

SOME PROPERTIES OF m -PROJECTIVE CURVATURE TENSOR IN A P -SASAKIAN MANIFOLD

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Abstract: The object of present paper is to study some properties of m –projective curvature tensor in a P -Sasakian manifold.

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1. Introduction

Sato[12,13] introduced the notation of an (almost) para-contact structure, either P -Sasakian manifold or SP -Sasakian manifold and obtain very interesting results about such manifolds. In 1970, Pokhariyal and Mishra [10] defined tensor fields W^* in the Riemannian manifold known as m -projective curvature tensor. Later, Ojha [7,8] defined and studied some properties of this curvature tensor in a Sasakian manifolds and Kähler manifolds. Singh[14], Taleshian and N. Asghari [12] studied m -projective curvature tensor in P -Sasakian manifold. Also m -projective curvature tensor have been studied by Chaubey and Ojha [4], Chaubey [3], Prakash et al. [11], Singh et al. 15,16] and many others in different structures.

The paper is organized as follows. In section 2, we give a brief account of P - Sasakian manifold. In section 3 we study m -projectively semi-symmetric P -Sasakian manifold. Section 4 deals with ξ - m -projectively flat P -Sasakian manifold. In section 5 we study quasi m -projectively flat P -Sasakian manifold. In the next section we proved that m -projectively recurrent P - Sasakian manifold is m -projectively semi-symmetric P - Sasakian manifold and hence P - Sasakian manifold is SP -Sasakian manifold. The last section deals with an Einstein P -Sasakian manifold satisfying $(div W^*)(X, Y)Z = 0$.

2. P-Sasakian Manifold

Let M be an n -dimensional differentiable manifold on which there exists a $(1,1)$ tensor field ϕ , a vector field ξ and 1-form η satisfying

$$\phi^2 = X - \eta(X)\xi, \quad (1)$$

$$\eta(\xi) = 1, \quad (2)$$

$$\eta \circ \phi = 0, \quad (3)$$

$$\phi\xi = 0 \quad (4)$$

is called an almost para contact manifold and the structure (ϕ, ξ, η) is called an almost para contact structure.

The first and one of the remaining last three above relations imply the other two relations. Let g be a compatible Riemannian metric with (ϕ, ξ, η) -structure such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (5)$$

or, equivalently,

$$g(\phi X, Y) = g(X, \phi Y) \quad \text{and} \quad g(X, \xi) = \eta(X) \quad (6)$$

for all $X, Y \in TM$.

Then M is called an almost para contact Riemannian manifold or an almost para contact metric manifold with an almost para contact Riemannian structure- (ϕ, ξ, η, g) .

Definition: An almost para contact Riemannian manifold is called P-Sasakian manifold if

$$(\nabla_X \phi)(Y) = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \quad (7)$$

for all $X, Y \in TM$.

where ∇ denotes the operator of co-variant differentiation with respect to Riemannian metric g .

On P-Sasakian manifold, we have

$$(\nabla_X \eta)(Y) = g(\phi X, Y) = (\nabla_Y \eta)(X) \quad (8)$$

$$(\nabla_X \eta)(Y) = \Phi(X, Y) \quad \text{where} \quad \Phi(X, Y) \stackrel{\text{def}}{=} g(\phi X, Y) \quad (9)$$

$$(\nabla_X \xi) = \phi X \quad (10)$$

Also in an P- Sasakian manifold M , the curvature tensor R , the Ricci tensor S , and the Ricci operator Q satisfy

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X \quad (11)$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi \quad (12)$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \quad (13)$$

$$S(X, \xi) = -(n-1)\eta(X) \quad (14)$$

$$Q\xi = -(n-1)\xi \quad (15)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y) \quad (16)$$

$$S(X, \phi Y) = S(X, Y) \quad (17)$$

Definition: An almost paracontact Riemannian manifold is said to be η -Einstein [17] if the Ricci tensor S satisfy

$$S(X, Y) = a g(X, Y) + b \eta(X)\eta(Y) \quad (18)$$

Where a and b are smooth functions on the manifold .In particular, if $b = 0$, then M is an Einstein manifold.

Pokhariyal[9], Pokhariyal and Mishra [10] have defined a tensor field W^* on a Riemannian manifold as

$$W^*(X, Y)Z = R(X, Y)Z + \frac{1}{2(n-1)} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \quad (19)$$

for arbitrary vector fields X, Y, Z where S is Ricci tensor of type $(0,2)$ so that

$$W^*(X, Y, Z, U) = g(W^*(X, Y)Z, U) = W^*(Z, U, X, Y)$$

Such a tensor field W^* is called m –projective curvature tensor.

It can be easily shown that in a P -Sasakian manifold the m – projective curvature tensor W^* satisfies the following relations

$$W^*(X, Y)\xi = \frac{1}{2} [\eta(X)Y - \eta(Y)X] - \frac{1}{2(n-1)} [\eta(Y)QX - \eta(X)QY] \quad (20)$$

$$\eta(W^*(X, Y)\xi) = 0 \quad (21)$$

$$W^*(\xi, Y)Z = \frac{1}{2} [\eta(Z)Y - g(Y, Z)\xi] - \frac{1}{2(n-1)} [S(Y, Z)\xi - \eta(Z)QY] \quad (22)$$

$$\eta(W^*(\xi, Y)Z) = -\frac{1}{2} \left[g(Y, Z) + \frac{1}{(n-1)} S(Y, Z) \right] \quad (23)$$

$$\eta(W^*(X, Y)Z) = \frac{1}{2} [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)] - \frac{1}{2(n-1)} [S(Y, Z)\eta(X) - S(X, Z)\eta(Y)] \quad (24)$$

3. m -Projectively Semi-symmetric P -Sasakian Manifold

Definition: An n -dimensional P -Sasakian manifold is called W^* -semi-symmetric if it satisfies

$$R(X, Y).W^* = 0 \quad (25)$$

Where R is the Riemannian curvature tensor and to be considered as a derivation of the tensor algebra at each point of the manifold for tangents vectors X, Y .

From (25), we have

$$(R(\xi, X). W^*)(Y, Z) = 0 \quad (26)$$

The above equation can be written as

$$R(\xi, X)W^*(Y, Z)U - W^*(R(\xi, X)Y, Z)U - W^*(Y, R(\xi, X)Z)U - W^*(Y, Z)R(\xi, X)U = 0 \quad (27)$$

Using (12) in (27), we have

$$\eta(W^*(Y, Z)U)X - W^*(Y, Z, U, X)\xi - \eta(Y)W^*(X, Z)U + g(X, Y)W^*(\xi, Z)U - \eta(Z)W^*(Y, X)U + g(X, Z)W^*(Y, \xi)U - \eta(U)W^*(Y, Z)X + g(X, U)W^*(Y, Z)\xi = 0 \quad (28)$$

Taking inner product of above equation with ξ and using (2), we have

$$\eta(W^*(Y, Z)U)\eta(X) - W^*(Y, Z, U, X) - \eta(Y)\eta(W^*(X, Z)U) + g(X, Y)\eta(W^*(\xi, Z)U) - \eta(Z)\eta(W^*(Y, X)U) + g(X, Z)\eta(W^*(Y, \xi)U) - \eta(U)\eta(W^*(Y, Z)X) + g(X, U)\eta(W^*(Y, Z)\xi) = 0 \quad (29)$$

using (19), (21), (23) and (24), we have

$$-R(Y, Z, U, X) - \frac{1}{2}[g(X, Y)g(Z, U) - g(X, Z)g(Y, U) + g(X, Y)\eta(Z)\eta(U) - g(X, Z)\eta(Y)\eta(U)] + \frac{1}{2(n-1)}[g(Z, U)S(Y, X) - g(Y, U)S(Z, X) + S(X, Z)\eta(Y)\eta(U) - S(X, Y)\eta(Z)\eta(U)] = 0 \quad (30)$$

Putting $Z = U = e_i$ in the above equation and taking summation over $i, 1 \leq i \leq n$,

We get

$$S(X, Y) = (1 - n)g(X, Y) \quad (31)$$

Thus we can state the theorem:

Theorem 3.1: An m -projectively semi-symmetric P -Sasakian manifold is an Einstein manifold

Using (31) in(30),we have

$$R(Y, Z, U, X) = g(Y, U)g(X, Z) - g(Z, U)g(X, Y)$$

This gives

$$R(Y, Z)U = -[g(Z, U)Y - g(Y, U)Z] \quad (32)$$

The above equation show that M is of constant curvature -1 and consequently it is locally isometric with the hyperbolic space $H^n(-1)$. Thus we have the following theorem:

Theorem 3.2: An n -dimensional P -Sasakian manifold M is m -projectively semi-symmetric, then it is locally isometric to the hyperbolic space $H^n(-1)$.

From (32) m -projectively semi-symmetric P -Sasakian manifold is of constant curvature. But it is known [1,2] that if a P -Sasakian manifold is of constant curvature, then it is an SP -Sasakian manifold. Thus we can state theorem

Theorem 3.3: An m -projectively semi-symmetric P -Sasakian manifold is an SP -Sasakian manifold.

4. ξ - m -Projectively Flat P -Sasakian Manifold

Definition: An n -dimensional P -Sasakian manifold M is said to be ξ - m -projectively flat [18] if $W^*(X, Y)\xi = 0$ for all $X, Y \in TM$.

Let $W^*(X, Y)\xi = 0$, then from (20), we obtain

$$\frac{1}{2}[\eta(X)Y - \eta(Y)X] - \frac{1}{2(n-1)}[\eta(Y)QX - \eta(X)QY] = 0 \quad (33)$$

Putting $Y = \xi$ in (33), we have

$$QX = -(n-1)X \quad (34)$$

Now taking inner product of above equation with U , we get

$$S(X, U) = (1-n)g(X, U) \quad (35)$$

Thus M is Einstein manifold. Conversely suppose that (35) is satisfied. Then, by virtue of (34) and (20), we have $W^*(X, Y)\xi = 0$. Thus we have a theorem

Theorem 4.1: An n -dimensional P -Sasakian manifold M is ξ - m -projectively flat if and only if it is an Einstein manifold.

5. Quasi m -Projectively flat P -Sasakian Manifold

Definition: An n -dimensional P -Sasakian manifold M is said to be quasi m -projectively flat, if

$$g(W^*(X, Y)Z, \phi U) = 0 \quad (36)$$

for any vector fields X, Y, Z, U .

From (19), we have

$$g(W^*(X, Y)Z, \phi U) = g(R(X, Y)Z, \phi U) - \frac{1}{2(n-1)}[S(Y, Z)g(X, \phi U) - S(X, Z)g(Y, \phi U) + g(Y, Z)S(X, \phi U) - g(X, Z)S(Y, \phi U)] \quad (37)$$

Let $\{e_1, e_2, e_3, \dots, e_n, \xi\}$ be a local orthonormal basis of vector fields in M , by using the fact that $\{\phi e_1, \phi e_2, \phi e_3, \dots, \phi e_n, \xi\}$ is also a local orthonormal basis, if we put $X = \phi e_i, U = e_i$ in (5.2) and sum up with respect to i , then we have

$$\sum_{i=1}^{n-1} g(W^*(\phi e_i, Y)Z, \phi e_i) = \sum_{i=1}^{n-1} g(R(\phi e_i, Y)Z, \phi e_i) - \frac{1}{2(n-1)} \sum_{i=1}^{n-1} [S(Y, Z)g(\phi e_i, \phi e_i) - S(\phi e_i, Z)g(Y, \phi e_i) + g(Y, Z)S(\phi e_i, \phi e_i) - g(\phi e_i, Z)S(Y, \phi e_i)] \quad (38)$$

On a P -Sasakian manifold, given by De and Sarkar [5], Matsumoto [6], we have

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z) \quad (39)$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r + (n - 1) \quad (40)$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) = S(\phi Y, \phi Z) \quad (41)$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = (n - 1) \quad (42)$$

$$\sum_{i=1}^{n-1} S(\phi e_i, Z)g(Y, \phi e_i) = S(Y, Z) - S(Z, \xi)\eta(Y) = S(Y, Z) + (n - 1)\eta(Y)\eta(Z) \quad (43)$$

$$\sum_{i=1}^{n-1} 'R(e_i, Y, Z, e_i) = \sum_{i=1}^{n-1} 'R(\phi e_i, Y, Z, \phi e_i) = S(Y, Z) + g(\phi Y, \phi Z) \quad (44)$$

Using (39) – (44) in (38), we get

$$\sum_{i=1}^{n-1} g(W^*(\phi e_i, Y)Z, \phi e_i) = \frac{1}{2(n-1)} \{(n + 1)S(Y, Z) + (n - 1 - r)g(Y, Z)\} \quad (45)$$

If M is quasi m –projectively flat, then (35) reduces to

$$(n + 1)S(Y, Z) = [r + (n - 1)]g(Y, Z) \quad (46)$$

Putting $Z = \xi$ in (46) and then using (6) and (11), we have

$$r = -n(n - 1) \quad (47)$$

Using (47) in (46), we get

$$S(Y, Z) = (1 - n)g(Y, Z) \quad (48)$$

Therefore M is Einstein manifold. Thus we have a theorem

Theorem 5.1: A quasi m –projectively flat P -Sasakian manifold is Einstein manifold.

Now using (48) in (19), we get

$$W^*(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y \quad (49)$$

If P -Sasakian manifold is m –projectively flat, *i. e.*, $W^*(X, Y)Z = 0$ then from (49), we have

$$R(X, Y)Z = -[g(Y, Z)X - g(X, Z)Y]$$

Hence we can state the following theorem

Theorem 5.2: A quasi m –projectively flat P -Sasakian manifold is locally isometric to the hyperbolic space $H^n(-1)$ if and only if M is m –projectively flat.

6. m –Projectively Recurrent P -Sasakian Manifold

Definition: An n -dimensional P -Sasakian manifold M is said to be m –projectively recurrent if it satisfies

$$(\nabla_U W^*)(X, Y)Z = A(U)W^*(X, Y)Z \quad (50)$$

for some non-zero 1 –form A .

We define a function

$$f^2 = g(W^*, W^*) \quad (51)$$

Now using the fact that $\nabla_U g = 0$, (51) gives that $f(Uf) = f^2(A(U))$

Since $f \neq 0$, we have

$$Uf = f(A(U)) \quad (52)$$

From the above equation, we have

$$X(Uf) - U(Xf) = \{XA(U) - UA(X)\}f \quad (53)$$

Therefore

$$(\nabla_X \nabla_U - \nabla_U \nabla_X - \nabla_{[X, U]})f = \{XA(U) - UA(X) - A([X, U])\}f = [dA(X, U)]f \quad (54)$$

Since the left hand side of (54) is zero and $f \neq 0$, we obtain

$$dA(X, U) = 0 \quad (55)$$

that is 1 –form A is closed.

From (50), we have

$$(\nabla_V \nabla_U W^*)(X, Y)Z = [VA(U) + A(V)A(U)]W^*(X, Y)Z \quad (56)$$

In view of (55) and (56), we have

$$(\nabla_V \nabla_U W^*)(X, Y)Z - (\nabla_U \nabla_V W^*)(X, Y)Z - (\nabla_{[V, U]} W^*)(X, Y)Z = 2dA(V, U)W^*(X, Y)Z = 0$$

Hence $(R(V, U).W^*)(X, Y)Z = 0$, where $R(V, U)$ is to be considered as a derivation of tensor algebra at each point of the manifold for tangent vectors V, U . Thus the m – projectively recurrent is m –projectively semi-symmetric. Thus from theorem [3.3], we state the following theorem

Theorem 6.1: An m –projectively recurrent P -Sasakian manifold is an SP -Sasakian manifold.

7. Einstein P -Sasakian manifold satisfying $(div W^*)(X, Y)Z = 0$

From (19), we have

$$\begin{aligned} W^*(X, Y, Z, V) = & 'R(X, Y, Z, V) - \frac{1}{2(n-1)} [S(Y, Z)g(X, V) - S(X, Z)g(Y, V) + \\ & g(Y, Z)S(X, V) - g(X, Z)S(Y, V)] \end{aligned} \quad (57)$$

Now differentiating above equation covariantly, we have

$$(\nabla_U W^*)(X, Y, Z, V) = (\nabla_U R)(X, Y, Z, V) - \frac{1}{2(n-1)} [(\nabla_U S)(Y, Z)g(X, V) - (\nabla_U S)(X, Z)g(Y, V) + g(Y, Z)(\nabla_U S)(X, V) - g(X, Z)(\nabla_U S)(Y, V)] \quad (58)$$

Putting $U = V = e_i$ in (58) and summing over $i, 1 \leq i \leq n$, we have

$$(\operatorname{div} W^*)(X, Y)Z = (\operatorname{div} R)(X, Y)Z - \frac{1}{2(n-1)} [(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) + \frac{1}{2}g(Y, Z)dr(X) - \frac{1}{2}g(X, Z)dr(Y)] \quad (59)$$

On a P -Sasakian manifold

$$(\operatorname{div} R)(X, Y)Z = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) \quad (60)$$

Using (60) in (59), we have

$$(\operatorname{div} W^*)(X, Y)Z = \frac{(2n-3)}{2(n-1)} [(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)] - \frac{1}{4(n-1)} [g(Y, Z)dr(X) - g(X, Z)dr(Y)] \quad (61)$$

For Einstein P -Sasakian manifold, we have

$$(\nabla_X S)(Y, Z) = 0 \quad (62)$$

Using (62) in (61), we have

$$(\operatorname{div} W^*)(X, Y)Z = -\frac{1}{4(n-1)} [g(Y, Z)dr(X) - g(X, Z)dr(Y)] \quad (63)$$

If $\operatorname{div} W^* = 0$, then from (63), we get

$$g(Y, Z)dr(X) - g(X, Z)dr(Y) = 0$$

This shows that $g(X, Z)dr(Y) = 0$, therefore $dr = 0$, that is r is constant.

Conversely, if r is constant then from (63), we get

$$(\operatorname{div} W^*)(X, Y)Z = 0$$

Thus we can state theorem

Theorem 7.1: An Einstein P -Sasakian manifold satisfying $(\operatorname{div} W^*)(X, Y)Z = 0$ if and only if the scalar curvature r is covariant constant.

Since a manifold is said to be m -projectively conservative if $\operatorname{div} W^* = 0$

Thus we have the following theorem as a corollary of the theorem [7.1]

Theorem 7.2: An Einstein P -Sasakian manifold M of dimension $n(n > 3)$ is m -projectively conservative if and only if the scalar curvature is constant.

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