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MATRIX GEOMETRIC SOLUTION OF UNRELIABLE SERVER M/M/1 QUEUEING SYSTEM WITH SECOND OPTIONAL SERVICE

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Abstract : An M/M/1 queueing system with second optional service and unreliable server is studied in existing paper. The customers arrive to the system according to Poisson process with state dependent rates depending upon the server's status. All customers demand the first essential service whereas only some of them demand the second optional service. A customer either may leave the system after the first essential service with probability $(1-r)$, or at the completion of the first essential service go for second optional service with probability r $(0 \le r \le 1)$. The matrix geometric technique is used for the analysis of the concerned queueing system. The sensitive analysis is also performed to examine the variation of the system performance characteristics with various input parameters.

Keywords : Second optional service, matrix geometric technique, server breakdown, repair, queue size, throughput.

2010 Mathematics Subject Classification : 60J05

1. Introduction

Most studies on queueing models have considered the main service, however in many real service systems some customers require the main as well as subsidiary services provided by the server. In many real cases, the server may experience breakdowns, so that a more realistic queueing model is that which incorporates the assumption of unreliable server. Also many authors have paid attention on matrix geometric approach which is used to solve the more complex queueing problems (cf. Netus, [4]).

$M^x/G/1$ queueing system was studied with additional second phase of optional service and unreliable server by Choudhury et al. [1]. Yang and Alfa [5] studied a multi server queueing system with identical unreliable server with phase type distributed service time. A single server working vacation queueing model with multiple types of server breakdown via matrix geometric approach was described by Jain and Jain [3]. M/G/1 unreliable server queue with two phases of services are studied by Choudhury and Tadj [2].

In this paper we develop a M/M/1 queueing model with second optional service and unreliable server. The paper is organized as follows. In next section 2, we outline the underlying assumptions and notations to develop mathematical model under study. The governing steady state equations are constructed by taking appropriate transition rates. In section 3, matrix geometric solution of the system is given. In section 4, various performance characteristics of the system are formulated explicitly in terms of steady sate probabilities. In next section 5, we perform comparative study of the system characteristics for various input parameters. In the last section 6, conclusions and future scopes of the work are provided.

2. Mathematical Modeling

We consider a single server vacation model with second optional service and unreliable server. The following assumptions are made to describe the model;

The customers arrive to the system according to Poisson process with rate λB.

Two types of services are provided to the customers, the first essential service is needed to all arriving customers with mean rate μ_1. As soon as the first service of a customer is completed, then with probability r, he may opt for the second service or else with probability $(1-r)$, he may opt to leave the system. The second service times are assumed to be exponentially distributed with mean service rate μ_2.

Assume that the life time of the server is exponentially distributed with mean $1/\alpha_1$ in first essential service. In second optional service, the server may fails according to exponential distribution with rate α_2.

After breakdown, the server immediately sent for repair. The repair time distributions while server fails during essential and optional service phases are exponentially distributed with mean rate β_1 and β_2, respectively.

Immediately after the server is fixed, it starts to serve the customers.

For mathematical formulation purpose, we define the following steady state probabilities:

$P(1,n,B)$: Probability that there are n customers in the system when the server is rendering first essential service

$P(2,n,B)$: Probability that there are n customers in the system when the server is rendering second optional service

$P(3,n,B)$: Probability that there are n customers in the system and the server is in broken down state while rendering first essential service

$P(4,n,B)$: Probability that there are n customers in the system and the server is in broken down state while rendering second optional service

The steady state equations governing the model are constructed as follows:

$$(\alpha_1+\lambda_B+(1-r)\mu_1+r\mu_1)P(1,1,B)=\beta_1P(3,1,B)+\mu_2P(2,2,B)+(1-r)\mu_1P(1,2,B) \quad ...(1)$$

$$(\alpha_1+\lambda_B+(1-r)\mu_1+r\mu_1)P(1,n,B)=\lambda_BP(1,n-1,B)+\beta_1P(3,n,B)+\mu_2P(2,n+1,B)$$
$$+(1-r)\mu_1P(1,n+1,B), n<1 \quad ...(2)$$

$$(\lambda_B+\alpha_2+\mu_2)P(2,1,B)=r\mu_1P(1,1,B)+\beta_2P(4,1,B) \quad ...(3)$$

$$(\lambda_B+\alpha_2+\mu_2)P(2,n,B)=r\mu_1P(1,n,B)+\beta_2P(4,n,B)+\lambda_BP(2,n-1,B), n>1 \quad ...(4)$$

$$(\lambda_B+\beta_1)P(3,1,B)=\alpha_1P(1,1,B) \quad ...(5)$$

$$(\lambda_B+\beta_1)P(3,n,B)=\alpha_1P(1,n,B)+\lambda_BP(3,n-1,B), \quad n>1 \quad ...(6)$$

$$(\lambda_B+\beta_2)P(4,1,B)=\alpha_2P(2,1,B) \quad ...(7)$$

$$(\lambda_B+\beta_2)P(4,n,B)=\alpha_2P(2,n,B)+\lambda_BP(4,n-1,B), \quad n>1 \quad ...(8)$$

3. Matrix Geometric Solution

The theory of matrix geometric approach was developed (cf.[4]) to solve the stationary state probabilities for the vector state Markov Process with repetitive structure. Consider the generator matrix Q as shown below:

$$Q = \begin{bmatrix} B_{00} & B_{01} & & & & & & & \\ B_{10} & A_1 & A_0 & & & & & & \\ & A_2 & A_1 & A_0 & & & & & \\ & & & & --- & & & & \\ & & & & & --- & & & \\ & & & & & A_2 & A_1 & A_0 & \\ & & & & & & A_2 & A_1 & A_0 \\ & & & & & & & & --- \end{bmatrix}$$

The matrix can be decomposed in to sub matrices $B_{00}, B_{01}, B_{10}, A_0, A_1, A_2$ are as follows:

$$B_{00} = [-\lambda_B]\,; \qquad B_{01} = [\lambda_B \ 0 \ 0 \ 0]$$

$$B_{10} = \begin{bmatrix} (1-r)\mu_1 \\ \mu_2 \\ 0 \\ 0 \end{bmatrix}; \qquad A_0 = \begin{bmatrix} \lambda_B & 0 & 0 & 0 \\ 0 & \lambda_B & 0 & 0 \\ 0 & 0 & \lambda_B & 0 \\ 0 & 0 & 0 & \lambda_B \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -(\lambda_B + \alpha_1 + \mu_1) & r\mu_1 & \alpha_1 & 0 \\ 0 & -(\lambda_B + \alpha_1 + \mu_1) & 0 & \alpha_2 \\ \beta_1 & 0 & -(\lambda_B + \beta_1) & 0 \\ 0 & \beta_2 & 0 & -(\lambda_B + \beta_1) \end{bmatrix}$$

$$A_2 = \begin{bmatrix} (1-r)\mu_1 & 0 & 0 & 0 \\ \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let X be the vector of the steady state probabilities with Q as coefficient matrix, such that $XQ = 0$, and the normalizing condition is $Xe = 1$, where e is the column vector of appropriate dimension with all elements equal to 1.

Let us partition X as $X = [X_1, X_2, ...]$, where

$X_i = [X_{1,i,B}, X_{2,i,B}, X_{3,i,B}, X_{4,i,B}], \ i \geq 1$

We examine the existence of a solution of the form

$$X_i = X_{i-1}R \quad \text{or} \quad X_i = X_1 R^{i-1}, i \geq 1 \qquad ...(9)$$

Because Xi depends only on the state transition between level (i-1) and level i, the balance equations are given by

$$X_0 B_{00} + X_1 B_{10} = 0 \qquad ...(10)$$

$$X_0 B_{01} + X_1 A_1 + X_2 A_2 = 0 \qquad ...(11)$$

$$X_{i-1} A_0 + X_i A_1 + X_{i+1} A_2 = 0, \ i > 1 \qquad ...(12)$$

In equation (9), R is a square matrix and is the unique minimal non negative solution to the non linear matrix equation

$$A_0 + RA_1 + R^2 A_2 = 0 \qquad ...(13)$$

The matrix R can be computed by successive substitution in the recurrence relation $R(0)=0$

$$R(n+1) = -A_0 A_1^{-1} - R^2(n) A_2 A_1^{-1}, \qquad n \geq 0 \qquad ...(14)$$

Finally we are interested to calculate the vector $X = [X_0, X_1, X_2, ...]$ for this purpose. The balance equations for the boundary states given by equations (11) and (12) can be written in matrix form as

$$X_0(A_1 + RA_2) = 0 \qquad ...(15)$$

$$X_0(I - R)^{-1} e = 1 \qquad ...(16)$$

where e is the column matrix of suitable dimension having all elements 1; this gives a unique solution for $X = [X_0, X_1, X_2, ...]$.

4. Performance Measures

The validity of the model and the system performance characteristics can be analyzed by computing the system performance characteristics, in terms of the steady state probabilities explicitly. Some of the system performance indices are as follows:

Probability that the server is busy with first essential service is

$$P(B_1) = \sum_{i=1}^{\infty} P(1,i,B) \qquad ...(17)$$

Probability that the server is busy with second optional service is

$$P(B_2) = \sum_{i=1}^{\infty} P(2,i,B) \qquad ...(18)$$

Probability that the server is in broken down state while failed during first essential service is

$$P(D_1) = \sum_{i=1}^{\infty} P(3,i,D) \qquad ... (19)$$

Probability that the server is in broken down state while failed during second optional service is

$$P(D_2) = \sum_{i=1}^{\infty} P(4,i,D) \qquad ... (20)$$

Average number of customers in the system is

$$E[N] = \sum_{i=1}^{\infty} i\left[P(1,i,B) + P(2,i,B) + P(3,i,D) + P(4,i,D)\right] \qquad ... (21)$$

Throughput is given by

$$T(P) = \sum_{i=1}^{\infty} [\mu_1 P(1,i,B) + \mu_2 P(2,i,B)] \qquad ... (22)$$

5. Sensitivity Analysis

In order to determine the performance of the queueing system with second optional service and unreliable server, we perform a computational experiment by a program developed in MATLAB using matrix geometric technique discussed in section 3. The table 1 depicts the variation of the system performance characteristics with respect to the various input parameters. We assume the following basic input data for table:

Table 1: $\mu_1 = 6,\ \mu_2 = 4,\ \alpha_1 = 0.05,\ \alpha_2 = 0.03,\ \beta_1 = 3.5,\ \beta_2 = 2.5,\ r = 0.2$

Table 1 shows the variation in long run probabilities of different states of the server by varying arrival rates. It is clear from tables that PB1, PB2, PD1, PD2 have increased values for higher arrival rates.

λ_B	P_{B1}	P_{B2}	P_{D1}	P_{D2}
0.1	0.0016	0.0519	0.0172	0.0233
0.2	0.0034	0.0819	0.0212	0.0476
0.3	0.0055	0.1191	0.0264	0.0743
0.4	0.0080	0.1641	0.0322	0.1054
0.5	0.0111	0.2179	0.0381	0.1434
0.6	0.0150	0.2819	0.0437	0.1917
0.7	0.0202	0.3582	0.0486	0.2546
0.8	0.0269	0.4500	0.0524	0.3385
0.9	0.0358	0.5630	0.0551	0.3520

Table 1: Long run probabilities of the server status by varying λ_B

6. Concluding Remarks

In many stochastic systems there may occur a situation in which the first service is essential to all arrivals whereas second service is needed by only some of them. Similarly breakdown is a remarkable and unavoidable phenomenon in the service facility of a queueing system, because system performance deteriorates seriously by the server breakdown and limitations of the repair capacity.

The inclusion of realistic factors such as unreliable server, optional service etc. makes our model more versatile from application point of view. By using matrix geometric solution we have obtained some important performance measures, which may be useful for the system designers and practitioners involved in many industrial organizations operating in congestion scenario.

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