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SOME COMPOSITE CONSTRUCTION OF GROUP DIVISIBLE DESIGNS

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Abstract : Some new methods of constructing group divisible designs are presented. The methods are based upon the idea of augmentation of incidence matrices of incomplete block designs. A table of such designs is also given. Plans of the tabulated designs can be obtained using available sources.

Keywords : GD design, composite GD design.

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1. Introduction

Since the introduction of balanced incomplete block designs by Yates (1936), a wide variety of partially balanced incomplete block designs have been suggested in the literature. Group divisible (GD) designs are the most important type of 2-associate partially balanced incomplete block (PBIB) designs. A GD design is an arrangement of $v = mn$ treatments in b blocks such that each block contains k ($<v$) distinct treatments; each treatment is replicated r times, and the set of treatments can be partitioned into $m(\geq 2)$ groups of $n(\geq 2)$ treatments each, any two distinct

treatments occurring together in $\lambda_1$blocks if they belong to the same group, and in λ_2 blocks if they belong to different groups. Furthermore, if $r - \lambda_1 = 0$, the GD design is said to be singular; if $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$, it is called semi-regular (SR); and if $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$, it is called regular (R).

Clatworthy [6] tabulated 443 parameters combinations of GD designs with their solutions. Since then Freeman [8], Kageyama and Tanaka [9], Bhagwandas and Parihar [5], Banerjee et al. [1 & 2], Bhagwandas et al. [4], Dey and Nigam [7], Banerjee and Kageyama [3], Sinha and Kageyama [11] and Sinha [10] have given several methods of constructing GD designs.

In this paper, some new composition approaches of constructing GD designs are described, with the composition of a balanced incomplete block (BIB) design and a GD design, by making use of augmentation process of incidence matrices. The search for new designs has been limited to the range of parameters $r, k \leq 15$.

2. Construction

In this section, we describe the composition process of obtaining regular group divisible designs by making use of the incidence matrices of BIB designs and GD designs.

Theorem 2.1 *The existence of a BIBD design with parameters*

$$v, b, r, k, \lambda_1, \lambda_2; m, n \qquad .....(1)$$

and a semi-regular group divisible design with parameters

$$v, b', r', k, \lambda_1', \lambda_2'; m, n \qquad ..... (2)$$

implies the existence of a regular group divisible design with parameter

$$v^* = v, b^* = b + b', r^* = r + r', k^* = k, \lambda_1^* = \lambda_1' + \lambda, \lambda_2^* = \lambda_2' + \lambda; m, n \qquad ..... (3)$$

Proof: Let S be the incidence matrix of BIBD with the parameters given in (1) and S^* be the incidence matrix of SRGD design with the parameters given in (2). Then by the process of composition, we obtain the incidence structure

$$M=(S{:}S^*)$$

which is the incidence matrix of a RGD design with the parameters given in (3).

Let S' denotes the transpose of S, without loss of generality SS' can be written as

$$SS' = \begin{pmatrix} r-\lambda & \lambda & \dots & \lambda \\ \lambda & r-\lambda & \dots & \lambda \\ . & . & \dots & . \\ \lambda & \lambda & \dots & r-\lambda \end{pmatrix} \qquad \text{..... (4)}$$

Also denote $S^{*'}$ as the transpose of S^*, then $S^{*'}\ S^{*'}$ can be written as

$$S^*S^{*'} = \begin{pmatrix} B_{11}^* & B_{12}^* & \dots & B_{1m}^* \\ B_{21}^* & B_{22}^* & \dots & B_{2m}^* \\ . & . & \dots & . \\ B_{m1}^* & B_{m2}^* & \dots & B_{mm}^* \end{pmatrix} \qquad \text{..... (5)}$$

where B_{ij}^* is an $n \times n$ sub matrix (*i, j*=1, 2... m) given by

$$B_{ij}^* = \left\{\left(r' - \lambda_1'\right)I_n + \left(\lambda_1' - \lambda_2'\right)J_n\right\}\delta_{ij} + \lambda_2' J_n \qquad \text{..... (6)}$$

where δ_{ij} being Kronecker delta, takes the values 1 if $i = j$ and 0 if $i \neq j$.

If $M=(S{:}S^*)$ be a (0,1) incidence matrix of order $v^* \times (b+b')$, then $MM' = SS' + S^*S^{*'}$. Hence from (4), (5) and (6), we get

$$MM' = \begin{pmatrix} A & B & \dots & B \\ B & A & \dots & B \\ . & . & \dots & . \\ B & B & \dots & A \end{pmatrix} \qquad \text{...(7)}$$

where $A = \{(r + r') - (\lambda + \lambda_1')\} I_n + (\lambda + \lambda_1') J_n,$ ...(8)

$$B = (\lambda + \lambda_2') J_n \quad ...(9)$$

M is a (0, 1) matrix and can be identified as the incidence matrix of a GD design D^* in which rows correspond to treatments and columns correspond to blocks. Since in *M* the row sum is $r + r'$ and column sum is k, therefore in $D^*, r^* = r + r'$ and $k^*=k$.

MM' being treatment structure matrix of D^*, its form (7) shows that treatments can be grouped into m groups in such a way that any two treatments belonging to different groups occur together in $\lambda_2^* = \lambda_2' + \lambda$ blocks, whereas any two treatments belonging to the same group (as evidenced by diagonal sub matrices of *MM'*) occur together in $\lambda_1^* = \lambda_1' + \lambda$ blocks. Hence M is the incidence matrix of a GD design D^* with the parameters given in (3).

It is to be noted that GD design D^* thus constructed is always regular GD design, since $r + r' > \lambda + \lambda_1'$, and $r^* k^* > v^* \lambda_2^*$ are satisfied.

As an illustration, consider BIBD (6, 10, 5, 3, 2) and SR 19 (6, 8, 4, 3, 0, 2; *m*=3, *n*=2), we obtain RGD design with parameters $v^* = 6, b^* = 18, r^* = 9, k^* = 3,$ $\lambda_1^* = 2, \lambda_2^* = 4;\ m = 3, n = 2.$ This is the same design as R51 in Clatworthy [6], our solution is new non-isomorphic to the solution presented in Clatworthy [6]. In Clatworthy solution some blocks are repeated two times but in our solution it is not the case. The efficiency of the design E=0.79.

Theorem 2.2 *The existence of a BIBD* (v, b, r, k, λ) *and a RGD design* $(v', b', r', k', \lambda_1', \lambda_2'; m, n)$ *implies the existence of regular GD design with parameter*

$$v^* = v = v', b^* = b + b', r^* = r + r', k^* = k = k', \lambda_1^* = \lambda_1' + \lambda, \lambda_2^* = \lambda_2' + \lambda; m, n$$

Proof: Obvious.

As an illustration consider BIBD (6, 10, 5, 3, 2) and R42 (6, 6, 3, 3, 2, 1; m=3, n=2), we obtain regular GD design with the parameters

$$v^* = 6, b^* = 16, r^* = 8, k^* = 3, \lambda_1^* = 4, \lambda_2^* = 3;\ m = 3, n = 2.$$

This is the same design as R48 in Clatworthy (1973). It is obvious that the present solution is non-isomorphic to the reported solution in Clatworthy (1973). In Clatworthy solution no any block is repeated whereas in our solution some blocks are repeated. Furthermore, the solution reported in Clatworthy (1973) is 2-resolvable, but our solution is nonresolvable.

Theorem 2.3 *The existence of a SRGD design with parameters*

$$v = mn, b, r, k, \lambda_1, \lambda_2; m, n \qquad \text{..... (10)}$$

for positive integers $m(\geq 2), n(\geq 2)$, *and a RGD design with parameters*

$$v' = mn, b', r', k' = k, \lambda_1', \lambda_2'; m, n \qquad \text{..... (11)}$$

implies the existence of a RGD design with parameters

$$v^* = v = v', b^* = b + b', r^* = r + r', k^* = k, \lambda_1^* = \lambda_1' + \lambda_1, \lambda_2^* = \lambda_2' + \lambda_2; m, n \qquad \text{...(12)}$$

Proof: Let N be the incidence matrix of SRGD design with the parameters given in (10) and N^* be the incidence matrix of RGD design with the parameters given in (11). Then by the composition method, we have incidence structure as $M = (N : N^*)$ which is the incidence matrix of a RGD design with the parameters given in (12).

Suppose N' be the transpose of N. Then NN' can be written as

$$NN' = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ . & . & \cdots & . \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{pmatrix} \qquad \text{..... (13)}$$

where B_{ij} is an $n \times n$ sub matrix (i, j = 1, 2… m) given by

$$B_{ij} = \{(r - \lambda_1)I_n + (\lambda_1 - \lambda_2)J_n\}\delta_{ij} + \lambda_2 J_n, \qquad \text{.....(14)}$$

where δ_{ij} being Kronecker delta, takes the values 1 if $i = j$ and 0 if $i \neq j$.

Denoting $N^{*'}$ the transpose of N^*. Without loss of generality $N^* N^{*'}$ can be written as

$$N^*N^{*'} = \begin{pmatrix} B^*_{11} & B^*_{12} & \cdots & B^*_{1m} \\ B^*_{21} & B^*_{22} & \cdots & B^*_{2m} \\ . & . & \cdots & . \\ B^*_{m1} & B^*_{m2} & \cdots & B^*_{mm} \end{pmatrix} \qquad \text{..... (15)}$$

where B_{ij} is an $n \times n$ sub matrix (i, j = 1, 2… m) given by

$$B^*_{ij} = \{(r' - \lambda'_1)I_n + (\lambda'_1 - \lambda'_2)J_n\}\delta_{ij} + \lambda'_2 J_n$$

where δ_{ij} is 1 if $i = j$ and 0 if $i \neq j$.

If M=(N:N*) be a (0,1) incidence matrix of order $v^* \times (b + b')$, then $MM' = NN' + N^*N^{*'}$. Hence

$$MM' = \begin{bmatrix} A & B & B & \cdots & B \\ B & A & B & \cdots & B \\ B & B & A & \cdots & B \\ . & . & . & \cdots & . \\ B & B & B & \cdots & A \end{bmatrix} \qquad \text{..... (16)}$$

where $A=\left\{\left(r+r^{'}\right)-\left(\lambda_1+\lambda_1^{'}\right)\right\}I_n+\left(\lambda_1+\lambda^{'}\right)J_n,$ ..... (17)

$$B=\left(\lambda_2+\lambda_2^{'}\right)J_n. \quad .... (18)$$

M can be identified as the incidence matrix of a GD design D^* in which row sum is $r+r^{'}$ and column sum is k, therefore in $D^*, r^*=(r+r^{'})$ and $k^*=k.$

From (17) it is evident that $\lambda_1^*=\lambda_1^{'}+\lambda_1$ and from (18), $\lambda_2^*=\lambda_2^{'}+\lambda_2$. The other parameters are obvious.

Now we shall show that GD design D^* thus constructed is always regular GD design, since $r+r^{'}>\lambda_1^{'}+\lambda_1$ is satisfied. Also, we have,

$$\left(r+r^{'}\right)k^*-v^*\left(\lambda_2+\lambda_2^{'}\right)=\left(rk^*-v^*\lambda_2\right)+\left(r^{'}k^*-v^*\lambda_2^{'}\right) \quad ...(19)$$

If N be the incidence matrix of semi regular GD design, then first bracket on the right hand side of (19) is zero, the second is positive and therefore it is always positive. Hence GD design D^* will always be regular.

As an application of Theorem 2.3, consider SR 24 and R 59. The resulting GD design is of regular type with the parameters $33, r^*=11, k^*=3,$ $v^*=9, b^*=\lambda_1^*=2, \lambda_2^*=3;\ m=3, n=3.$ This regular GD design seems to be new because it is not available in the existing lists.

Corollary 2.1: If N is the incidence matrix of a regular GD design with parameters $(v\ , b, r\ , k\ , \lambda_1\ , \lambda_2\ ; m, n)$ and N^* be the incidence matrix of another regular GD design with parameters $(v^{'}, b^{'}, r^{'}, k^{'}, \lambda_1^{'}, \lambda_2^{'}; m, n)$, then the incidence pattern

$$M=(N:N^*)$$

yields the regular GD design with parameters

$$v^* = v = v', b^* = b + b', r^* = r + r', k^* = k, \lambda_1^* = \lambda_1' + \lambda_1, \lambda_2^* = \lambda_2' + \lambda_2; m, n \quad \dots (20)$$

The application of corollary 2.1 to the RGD R54 and another RGD R55 gives a RGD R57 with parameters

$$v^* = 8, b^* = 24, r^* = 9, k^* = 3, \lambda_1^* = 0, \lambda_2^* = 3; m = 4, n = 2.$$

The methods described in Theorems 2.1, 2.2 and 2.3 and corollary 2.1 are useful for the combinatorial construction of GD designs, but they may produce designs with relatively large parameter values. In this table, we have listed regular GD designs in the range $r, k \le 15$. The references to design numbers are from Clatworthy [6].

Table: Regular GD designs with $r, k \le 15$

S.No.	v^*	b^*	r^*	k^*	λ_1^*	λ_2^*	m	n	source
1	9	42	14	3	5	3	3	3	Th. 2.2
2	9	42	14	3	8	2	3	3	Th. 2.2
3	9	33	11	3	5	2	3	3	Th. 2.2
4	9	33	11	3	2	3	3	3	Th. 2.2
5	9	36	12	3	6	2	3	3	Th. 2.2
6	9	39	13	3	7	2	3	3	Th. 2.2
7	9	39	13	3	4	3	3	3	Th. 2.2
8	8	22	11	4	3	5	4	2	Th. 2.1
9	8	26	13	4	3	6	4	2	Th. 2.1
10	8	26	13	4	5	6	2	4	Th. 2.1
11	8	30	15	4	3	7	4	2	Th. 2.1
12	20	70	14	4	3	2	4	5	Th. 2.3
13	16	56	14	4	6	2	4	4	Th. 2.3
14	20	56	14	5	8	2	5	4	Th. 2.3
15	18	42	14	6	5	4	6	3	Th. 2.3
16	18	30	15	9	8	7	9	2	Th. 2.3

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