(1-\beta+k\beta)]k^n b_k} z^k$$

$$\left(b_k \geq 0, 0 \leq \beta \leq 1, 1 \leq \alpha < \frac{4}{3}, n \in N\right),$$

Then $f(z) \in \mathcal{A}_g^*(n, \alpha, \beta)$ if and only if it can be expressed in the following form :

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z),$$

where$\lambda_k \geq 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) = z + \sum_{k=2}^{\infty} \lambda_k \frac{(\alpha - 1)}{[(k-\alpha)(1-\beta+k\beta)]k^n b_k} z^k.$$

Then from Theorem 2.1, we have

$$\sum_{k=2}^{\infty} [(k-\alpha)(1-\beta+k\beta)]k^n b_k \lambda_k \frac{(\alpha - 1)}{[(k-\alpha)(1-\beta+k\beta)]k^n b_k} b_k$$

$$= \sum_{k=2}^{\infty} (\alpha - 1)\lambda_k = (1-\lambda_1)(\alpha - 1) \leq (\alpha - 1).$$

Thus, in view of Theorem 2.1, we find that $f(z) \in \mathcal{A}_g^*(n, \alpha, \beta)$.

Conversely, suppose that $f(z) \in \mathcal{A}_g^*(n, \alpha, \beta)$.Then, since

$$a_k \leq \frac{(\alpha - 1)}{[(k-\alpha)(1-\beta+k\beta)]k^n b_k},$$

we may set

$$\lambda_k = \frac{[(k-\alpha)(1-\beta+k\beta)]k^n b_k}{(\alpha - 1)} a_k$$

and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k.$$

Thus, clearly, we have

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z).$$

This completes the proof of theorem.

Corollary 4.2 The extreme points of the class $\mathcal{A}_g^*(n,\alpha,\beta)$ are given by

$$f_1(z)=z$$

and

$$f_k(z)=z+\frac{(\alpha-1)}{[(k-\alpha)(1-\beta+k\beta)]k^n b_k}z^k, \qquad (k\geq 2). \qquad (10)$$

Theorem 4.3 The class $\mathcal{A}_g^*(n,\alpha,\beta)$ is a convex set.

Proof : Suppose that each of the functions $f_i(z)$, $(i=1,2)$given by

$$f_i(z)=z+\sum_{k=2}^{\infty}a_{k,i}z^k, \qquad (a_{k,i}\geq 0)$$

is in the class $\mathcal{A}_g^*(n,\alpha,\beta)$. It is sufficient to show that the function $g(z)$ defined by

$$g(z)=\eta f_1(z)+(1-\eta)f_2(z), \qquad (0\leq\eta<1)$$

is also in the class $\mathcal{A}_g^*(n,\alpha,\beta)$. Since

$$g(z)=\eta\left(z+\sum_{k=2}^{\infty}a_{k,1}z^k\right)+(1-\eta)\left(z+\sum_{k=2}^{\infty}a_{k,2}z^k\right)$$

$$=z+\sum_{k=2}^{\infty}[\eta a_{k,1}+(1-\eta)a_{k,2}]z^k$$

with the aid of Theorem 2.1, we have

$$\sum_{k=2}^{\infty}(k-\alpha)(1-\beta+k\beta)k^n\,[\eta a_{k,1}+(1-\eta)a_{k,2}]b_k$$

$$=\eta\sum_{k=2p+1}^{\infty}[(k-\alpha)(1-\beta+k\beta)]k^n a_{k\,,1}b_k+(1-\eta)\sum_{k=2p+1}^{\infty}[(k-\alpha)(1-\beta+k\beta)]k^n a_{k,2}b_k$$

$$\leq\eta(\alpha-1)+(1-\eta)(\alpha-1)=(\alpha-1).$$

Which completes the proof of the theorem.

5. Integral Means Inequalities

In 1925, Littelewood prove the following subordination theorem.

Theorem 5.1 (Littlewood [10]) If f and g are analytic in U with $f \prec g$, then for $\mu > 0$ and

$$z = re^{i\theta} (0 < r < 1)$$

$$\int_0^{2\pi} |f(z)|^\mu \, d\theta \ \leq \int_0^{2\pi} |g(z)|^\mu \, d\theta.$$

We will make use of Theorem 5.1 to prove

Theorem 5.2 Let $f(z) \in \mathcal{A}_g^*(n,\alpha,\beta)$ and $f_k(z)$ is defined by (10). If there exist an analytic function $w(z)$ given by

$$[w(z)]^{k-p} = \frac{[(k-\alpha)(1-\beta+k\beta)]k^n b_k}{(\alpha-1)} \sum_{k=2}^{\infty} a_k z^{k-1},$$

then for $z = re^{i\theta} (0 < r < 1)$

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\mu d\theta \ \leq \int_0^{2\pi} \left|g(re^{i\theta})\right|^\mu d\theta. \qquad (\mu > 0).$$

Proof . We must show that

$$\int_0^{2\pi} \left|1 + \sum_{k=2}^{\infty} a_k z^{k-1}\right|^\mu d\theta \ \leq \int_0^{2\pi} \left|1 + \frac{(\alpha-1)}{[(k-\alpha)(1-\beta+k\beta)]k^n b_k} z^{k-1}\right|^\mu d\theta.$$

By applying Littlewood's subordition theorem, it would suffice to show that

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 + \frac{(\alpha-1)}{[(k-\alpha)(1-\beta+k\beta)]k^n b_k} z^{k-1}.$$

By setting

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} = 1 + \frac{(\alpha-1)}{[(k-\alpha)(1-\beta+k\beta)]k^n b_k} [w(z)]^{k-1},$$

we find that

$$[w(z)]^{k-1} = \frac{[(k-\alpha)(1-\beta+k\beta)]k^n b_k}{(\alpha-1)} \sum_{k=2}^{\infty} a_k z^{k-1}$$

which readily yields $w(0) = 0$.

Furthermore, using (4) we obtain

$$|w(z)|^{k-1} \le \left| \frac{[(k-\alpha)(1-\beta+k\beta)]k^n b_k}{(\alpha-1)} \sum_{k=2}^{\infty} a_k z^{k-1} \right|$$

$$\le \frac{[(k-\alpha)(1-\beta+k\beta)]k^n b_k}{(\alpha-1)} \sum_{k=2}^{\infty} a_k |z|^{k-1}$$

$$\le |z|^{k-1} < 1$$

This completes the proof of the theorem.

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