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FRACTIONAL CALCULUS OF GENERALIZED K-MITTAG-LEFFLER FUNCTION

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Abstract: This paper deals with the derivation of the k-fractional differentiation and k-fractional integration of the generalized k-Mittag-Leffler function defined and studied by Saxena et al. [13]. The results derived in this paper provide extension of the results given by Kilbas et al. [4,5], Saxena [12], Saxena and Saigo [10] and Saxena et al [11]. The results obtained are useful in applied problems of science, engineering and technology.

Key words: Generalized k-Mittag-Leffler function, k-fractional differentiation, k-fractional integration, k-Pochhammer symbol, beta function.

Mathematics Subject Classification: 26A33, 33C60

1. Introduction

The k-fractional integral is defined and studied by Mubeen and Habibullah [7] in the form:

$$I_k^{\alpha}[f(x)] = \frac{1}{k\,\Gamma_k(\alpha)}\int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t)\,dt. \quad \mathrm{Re}(\alpha)>0. \tag{1}$$

The k-fractional differentiation is defined by Romero et al. [9] in the form:

$$D_k^{\beta} f(t) = \frac{d}{dt} I_k^{1-\beta} f(t), \qquad \beta \in R \;\; and \;\; 0<\beta\leq 1. \tag{2}$$

Or

$$D_k^{\beta} f(t) = \frac{d}{dt} I_k^{1-\beta} f(t) = \frac{d}{dt}\left(\frac{1}{k\,\Gamma_k(1-\beta)}\int_0^x (x-t)^{\frac{1-\beta}{k}-1} f(t)\,dt.\right) \;\; \mathrm{Re}(\alpha)>0. \tag{3}$$

and

$$D_k^{\beta} f(t) = \left(\frac{d}{dt}\right)^n \left[I_k^{n-\beta} f(t) \right] = \left(\frac{d}{dt}\right)^n \left(\frac{1}{k\,\Gamma_k(1-\beta)} \int_0^x (x-t)^{\frac{n-\beta}{k}-1} f(t)\,dt. \right) \tag{4}$$

The k-Pochhammer symbol has been introduced in [1] in the form:

$$(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k),$$

$$(x)_{(n+r)q,k} = (x)_{rq,k}\,(x+qrk)_{nq,k}, \tag{5}$$

where $x \in C;\ k \in R\ and\ n \in N.$

Proposition 1 Let $\gamma \in C\ and\ k, s \in R,$ then the following identity holds:

$$\Gamma_s(\gamma) = \left(\frac{s}{k}\right)^{\frac{\gamma}{s}-1} \Gamma_k\left(\frac{k\gamma}{s}\right), \tag{6}$$

and in particular

$$\Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\,\Gamma\left(\frac{\gamma}{k}\right), \tag{7}$$

Proposition 2. Let $\gamma \in C;\ k, s \in R\ and\ n \in N,$ then the following identity holds:

$$(\gamma)_{nq,s} = \left(\frac{s}{k}\right)^{nq} \left(\frac{k\gamma}{s}\right)_{nq,k}, \tag{8}$$

and in particular

$$(\gamma)_{nq,k} = k^{nq} \left(\frac{\gamma}{k}\right)_{nq}, \tag{9}$$

Note 1: For further details of k-Pochammer symbol, k-special functions and fractional Fourier transform one can refer to the papers by Romero et al [8] and Mubeen and Habibullah [9].

Definition 1: Let $k \in R;\ \alpha, \beta, \gamma \in C;\ \mathrm{Re}(\alpha) > 0\ and\ \tau \in C,$ then the generalized k-Mittag-Leffler function is given by Saxena et al. [13]

$$E_{k,\alpha,\beta}^{\gamma,\tau}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)} \frac{z^n}{n!} \tag{10}$$

where $(x)_\tau, (x, \tau \in C)$ denotes the Pochhammer symbol with $(1)_n = n!$ for $n \in N = N \cup \{0\},$

which is defined in terms of gamma function as (also see [16])

$$(x)_\tau = \frac{\Gamma(x+\tau)}{\Gamma(x)} = \begin{cases} 1 & (\tau=0;\ x\in C\setminus\{0\}) \\ x(x+1)\ldots(x+\tau-1) & (\tau=n\in N;\ x\in C) \end{cases}$$

Special cases of $E_{k,\alpha,\beta}^{\gamma,\tau}(z)$

(i) For τ = q, equation (7) yields generalized K-Mittag-Leffler function defined by Saxena et al [12]

$$E_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(n\alpha+\beta)} \frac{z^n}{n!} = E_{k,\alpha,\beta}^{\gamma,q}(z) \tag{11}$$

(ii) For k = 1, equation (11) yields generalized Mittag-Leffler function defined by Shukla and Prajapati [15]

$$E_{1,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma(n\alpha+\beta)} \frac{z^n}{n!} = E_{\alpha,\beta}^{\gamma,q}(z) \tag{12}$$

(ii) When q = 1, equation (11) gives the Mittag-Leffler function defined by Doorego & Cerutti [2].

$$E_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(n\alpha+\beta)} \frac{z^n}{n!} = E_{k,\alpha,\beta}^{\gamma}(z) \tag{13}$$

Note 2: A detailed account of Mittag-Leffler function and their application can be found in the survey papers by Haubold et al. [3], Mathai et al. [6], Saxena [12] and Saxena et al [14].

Theorem 1. If $k\in R$; $\alpha,\beta,\gamma\in C$;$\mathrm{Re}(\alpha)>0,\ \tau\in C$ *and* $n\in N=N\cup\{0\}$ then

$$\left\{I_r^\eta\left[t^{\beta/k-1}E_{k,\alpha,\beta}^{\gamma,\tau}(wt^{\alpha/k})\right]\right\}(x) = x^{\beta/k+\eta/r-1}\left(\frac{k}{r}\right)^{\eta/r} E_{k,\alpha,\beta+\frac{\eta}{r}k}^{\gamma,\tau}(wx^{\alpha/k}) \tag{14}$$

Proof : Using equation (1) and (10), it gives

$$\left\{I_r^\eta\left[t^{\beta/k-1}E_{k,\alpha,\beta}^{\gamma,\tau}(wt^{\alpha/k})\right]\right\}(x) = \frac{1}{r\Gamma_r(\eta)}\int_0^x (x-t)^{\frac{\eta}{r}-1} t^{\beta/k-1}E_{k,\alpha,\beta}^{\gamma,\tau}(wt^{\alpha/k})\,dt$$

$$= \frac{1}{r\Gamma_r(\eta)}\int_0^x (x-t)^{\frac{\eta}{r}-1} t^{\beta/k-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)}\frac{(wt^{\alpha/k})^n}{n!}\ dt$$

$$=\frac{1}{r\,\Gamma_r(\eta)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)}\frac{(w)^n}{n!}\int_0^x (x-t)^{\frac{\eta}{r}-1}\,t^{\beta/k+\alpha n/k-1}\,dt$$

We set $t=xu$

$$=\frac{1}{r\,\Gamma_r(\eta)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)}\frac{(w)^n}{n!}\int_0^1 (x-xu)^{\frac{\eta}{r}-1}\,(xu)^{\beta/k+\alpha n/k-1}\,dt$$

$$=\frac{x^{\beta/k+\eta/r-1}}{r\,\Gamma_r(\eta)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)}\frac{(wx^{\alpha/k})^n}{n!}\int_0^1 (1-u)^{\eta/r-1}\,u^{\beta/k+\alpha n/k-1}\,du$$

$$=\frac{x^{\beta/k+\eta/r-1}}{r\,\Gamma_r(\eta)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)}\frac{(wx^{\alpha/k})^n}{n!}\frac{\Gamma\left(\frac{\eta}{r}\right)\Gamma\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)}{\Gamma\left(\frac{\alpha n}{k}+\frac{\beta}{k}+\frac{\eta}{r}\right)}$$

By virtue of the relation (7), the above expression becomes

$$=\frac{x^{\beta/k+\eta/r-1}}{r\,\Gamma_r(\eta)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{k^{\alpha n/k+\beta/k-1}}\frac{(wx^{\alpha/k})^n}{n!}\frac{\Gamma\left(\frac{\eta}{r}\right)}{\Gamma\left(\frac{\alpha n}{k}+\frac{\beta}{k}+\frac{\eta}{r}\right)}$$

$$=x^{\beta/k+\eta/r-1}\left(\frac{k}{r}\right)^{\eta/r}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k\left(\alpha n+\beta+\frac{\eta}{r}k\right)}\frac{(wx^{\alpha/k})^n}{n!}$$

This completes the proof of Theorem 1.

Corollary 1.1 For $r=1$ equation (14) reduce in the Riemann-Liouville fractional integral

$$\left\{I_+^{\eta}\left[t^{\beta/k-1}E_{k,\alpha,\beta}^{\gamma,\tau}(wt^{\alpha/k})\right]\right\}(x)=x^{\beta/k+\eta-1}\,(k)^{\eta/r}\,E_{k,\alpha,\beta+\eta k}^{\gamma,\tau}(wx^{\alpha/k}) \tag{15}$$

Corollary 1.2 For $\tau=q,$ equation (14) reduces in the following form

$$\left\{I_r^{\eta}\left[t^{\beta/k-1}E_{k,\alpha,\beta}^{\gamma,q}(wt^{\alpha/k})\right]\right\}(x)=x^{\beta/k+\eta/r-1}\left(\frac{k}{r}\right)^{\eta/r}E_{k,\alpha,\beta+\frac{\eta}{r}k}^{\gamma,q}(wx^{\alpha/k}) \tag{16}$$

Corollary 1.3 When $k=1$ equation (16) gives

$$\left\{I_r^{\eta}\left[t^{\beta-1}E_{\alpha,\beta}^{\gamma,q}(wt^{\alpha})\right]\right\}(x)=\frac{x^{\eta/r+\beta-1}}{r^{-\eta/r}}\,E_{\alpha,\beta+\frac{\eta}{r}}^{\gamma,q}(wx^{\alpha}) \tag{17}$$

Corollary 1.4 When $q=k=1 and\ r=1$ equation (14) yields

$$\left\{I_{+}^{\eta}\left[t^{\beta-1}E_{\alpha,\beta}^{\gamma}(wt^{\alpha})\right]\right\}(x)=x^{\eta+\beta-1}E_{\alpha,\beta+\eta}^{\gamma}(wx^{\alpha}) \tag{18}$$

Theorem 2. Let $k\in R$; $\alpha,\beta,\gamma\in C$; $\mathrm{Re}(\alpha)>0,\ \tau\in C\ and\ n\in N=N\cup\{0\}$ then, we have

$$\left\{D_r^{\eta}\left[t^{\beta/k-1}E_{k,\alpha,\beta}^{\gamma,\tau}(wt^{\alpha/k})\right]\right\}(x)=\left(\frac{k}{r}\right)^{\frac{-\eta}{r}}k^{-k}\ x^{\beta/k-\eta/r-1}E_{k,\alpha,\beta-\frac{\eta}{r}k}^{\gamma,\tau}(wx^{\alpha/k}) \tag{19}$$

Proof : In view of (4) and (10) gives

$$\left\{D_r^{\eta}\left[t^{\beta/k-1}E_{k,\alpha,\beta}^{\gamma,\tau}(wt^{\alpha/k})\right]\right\}(x)=\left(\frac{d}{dx}\right)^{p}\left\{I_r^{p-\eta}\left[t^{\beta/k-1}E_{k,\alpha,\beta}^{\gamma,\tau}(wt^{\alpha/k})\right]\right\}(x)$$

$$=\left(\frac{d}{dx}\right)^{p}\frac{1}{r\Gamma_r(p-\eta)}\int_0^x(x-t)^{\frac{p-\eta}{r}-1}t^{\beta/k-1}E_{k,\alpha,\beta}^{\gamma,\tau}(wt^{\alpha/k})\,dt$$

$$=\left(\frac{d}{dx}\right)^{p}\frac{1}{r\Gamma_r(p-\eta)}\int_0^x(x-t)^{\frac{p-\eta}{r}-1}t^{\beta/k-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)}\frac{(wt^{\alpha/k})^n}{n!}\ dt$$

$$=\left(\frac{d}{dx}\right)^{p}\frac{1}{r\Gamma_r(p-\eta)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)}\frac{(w)^n}{n!}\int_0^x(x-t)^{\frac{(p-\eta)}{r}-1}t^{\beta/k+\alpha n/k-1}\,dt$$

We set $t=xu$

$$=\left(\frac{d}{dx}\right)^{p}\frac{1}{r\Gamma_r(p-\eta)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)}\frac{(w)^n}{n!}\int_0^x(x-xu)^{\frac{(p-\eta)}{r}-1}(xu)^{\beta/k+\alpha n/k-1}\,dt$$

$$=\left(\frac{d}{dx}\right)^{p}\frac{x^{\beta/k+\alpha n/k+(p-\eta)/r-1}}{r\Gamma_r(p-\eta)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)}\frac{(w)^n}{n!}\int_0^1(1-u)^{\frac{(p-\eta)}{r}-1}u^{\beta/k+\alpha n/k-1}\,du$$

$$=\left(\frac{d}{dx}\right)^{p}\frac{x^{\beta/k+\alpha n/k+(p-\eta)/r-1}}{r\,\Gamma_{r}\,(p-\eta)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_{k}\,(n\alpha+\beta)}\frac{(w)^{n}}{n!}\frac{\Gamma\left(\frac{(p-\eta)}{r}\right)\Gamma\left(\frac{\alpha n}{k}+\frac{\beta}{k}\right)}{\Gamma\left(\frac{\alpha n}{k}+\frac{\beta}{k}+\frac{(p-\eta)}{r}\right)}$$

Using eqn. (9), the above expression becomes

$$=\left(\frac{d}{dx}\right)^{p}\frac{x^{\beta/k+\alpha n/k+(p-\eta)/r-1}}{r^{\,(p-\eta)/r}}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{k^{\frac{\alpha n}{k}+\frac{\beta}{k}-1}\,\Gamma\left(\frac{\alpha n}{k}+\frac{\beta}{k}+\frac{(p-\eta)}{r}\right)}\frac{(w)^{n}}{n!}$$

$$=\left(\frac{d}{dx}\right)^{p}\frac{x^{\beta/k+\alpha n/k+(p-\eta)/r-1}}{r^{\,(p-\eta)/r}}\sum_{n=0}^{\infty}\frac{k^{\frac{(p-\eta)}{rk}}\,(\gamma)_{n\tau,k}}{k^{\frac{\alpha n}{k}+\frac{\beta}{k}+\frac{(p-\eta)}{rk}-1}\,\Gamma\left(\frac{\alpha n}{k}+\frac{\beta}{k}+\frac{(p-\eta)}{rk}k\right)}\frac{(w)^{n}}{n!}$$

$$=\left(\frac{d}{dx}\right)^{p}\left(\frac{k}{r}\right)^{\frac{(p-\eta)}{r}}k^{-k}\;x^{\beta/k+\alpha n/k+(p-\eta)/r-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_{k}\left(\alpha n+\beta+\frac{(p-\eta)}{r}k\right)}\frac{(w)^{n}}{n!}$$

On differentiating P times

$$=\left(\frac{k}{r}\right)^{\frac{-\eta}{r}}k^{-k}\;x^{\beta/k+\alpha n/k+(-\eta)/r-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_{k}\left(\alpha n+\beta+\frac{(-\eta)}{r}k\right)}\frac{(w)^{n}}{n!}$$

$$=\left(\frac{k}{r}\right)^{\frac{-\eta}{r}}k^{-k}\;x^{\beta/k-\eta/r-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_{k}\left(\alpha n+\beta+\frac{(-\eta)}{r}k\right)}\frac{(w\,x^{\alpha/k})^{n}}{n!}$$

This completes the proof of Theorem 2.

Corollary 2.1 When $r=1$ equation (19) reduce to Riemann-Liouville fractional integral

$$\left\{D_{+}^{\eta}\left[t^{\beta/k-1}E_{k,\alpha,\beta}^{\gamma,\tau}\,(wt^{\alpha/k})\right]\right\}(x)=k^{\frac{-\eta}{rk}}\;x^{\beta/k-\eta-1}\,E_{k,\alpha,\beta-\eta k}^{\gamma,\tau}\,(w\,x^{\alpha/k})\qquad(20)$$

Corollary 2.2 When $\tau = q,$ equation (19) reduces in the following form

$$\left\{D_r^{\eta}\left[t^{\beta/k-1}E_{k,\alpha,\beta}^{\gamma,q}(wt^{\alpha/k})\right]\right\}(x) = \left(\frac{k}{r}\right)^{\frac{-\eta}{r}} k^{-k}\; x^{\beta/k-\eta/r-1}\, E_{k,\alpha,\beta-\frac{\eta}{r}k}^{\gamma,q}(wx^{\alpha/k}) \tag{21}$$

Corollary 2.3 When $k = 1$ equation (19) reduces to

$$\left\{D_r^{\eta}\left[t^{\beta/k-1}E_{\alpha,\beta}^{\gamma,q}(wt^{\alpha})\right]\right\}(x) = \left(\frac{1}{r}\right)^{\frac{-\eta}{r}} x^{\beta-\eta/r-1}\, E_{\alpha,\beta-\frac{\eta}{r}}^{\gamma,q}(wx^{\alpha}) \tag{22}$$

Corollary 2.4 For $q = k = 1 and\ r = 1$ equation (19) reduces in the following form

$$\left\{D_+^{\eta}\left[t^{\beta-1}E_{\alpha,\beta}^{\gamma}(wt^{\alpha})\right]\right\}(x) = x^{\beta-\eta-1}\, E_{\alpha,\beta-\eta}^{\gamma}(wx^{\alpha}) \tag{23}$$

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