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GEOMETRIC PROPERTIES OF THE WRIGHT FUNCTIONS

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Abstract: In this paper, we obtain sufficient conditions for certain integral operators involving Wright function to be univalent in the open unit disk. Results are new and relevance with earlier results are pointed out.

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1. Introduction

The Wright function $W_{\lambda,\mu}(z)$ is defined by the series

$$W_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\,\Gamma(\lambda n+\mu)}\lambda>-1,\mu\in C \tag{1}$$

If $\lambda > -1$, the series (1) is absolutely convergent for all $z \in C$, while for $\lambda = -1$ this series is absolutely convergent for $|z| < 1$. Moreover, for $\lambda > -1$, $W_{\lambda,\mu}$ is entire function of z. The Wright functions was introduced by Wright in 1933 [28], and have been used widely in the asymptotic theory of partitions, in the Mikusinski operational calculus and in the theory of integral transforms of Hankel type. Recently these functions have appeared in the solution of partial differential equations of fractional order, it was found that the corresponding Green functions can be represented in terms of the Wright function (See [19,25]).

If λ is a positive rational numbers, then the Wright function $W_{\lambda,\mu}$ can be represented in terms of more familiar generalized hypergeometric function (see, [11, [1]). In particular, when $\lambda = 1$ and $\mu=\nu+\frac{b+1}{2}(\nu,b\in C)$, the function $W_{1,\nu+\frac{b+1}{2}}(-cz^2/4)$ can be expressed in terms of the generalized Bessel functions $W_{\nu,b,c}$, given as

$$W_{\nu,b,c}(z)=\left(\frac{z}{2}\right)^{\nu} W_{1,\nu+\frac{b+1}{2}}\left(\frac{-cz^2}{4}\right)=\sum_{n=0}^{\infty}\frac{(-c)^n (z/2)^{2n+\nu}}{n!\,\Gamma(n+\nu+\frac{b+1}{2})},(\nu,b,c,z\in C) \tag{2}$$

Note that generalized Bessel functions $W_{\nu,b,c}$ is the solution of differential equation

$$z^2w''(z)+bzw'(z)+(cz^2-\nu^2+(1-b)\nu)w(z)=0$$

for all $z \in C$, $z \neq 0$. Further, observe that function $W_{\nu,b,c}$ permits the study of Bessel, modified Bessel, spherical Bessel and modified spherical Bessel functions together. It is clear that for c = 1 and b = 1 the function $W_{\nu,b,c}$ reduces to J_ν, Bessel function of the first kind of order ν, when c = –1 and b = 1 the function $W_{\nu,b,c}$ becomes I_ν, is modified Bessel function of the first kind of order ν. Similarly, when c = 1 and b = 2 the function $W_{\nu,b,c}$ reduces to $2j_\nu/\sqrt{\pi}$ where j_ν is the spherical Bessel function of order ν, while if c = – 1 and b = 2, then $W_{\nu,b,c}$ becomes $2i_\nu/\sqrt{\pi}$ where i_ν is the modified spherical Bessel function of order ν (see[2]). Also the Wright function generalizes various simple functions like the Array function, Wittakar function, (Wright-type) entire auxiliary functions etc. For the details, we refer to [12,11].

Recently, several researchers studied classes of integral operators involving the Bessel functions, to find different conditions such that the integral operator have certain geometric properties like univalency, convexity and starlikeness. In this context many results are available in the literature (see [6,10,13,14,20,27]). In this article, we aim to study univalency of certain integral operators involving the Wright function.

We denote by A, the class of all analytic functions f in the open unit disk $D = \{z : |z| < 1\}$ having the normalization f(0) = 0; $f'(0) = 1$ and have the form:

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n, \qquad z\in D \tag{3}$$

A function $f \in A$ is called starlike (with respect to origin 0), denoted by $f \in S^*$ if $tw \in f(D)$ whenever $w \in f(D)$ and $t \in [0,1]$. A function $f \in A$ maps D onto a convex domain is called convex function and class of such functions is denoted by K. For a given $0 \leq \lambda < 1$, a function $f \in A$ is called starlike function of order λ, class of such function denoted by $S^*(\lambda)$, if $R(z f'(z)/f(z)) > \lambda$, $z \in D$. Further, for $0 \leq \lambda < 1$, a function $f \in A$ is called convex function of order λ, class of such function denoted by $K(\lambda)$, if $1+R(zf''(z)/f'(z)) > \lambda$, $z \in D$. It is well known that $S^*(0) = S^*$ and $K(0) = K$. A function $f \in A$ is called close-to-convex in D, if the range f(D) is close-to-convex, i.e. the complement of f(D) can be written as the union of non-intersecting half-times. Moreover, a function $f \in A$ is said to be close-to-convex with respect to a fixed starlike function g (need not be normalized), denoted by C_g, if R (z f'(z)/g(z)) > 0, $z \in D$. For more details about these classes can be found in [9].

Recently, several researchers studied families of analytic functions involving special functions $f \subset A$, to find different conditions such that the members of F have certain geometric properties like univalency, starlikeness or convexity in D. In this context many results are available in the literature regarding the hypergeometric functions [16,24,23], Bessel functions [1,2,3,4,5,15,21]. Recently author [22,7], have obtained certain sufficient conditions for the Wright function and Mittag-Leffler functions to be starlike, convex and close-to-convex. In continuation, we define a new normalization of Wright function $W_{\lambda,\mu}$ ($\lambda > -1$, $\mu \in C$) in D. Consider

$$W_{\lambda,\mu}(z) \square = \Gamma(\mu) z W_{\lambda,\mu}(z/4) := \sum_{n=0}^{\infty} \frac{\Gamma(\mu)\, z^{n+1}}{4^n\, n!\,\Gamma(\lambda n + \mu)} (\lambda > -1, \mu > 0, z \in D). \tag{4}$$

Note that $W_{\lambda,\mu}(z) \in A$ and

$$W_{1,\nu+\frac{b+1}{2}}(-cz) = u_{\nu,b,c}(z) = 2^{\nu}\, \Gamma(\nu + (b+1)/2)\, z^{1-\nu/2}\, W_{\nu,b,c}(\sqrt{z}), \tag{5}$$

and

$$W_{1,\nu+1}(-z) = g_{\nu}(z) = 2^{\nu}\,\Gamma(\nu+1) z^{1-\nu/2}\, J_{\nu}(\sqrt{z}), \tag{6}$$

where, the function $u_{\nu,b,c}(z)$ studied recently in [1,15] and $g_\nu(z)$ is investigated in [4,21,26].

2. Starlikeness and Convexity

Theorem 1. If $\lambda \geq 1$ and $\mu \geq \frac{-1+\sqrt{33}}{8} \approx 0.59307$ then $W_{\lambda,\mu}$ is starlike in D.

Proof. Observe that under the hypothesis the inequality $\Gamma(\mu + n) \leq \Gamma(\lambda n + p), n \in N$ holds, which is equivalent to

$$\frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)} \leq \frac{1}{\mu(\mu+1)\dots(\mu + n - 1)},\ n \in N. \tag{7}$$

If $z \in D$, then using (4), we get

$$\left|W'_{\lambda,\mu}(z)-\frac{W_{\lambda,\mu}(z)}{z}\right|=\left|\sum_{n=1}^{\infty}\frac{n\,\Gamma(\mu)z^n}{4^n\,n\,!\Gamma(\lambda n+\mu)}\right|\leq\sum_{n=1}^{\infty}\frac{1}{4^n\mu(\mu+1)...(\mu+n-1)}$$

$$\leq\frac{1}{4\mu}\sum_{n=0}^{\infty}\left(\frac{1}{4(\mu+1)}\right)^n=\frac{\mu+1}{4\mu^2+3\mu}. \tag{8}$$

and

$$\left|\frac{W_{\lambda,\mu}(z)}{z}\right|\geq 1-\left|\sum_{n=1}^{\infty}\frac{\Gamma(\mu)z^n}{4^n\,n\,!(\lambda n+\mu)}\right|\geq 1-\left|\sum_{n=1}^{\infty}\frac{1}{4^n\mu(\mu+1)...(\mu+n-1)}\right|$$

$$\geq 1-\frac{1}{4\mu}\sum_{n=0}^{\infty}\left(\frac{1}{4(\mu+1)}\right)^n=\frac{4\mu^2+2\mu-1}{4\mu^2+3\mu}. \tag{9}$$

Using (9), we obtain

$$\left|\frac{W'_{\lambda,\mu}(z)}{W_{\lambda,\mu}(z)}-1\right|=\left|W'_{\lambda,\mu}(z)-\frac{W_{\lambda,\mu}(z)}{z}\right|\left|\frac{z}{W_{\lambda,\mu}(z)}\right|\leq\frac{\mu+1}{4\mu^2+2\mu-1} \tag{10}$$

Note that, under hypothesis $4\mu^2+\mu-2\geq 0,$ the along with (10) gives

$$\left|\frac{W'_{\lambda,\mu}(z)}{W_{\lambda,\mu}(z)}-1\right|<1,\ \ z\in D, \tag{11}$$

which is equivalent to $R(zW'_{\lambda,\mu}/W_{\lambda,\mu})>0.$ This shows that the function $W_{\lambda,\mu}$ is starlike in D.

Remark 1. Note that for $\lambda = 1$, $\mu = \nu+1$, Lemma 1 is corresponding to the known result [26, Theorem 8].

Theorem 2. If $\lambda \geq 1$, $\mu > 0$ are such that the inequality $\Gamma(\mu+n)\leq\Gamma(\lambda n+\mu), n\in N$ holds, then function $W_{\lambda,\mu}$ satisfying the inequality

$$\frac{4\mu^2+2\mu-1}{4\mu^2+3\mu}\leq|W_{\lambda,\mu}(z)|\leq\frac{4\mu^2+4\mu+1}{4\mu^2+3\mu},\ \ z\in D. \tag{12}$$

Proof. We observe that, the hypothesis (7) holds. Using (4), (7) and triangle inequality, we obtain

$$|W_{\lambda,\mu}(z)| \geq 1 - = \left|\sum_{n=1}^{\infty} \frac{n\,\Gamma(\mu)z^{n}}{4^{n}\, n!\,\Gamma(\lambda n+\mu)}\right| \geq 1 - \sum_{n=1}^{\infty} \frac{1}{4^{n}\,\mu(\mu+1)\ldots(\mu+n-1)}$$

$$\geq \frac{1}{4\mu}\sum_{n=0}^{\infty}\left(\frac{1}{4(\mu+1)}\right)^{n} = \frac{4\mu^{2}+2\mu-1}{4\mu^{2}+3\mu} \tag{13}$$

which is positive under the given condition on μ. Similarly

$$|W_{\lambda,\mu}(z)| \leq 1 + \frac{1}{4\mu}\sum_{n=0}^{\infty}\left(\frac{1}{4(\mu+1)}\right)^{n} = \frac{4\mu^{2}+4\mu+1}{4\mu^{2}+3\mu}. \tag{14}$$

which is positive. Thus the proof is complete.

Theorem 3. If $\lambda \geq 1$, $\mu > 0$ are such that the inequality $\Gamma(\mu+n) \leq \Gamma(\lambda n+\mu), n \in N$ holds, then

$$|z\,W'_{\lambda,\mu}(z)| < \frac{4\mu^{2}+5\mu+2}{4\mu^{2}+3\mu},\ \ z\ \in D. \tag{15}$$

Proof. Using (4), (7) and triangle inequality, we get

$$|zW'_{\lambda,\mu}(z)|\ = \left|\sum_{n=1}^{\infty} \frac{(n+1)\,\Gamma(\mu)z^{n+1}}{4^{n}\ n!\,\Gamma(\lambda n+\mu)}\right| \leq 1 + \sum_{n=1}^{\infty} \frac{n+1}{4^{n}\ n!\,\mu(\mu+1)\ldots(\mu+n-1)}$$

$$\leq 1 + \frac{1}{2\mu}\sum_{n=0}^{\infty}\left(\frac{1}{4(\mu+1)}\right)^{n} = \frac{4\mu^{2}+5+2}{4\mu^{2}+3\mu}. \tag{16}$$

3. Integral Operators Involving Wright Functions

Lemma 1. (see[17]). Let $\beta \in C$ and $R(\beta) > 0, c \in C$ with $|c| \leq 1, c \neq -1$. If $h \in A$ satisfies

$$\left| c\,|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)} \right| \leq 1,$$

for all $z \in D$, then the integral operator defined by

$$F_{\beta}(z) = \left\{\beta\int_{0}^{z} t^{\beta-1}h'(t)\,dt\right\}^{1/\beta}, \tag{17}$$

is analytic and univalent in D.

Lemma 2. (see[18]). Let $\alpha \in C$ such that $\mathrm{Re}(\alpha) > 0$. If $h \in A$ satisfied

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right| \le 1,$$

for all $z \in D$, then for all $\beta \in C$ such that $\mathrm{Re}(\beta) \ge \mathrm{Re}(\alpha)$ the integral operator $F_\beta(z)$ defined by (17) is analytic and univalent in D.

We observe that, for $\alpha = 1$, Lemma 4 is equivalent to the Becker's criterion for univalency [8], which shows that, if $f \in A$ satisfying the inequality $(1-|z|^2)\,|z f''(z)/f'(z)| \le 1$ for each $z \in D$, then f is univalent in D.

In next theorem, we obtain conditions so that F_β is univalent in D.

Theorem 1. If $\lambda > -1, \mu > \frac{-1+\sqrt{5}}{4}$ are such that the inequality $\Gamma(\mu+n) \le \Gamma(\lambda n+\mu)$, $n \in N$ holds. Suppose also that $\lambda > -1$, $\mu > 0.46166$, $\beta \in C$ with $\mathrm{Re}(\beta) > 0$, $c \in C$ with $c \ne 1$ and $\alpha \in C \setminus \{0\}$. Suppose also that these numbers satisfy the following inequality

$$|c| + \frac{\mu+1}{4\mu^2+2\mu-1\alpha\beta} \le 1.$$

Then the function $F_{\mu,\nu,\alpha,\beta} : D \to C,$ defined by

$$F_{\mu,\nu,\alpha,\beta}(z) = \left\{\beta\int_0^z t^{\beta-1}\left(\frac{W_{\lambda,\mu}(t)}{t}\right)^{1/\alpha} dt\right\}^{1/\beta}, \ z \in D. \tag{18}$$

is univalent in D.

Proof. Let us consider a function $F_{\mu,\nu,\alpha,\beta} : D \to C,$ defined by

$$F_{\mu,\nu,\alpha,\beta}(z) = \int_0^z \left(\frac{W_{\lambda,\mu}(t)}{t}\right)^{1/\alpha} dt, \ z \in D.$$

Observe that $F_{\mu,\nu,\alpha,\beta} \in A.$ Differentiating F with respect to z, we get

$$\frac{zF''_{\mu,\nu,\alpha}(z)}{F'_{\mu,\nu,\alpha}(z)} = \frac{1}{\alpha}\left(\frac{zW'_{\lambda,\mu}(z)}{W_{\lambda,\mu}(z)} - 1\right)$$

Now using Lemma 1 and the triangle inequality, we get

$$\left| c\,|z|^{2\beta} + 1(1-|z|^{2\beta})\frac{zF''_{\mu,\nu,\alpha}(z)}{\beta F'_{\mu,\nu,\alpha}(z)} \right| \leq |c|\,|z|^{2\beta} + \frac{1-|z|^{2\beta}}{\alpha\beta}\left(\frac{zW'_{\lambda,\mu}(z)}{W_{\lambda,\mu}(z)} - 1 \right)$$

$$\leq |c| + \frac{\mu+1}{4\mu^2+2\mu-1\alpha\beta} \leq 1.$$

which in view of Lemma 3 implies that, $F_{\mu,\nu,\alpha,\beta}$ is univalent in D.

Theorem 2. Let $\lambda > -1$, $\mu > 0.46166$, $\alpha \in C$ with $\text{Re}(\alpha) > 0$. Suppose also that these numbers satisfying the inequality

$$|\alpha| \leq \frac{4\mu^2+2\mu-1}{\mu+1}\,\text{Re}(\alpha).$$

Then the function $G_{\mu,\nu,\alpha} : D \to C,$ defined by

$$G_{\mu,\nu,\alpha}(z) = \left\{ (\alpha+1)\int_0^z (W_{\lambda,\mu}(t))^{\alpha}\,dt \right\}^{1/(\alpha+1)}, \; z \in D. \tag{19}$$

is univalent in D.

Proof. Let us consider a function $G_{\mu,\nu,\alpha} : D \to C,$ defined by

$$G_{\mu,\nu,\alpha}(z) = \int_0^z \left(\frac{W_{\lambda,\mu}(t)}{t} \right)^{\alpha} dt, \; z \in D.$$

Observe that $G_{\mu,\nu,\alpha} \in A$. Now using Lemma 1 and the triangle inequality, we obtain for all $z \in D$

$$\frac{1-|z|^{2\text{Re}(\alpha)}}{\text{Re}(\alpha)} \left| \frac{zG^{n}_{\mu,\nu,\alpha}(z)}{\beta G'_{\mu,\nu,\alpha}(z)} \right| \leq \frac{|\alpha|}{\text{Re}(\alpha)} \left| \frac{zW'_{\lambda,\mu}(z)}{W_{\lambda,\mu}(z)} - 1 \right| \leq \frac{|\alpha|}{\text{Re}(\alpha)} \frac{\mu+1}{4\mu^2+2\mu-1} \leq 1.$$

which in view of Lemma 4 implies that, $G_{\mu,\nu,\alpha}$ is univalent in D.

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