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ON SOME CAUCHY SEQUENCES DEFINED IN l^p CONSIDERED AS n-NORMED SPACE

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Abstract: Singh and Srivastava [9] have studied n-normed structure of l^p by introducing a new n-norm $\overline{\overline{\|.,.,\dots,.\|}}_p$ on it and have observed that $\left(l^p, \overline{\overline{\|.,.,\dots,.\|}}_p\right)$is not complete in general. Here we shall investigate some sufficient condition for a Cauchy sequence to be convergent in the n-normed space $\left(l^p, \overline{\overline{\|.,.,\dots,.\|}}_p\right)$and some new results on Cauchy sequences.

Key words: l^pspace, normed space, n-normed space, parallel re-arranged sequences, Cauchy sequence, completeness.

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1. Introduction

The concept of 2-normed spaces was initially investigated by Gähler [3] in the mid of 1960's. Since then this theory has been developed in many directions including its generalization to n-normed spaces by endowing a linear space with n-norm. For instance see Misiak [8], Malćeski [7], Gunawan [4,5,6] and Acikgöz [1].

Definition 1.1: Let $\boldsymbol{X}$ be a vector space over $\mathbb{K}(=\mathbb{R}\ or\ \mathbb{C})$ of dimension $d \geq n\ (n \geq 2)$. A non-negative real valued function $\|.,.,\dots,.\|$ defined on $\boldsymbol{X}^n$ satisfying the four conditions:

(N_1) $\|\boldsymbol{x^1}, \boldsymbol{x^2}, \dots, \boldsymbol{x^n}\| = 0$ if and only if $\boldsymbol{x^1}, \boldsymbol{x^2}, \dots, \boldsymbol{x^n}$ are linearly dependent ;

(N_2) $\|\boldsymbol{x^1}, \boldsymbol{x^2}, \dots, \boldsymbol{x^n}\|$ is invarient under any permutation of $\boldsymbol{x^1}, \boldsymbol{x^2}, \dots, \boldsymbol{x^n}$;

(N_3) $\|\boldsymbol{\alpha x^1}, \boldsymbol{x^2}, \dots, \boldsymbol{x^n}\| = |\boldsymbol{\alpha}|\|\boldsymbol{x^1}, \boldsymbol{x^2}, \dots, \boldsymbol{x^n}\|$;

(N_4) $\|\boldsymbol{x^1} + \boldsymbol{y}, \boldsymbol{x^2}, \dots, \boldsymbol{x^n}\| \leq \|\boldsymbol{x^1}, \boldsymbol{x^2}, \dots, \boldsymbol{x^n}\| + \|\boldsymbol{y}, \boldsymbol{x^2}, \dots, \boldsymbol{x^n}\|$**;** for all $\boldsymbol{x^1}, \boldsymbol{x^2}, \dots, \boldsymbol{x^n}, \boldsymbol{y} \in \boldsymbol{X}$ and for all $\boldsymbol{\alpha} \in \mathbb{K}$ is called an **n-norm** on $\boldsymbol{X}$, and the pair $(\boldsymbol{X}, \|.,.,\dots,.\|)$ is called an **n-normed space.**

Definitions 1.2: Let $(\boldsymbol{X}, \|.,.,\dots,.\|)$ be an n-normed space. A sequence $(\boldsymbol{x^l})_{l=0}^{\infty}$ in $\boldsymbol{X}$ is said to be **a Cauchy sequence** in **X** if $\|\boldsymbol{x^l} - \boldsymbol{x^{l'}}, \boldsymbol{a^1}, \boldsymbol{a^2}, \dots, \boldsymbol{a^{n-1}}\| \to 0$ as $\boldsymbol{l}, \boldsymbol{l'} \to \infty$ for all $\boldsymbol{a^1}, \boldsymbol{a^2}, \dots, \boldsymbol{a^{n-1}} \in \boldsymbol{X}$.

$(\boldsymbol{x^l})_{l=0}^{\infty}$ is said to be **convergent at** $\boldsymbol{x} \in \boldsymbol{X}$ ($\boldsymbol{x}$ is called the limit point of the sequence) if

$\|\boldsymbol{x^l} - \boldsymbol{x}, \boldsymbol{a^1}, \boldsymbol{a^2}, \dots, \boldsymbol{a^{n-1}}\| \to 0$ as $\boldsymbol{l} \to \infty$ for all $\boldsymbol{a^1}, \boldsymbol{a^2}, \dots, \boldsymbol{a^{n-1}} \in \boldsymbol{X}$.

An **n-*normed space*** $(\boldsymbol{X}, \|.,.,\dots,.\|)$ is called an ***n-Banach space*** (***or complete n-normed space***) if every Cauchy sequence in $\boldsymbol{X}$ converges to an element of $\boldsymbol{X}$.

Here, we shall consider the well known sequence space $\boldsymbol{l^p}$, $1 \le p < \infty$; where

$$\boldsymbol{l^p} = \left\{ \boldsymbol{x} = (x_i)_{i=0}^{\infty} \;\middle|\; \sum_{i=0}^{\infty} |x_i|^p < \infty \; and \; x_i \in \mathbb{K};\; i = 0,1,2,\dots \right\}$$

With norms

$$\|\boldsymbol{x}\|_p = \left(\textstyle\sum_{i=0}^{\infty} |x_i|^p\right)^{1/p} \tag{1}$$

and

$$\|\boldsymbol{x}\|_\infty = \sup_{0 \le i < \infty} |x_i| \tag{2}$$

we know that $(\boldsymbol{l^p}, \|.\|_p)$ is a Banach space while $(\boldsymbol{l^p}, \|.\|_\infty)$ is not a simply normed space.

For our convenience and need, we shall denote the set of whole numbers as $\mathbb{N} = \{0,1,2,3,4,\dots\}$ will also be written in the form of a sequence $\mathbb{N} = (0,1,2,3,4,\dots)$ as well as in the form of n − consecutive terms notation [Moreover this paper is in continuation of [9] therefore we shall follow the notation used therein] as:

$$\mathbb{N} = \left(nl, nl+1, \dots, nl+(n-1)\right)_{l=0}^{\infty}$$

Where "n" is fixed positive integer and refer to the integer "n" of n-*normed space*.

Let $\overline{\overline{\mathbb{N}}} = (\overline{\overline{m}}_{nk}, \overline{\overline{m}}_{nk+1}, \dots, \overline{\overline{m}}_{nk+(n-1)})_{k=0}^{\infty}$ be a rearrangement of the sequence $\mathbb{N}$. In [9], we have seen that $(\boldsymbol{l^p}, \overline{\overline{\|.,.,\dots,.\|_p}})$ is an n-normed space, but not complete where

$$\overline{\overline{\|\boldsymbol{x^1}, \boldsymbol{x^2}, \dots, \boldsymbol{x^n}\|_p}} = sup\left\{ \left|\overline{\overline{\boldsymbol{x}}}^{\boldsymbol{1}}, \overline{\overline{\boldsymbol{x}}}^{\boldsymbol{2}}, \dots, \overline{\overline{\boldsymbol{x}}}^{\boldsymbol{n}}\right| : \overline{\overline{\boldsymbol{x}}}^{\boldsymbol{1}}, \overline{\overline{\boldsymbol{x}}}^{\boldsymbol{2}}, \dots, \overline{\overline{\boldsymbol{x}}}^{\boldsymbol{n}} \text{are parallel rearrangements of } \boldsymbol{x^1}, \boldsymbol{x^2}, \dots \boldsymbol{x^n} \text{ respectively} \right\}. \tag{3}$$

and $$\left|\overline{\overline{x}}^{\mathbf{1}},\overline{\overline{x}}^{\mathbf{2}},\dots,\overline{\overline{x}}^{\boldsymbol{n}}\right| = \left(\sum_{k=0}^{\infty}\left|det\begin{pmatrix} x^1_{\overline{\overline{m}}_{nk}} & x^1_{\overline{\overline{m}}_{nk+1}} & \dots & x^1_{\overline{\overline{m}}_{nk+(n-1)}} \\ x^2_{\overline{\overline{m}}_{nk}} & x^2_{\overline{\overline{m}}_{nk+1}} & \dots & x^2_{\overline{\overline{m}}_{nk+(n-1)}} \\ \dots & \dots & \dots & \dots \\ x^n_{\overline{\overline{m}}_{nk}} & x^n_{\overline{\overline{m}}_{nk+1}} & \dots & x^n_{\overline{\overline{m}}_{nk+(n-1)}} \end{pmatrix}\right|^{\boldsymbol{p}}\right)^{\frac{1}{\boldsymbol{p}}}; \quad (4)$$

$$\overline{\overline{x}}^t = (x^t_{\overline{\overline{m}}_{nk}}, x^t_{\overline{\overline{m}}_{nk+1}}, \dots x^t_{\overline{\overline{m}}_{nk+(n-1)}})_{k=0}^{\infty};$$

$$\boldsymbol{x^t} = (x^t_{nl}, x^t_{nl+1}, \dots, x^t_{nl+(n-1)})_{l=0}^{\infty}; \; t = 1,2,\dots,n$$

As we know that expansion of a determinant of order n consists of sum of $n!$ terms, among which each term is again a product of n terms belonging to distinct rows and columns. Now using **(1)-(4)** together with Minkowski inequality, we get

$$\overline{\overline{\|\boldsymbol{x^1},\boldsymbol{x^2},\dots,\boldsymbol{x^n}\|_{\boldsymbol{p}}}} \leq n! \; \|\boldsymbol{x^{\pi_1}}\|_{\boldsymbol{p}}.\|\boldsymbol{x^{\pi_2}}\|_{\boldsymbol{p}/\infty} \dots \|\boldsymbol{x^{\pi_n}}\|_{\boldsymbol{p}/\infty} \quad (5)$$

where $\boldsymbol{\pi_1},\boldsymbol{\pi_2},\dots,\boldsymbol{\pi_n}$ is any permutation of $1,2,\dots,n$ and $\|\boldsymbol{x^{\pi_t}}\|_{\boldsymbol{p}/\infty}$ means either $\|\boldsymbol{x^{\pi_t}}\|_{\boldsymbol{p}}$ or $\|\boldsymbol{x^{\pi_t}}\|_{\infty}$ is taken freely.

During our study for $\boldsymbol{u} = (u_i)_{i=0}^{\infty}, \boldsymbol{v} = (v_i)_{i=0}^{\infty} \in \boldsymbol{l}^p$ we shall write $\boldsymbol{u}.\boldsymbol{v} = (u_i.v_i)_{i=0}^{\infty}$ which is the term wise (or co-ordinate wise) multiplication of sequences $\boldsymbol{u}$ and $\boldsymbol{v}$.

Main Results

We know that, whenever $(\boldsymbol{x^r})_{r=0}^{\infty}$ is a Cauchy sequence in $(\boldsymbol{l^p}, \|\,.\,\|_p)$, then for each fixed $\boldsymbol{i} \in \mathbb{N}$, the sequence $(x_i^r)_{r=0}^{\infty}$ is a Cauchy in $\mathbb{K}$, where x_i^r is the i^{th} term of $\boldsymbol{x^r}$. We are going to present a similar result in $(\boldsymbol{l^p}, \overline{\overline{\|\,.\,,.\,,\dots,.\,\|_{\boldsymbol{p}}}})$.

Theorem 2.1: If $(\boldsymbol{x^r})_{r=0}^{\infty}$ is a Cauchy sequence in the n-normed space $(\boldsymbol{l^p}, \overline{\overline{\|\,.\,,.\,,\dots,.\,\|_{\boldsymbol{p}}}})$, then for each fixed $\boldsymbol{i} \in \mathbb{N}$, the sequence $(x_i^r)_{r=0}^{\infty}$ is also Cauchy sequence in $\mathbb{K}$.

Proof: Let $\boldsymbol{a^1},\boldsymbol{a^2},\dots,\boldsymbol{a^{n-1}} \in \boldsymbol{l^p}$. For $j = 1,\dots,n-1$, $\boldsymbol{a^j} = (a^j_{nl},\dots,a^j_{nl+(n-1)})_{l=0}^{\infty}$ is defined as follows:

$$a_i^j = \begin{cases} 1 \;; & \text{if } i = j \text{ or } n+j \\ 0 \;; & \text{otherwise} \end{cases}$$

Since $(\boldsymbol{x^r})_{r=0}^{\infty}$ is a Cauchy sequence in the n-normed space $(\boldsymbol{l^p}, \overline{\overline{\|\,.\,,.\,,\dots,.\,\|_{\boldsymbol{p}}}})$ then for every given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that

$$\overline{\overline{\|\boldsymbol{x^r}-\boldsymbol{x^s},\boldsymbol{a^1},\dots,\boldsymbol{a^{n-1}}\|_{\boldsymbol{p}}}} < \boldsymbol{\varepsilon} \quad \text{for all } r,s \geq N.$$

But from **(3)**, we get:

$$\left|\overline{\overline{x}}^{\boldsymbol{r}}-\overline{\overline{x}}^{\boldsymbol{s}},\overline{\overline{a}}^{\mathbf{1}},\dots,\overline{\overline{a}}^{\boldsymbol{n-1}}\right| \leq \overline{\overline{\|\boldsymbol{x^r}-\boldsymbol{x^s},\boldsymbol{a^1},\dots,\boldsymbol{a^{n-1}}\|_{\boldsymbol{p}}}} < \boldsymbol{\varepsilon} \text{ for all } r,s \geq N \text{ and for all } \overline{\overline{\mathbb{N}}} \quad (6)$$

Let $i = nl + t\,; 0 \leq t \leq n-1$ is taken arbitrarily. Then if we consider $\overline{\overline{\mathbb{N}}}' = \{i = nl + t\,, t+1\,, t+2\,, \dots\,, n-1\,, n+1\,, \dots\,, n+t, \dots\}$, that is, in the rearrangement $\overline{\overline{\mathbb{N}}}'$ of $\mathbb{N}$, the first n − consecutive terms are $i = nl + t\,, t+1\,, t+2\,, \dots\,, n-1\,, n+1\,, \dots\,, n+t$. Now referring to **(6)** for all $r, s \geq N$ we get:

$$\left|\overline{\overline{x}}'^{r} - \overline{\overline{x}}'^{s}, \overline{\overline{a}}'^{1}, \dots, \overline{\overline{a}}'^{n-1}\right| = (|x_i^r - x_i^s|^p + \tau)^{1/p} < \varepsilon\,;\text{ where } \tau \geq 0.$$

which implies that $|x_i^r - x_i^s| < \varepsilon$ for all $r, s \geq N$

Bu $i = nl + t$, t was taken arbitrarily, therefore for all $\boldsymbol{i} \in \mathbb{N}$, taking $\overline{\overline{\mathbb{N}}}$ accordingly, and using **(6)** we can easily show that for each $\boldsymbol{i} \in \mathbb{N}$ we get the same $N \in \mathbb{N}$ for which

$$|x_i^r - x_i^s| < \varepsilon \text{ for all } r, s \geq N \tag{7}$$

This exhibits that for each fixed $\boldsymbol{i} \in \mathbb{N}$ the sequence $(x_i^r)_{r=0}^{\infty}$ is a Cauchy sequence in $\mathbb{K}$.

It is well known that if$(\boldsymbol{x^r})_{r=0}^{\infty}$ is a Cauchy sequence in$\left(\boldsymbol{l^p}, \|\,.\,\|_p\right)$ and $\boldsymbol{a} \in \boldsymbol{l^p}$ then $(\boldsymbol{ax^r})_{r=0}^{\infty}$ is a Cauchy sequence in$\left(\boldsymbol{l^p}, \|\,.\,\|_p\right)$. We now prove the similar result for the n-normed space $\left(\boldsymbol{l^p}, \overline{\overline{\|\,.\,,.\,, \dots, .\,\|}}_{\boldsymbol{p}}\right)$ with the help of following Lemma.

Lemma 2.2: If $(\boldsymbol{x^r})_{r=0}^{\infty}$ is a Cauchy sequence in the n-normed space $\left(\boldsymbol{l^p}, \overline{\overline{\|\,.\,,.\,, \dots, .\,\|}}_{\boldsymbol{p}}\right)$ then $\left(\boldsymbol{a^1}.\boldsymbol{a^2} \dots \boldsymbol{a^{n-1}}.\boldsymbol{x^r}\right)_{\boldsymbol{r=0}}^{\infty}$ is a Cauchy sequence in$\left(\boldsymbol{l^p}, \|\,.\,\|_p\right)$; for all $\boldsymbol{a^1}, \boldsymbol{a^2}, \dots, \boldsymbol{a^{n-1}} \in \boldsymbol{l^p}$, where $\boldsymbol{a^1}.\boldsymbol{a^2} \dots \boldsymbol{a^{n-1}}.\boldsymbol{x^r}$ is defined as coordinate wise multiplication of the vectors $\boldsymbol{a^1}, \boldsymbol{a^2}, \dots, \boldsymbol{a^{n-1}}, \boldsymbol{x^r}$.

Proof: Suppose $\boldsymbol{a^1}, \boldsymbol{a^2}, \dots, \boldsymbol{a^{n-1}}$ are arbitrary vectors in $\boldsymbol{l^p}$, for our convenience we denote $\boldsymbol{a}.\boldsymbol{x^r} = (\boldsymbol{a^1}.\boldsymbol{a^2} \dots \boldsymbol{a^{n-1}}.\boldsymbol{x^r}) = (a_i x_i^r)_{i=0}^{\infty}$; where
$\boldsymbol{a} = (a_{\boldsymbol{i}})_{i=0}^{\infty} = \left(a_{\boldsymbol{i}}^1.a_{\boldsymbol{i}}^2 \dots a_{\boldsymbol{i}}^{n-1}\right)_{i=0}^{\infty}$ for $\boldsymbol{a^t} = (a_{\boldsymbol{i}}^t)_{i=0}^{\infty}$ and $t = 1, 2, \dots, n-1$.

Now for $j = 0, 1, \dots, n-1$; define $\boldsymbol{u^{j_r}} = \left(u_i^{j_r}\right)_{i=0}^{\infty}$ as follows:

$$u_i^{j_r} = \begin{cases} a_i x_i^r & ; \quad \text{whenever } i \equiv j \ (mod\ n) \\ 0 & ; \quad \text{otherwise} \end{cases}.$$

This shows that

$$\boldsymbol{ax^r} = \boldsymbol{u^{0_r}} + \boldsymbol{u^{1_r}} + \dots + \boldsymbol{u^{n-1_r}} \tag{8}$$

and

$$\boldsymbol{ax^r} - \boldsymbol{ax^s} = \left(\boldsymbol{u^{0_r}} - \boldsymbol{u^{0_s}}\right) + \left(\boldsymbol{u^{1_r}} - \boldsymbol{u^{1_s}}\right) + \dots + \left(\boldsymbol{u^{n-1_r}} - \boldsymbol{u^{n-1_s}}\right) \tag{9}$$

We now prove that for each $j = 0, 1, \dots, n-1$, $\left(\boldsymbol{u^{j_r}}\right)_{r=0}^{\infty}$ is a Cauchy sequence in$\left(\boldsymbol{l^p}, \|\,.\,\|_p\right)$.

For any j, define $\boldsymbol{b^{j^t}} = \left(b_i^{j^t}\right)_{i=0}^{\infty}$ $t = 1, 2, \dots, n-1$ as follows:

for $t > j$,
$$b_i^{j^t} = \begin{cases} a_{(i-t)+j}^t & ; \quad \text{if } i \equiv t (mod\ n) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

and for $t \leq j$, $$b_i^{j^t} = \begin{cases} a^t_{(i-t)+j+1} \;; & if\ i+1 \equiv t (mod\ n) \\ 0 \;; & otherwise \end{cases} \tag{10}$$

Now, since $(\boldsymbol{x^r})_{r=0}^{\infty}$ is a Cauchy sequence therefore for a given $\varepsilon > 0$, $\exists\, N^j$ such that for all $r, s \geq N^j$, we have

$$\overline{\overline{\left\| \boldsymbol{x^r - x^s, b^{j^1}, b^{j^2}, \dots, b^{j^{n-1}}} \right\|_{\boldsymbol{p}}}} < \varepsilon \tag{11}$$

But, taking $\overline{\overline{\mathbb{N}}} = \mathbb{N} = (nl, nl+1, \dots, nl+(n-1))_{l=0}^{\infty}$ and considering **(3), (4), (10)** and **(11)** we see that for all $r, s \geq N^j$.

$$\left| \boldsymbol{x^r - x^s, b^{j^1}, b^{j^2}, \dots, b^{j^{n-1}}} \right|$$

$$= \left(\sum_{\boldsymbol{l=0}}^{\infty} \left| det \begin{pmatrix} x^r_{nl} - x^s_{nl} & \cdots & x^r_{nl+(n-1)} - x^s_{nl+(n-1)} \\ b^{\boldsymbol{j^1}}_{nl} & & b^{\boldsymbol{j^1}}_{nl+(n-1)} \\ \vdots & \ddots & \vdots \\ b^{\boldsymbol{j^{n-1}}}_{nl} & \cdots & b^{\boldsymbol{j^{n-1}}}_{nl+(n-1)} \end{pmatrix} \right|^{\boldsymbol{p}} \right)^{\boldsymbol{\frac{1}{p}}}$$

$$= \left(\sum_{\boldsymbol{l=0}}^{\infty} \left| \left(x^r_{nl+j} - x^s_{nl+j} \right) a^1_{nl+j}\, a^2_{nl+j} \dots a^{n-1}_{nl+j} \right|^{\boldsymbol{p}} \right)^{\boldsymbol{\frac{1}{p}}} = \left\| \boldsymbol{u^{j_r} - u^{j_s}} \right\|_p < \varepsilon.$$

This shows that $\left(\boldsymbol{u^{j_r}}\right)_{r=0}^{\infty}$ is a Cauchy sequence in $\left(\boldsymbol{l^p}, \|\,.\,\|_p\right)$ for every $j = 0, 1, \dots, n-1$.

Thus, for given $\delta > 0$ there exists N^j such that for all $r, s \geq N^j$, we have

$$\left\| \boldsymbol{u^{j_r} - u^{j_s}} \right\|_p < {}^{\delta}\!/_{n}\ .$$

Taking $N = max\{\, N^j : j = 0, 1, \dots, n-1\}$ and using **(9)** for all $r, s \geq N$ we have

$$\| \boldsymbol{ax^r - ax^s} \|_p \leq \sum_{j=0}^{n-1} \left\| \boldsymbol{u^{j_r} - u^{j_s}} \right\|_p < \delta$$

and hence, $\boldsymbol{a.x^r} = \left(\boldsymbol{a^1.a^2 \dots a^{n-1}.x^r}\right)_{\boldsymbol{r=0}}^{\infty}$ is a Cauchy sequence in $\left(\boldsymbol{l^p}, \|\,.\,\|_p\right)$.

Here, we state the result of [9] as following Lemma:

Lemma 2.3: If $\left(\boldsymbol{x^l}\right)_{l=0}^{\infty}$ is a Cauchy sequence in $\left(\boldsymbol{l^p}, \|\,.\,\|_p\right)$ then $\left(\boldsymbol{x^l}\right)_{l=0}^{\infty}$ is a Cauchy sequence in n-normed space $\left(\boldsymbol{l^p}, \overline{\overline{\|\,.\,,.\,,\dots,.\,\|_{\boldsymbol{p}}}}\right)$ also.

Theorem 2.4: If $(\boldsymbol{x^r})_{r=0}^{\infty}$ is a Cauchy sequence in the n-normed space $\left(\boldsymbol{l^p}, \overline{\overline{\|\,.\,,.\,,\dots,.\,\|_{\boldsymbol{p}}}}\right)$ then, $\left(\boldsymbol{a^1.a^2 \dots a^{n-1}.x^r}\right)_{\boldsymbol{r=0}}^{\infty}$ is a Cauchy sequence in the n-normed space $\left(\boldsymbol{l^p}, \overline{\overline{\|\,.\,,.\,,\dots,.\,\|_{\boldsymbol{p}}}}\right)$ for all $\boldsymbol{a^1, a^2, \dots, a^{n-1}} \in \boldsymbol{l^p}$.

Proof: In view of Lemma 2.2 and Lemma 2.3 proof easily follows.

To discuss about the converse of Lemma 2.2 and Theorem2.4, consider the following example.

Example 2.5: Let us define a sequence $(\boldsymbol{x^r})_{\boldsymbol{r=0}}^{\infty}$ in $\boldsymbol{l^p}$ where for all $\boldsymbol{r} \in \mathbb{N}$;

$\boldsymbol{x^r} = (\delta_i^r)_{i=0}^{\infty}$ and $\|\boldsymbol{x^r}\|_{\boldsymbol{p}} = \mathbf{1}$. (12)

Suppose $\boldsymbol{a^1}, \boldsymbol{a^2}, \dots, \boldsymbol{a^{n-1}}$ are arbitrary vectors in $\boldsymbol{l^p}$. For our convenience, we denote $\boldsymbol{a}.\boldsymbol{x^r} = (\boldsymbol{a^1}.\boldsymbol{a^2} \dots \boldsymbol{a^{n-1}}.\boldsymbol{x^r}) = (a_i x_i^r)_{i=0}^{\infty}$; where
$\boldsymbol{a} = (a_{\boldsymbol{i}})_{i=0}^{\infty} = \left(a_{\boldsymbol{i}}^1 . a_{\boldsymbol{i}}^2 \dots a_{\boldsymbol{i}}^{n-1}\right)_{i=0}^{\infty}$ for $\boldsymbol{a^t} = (a_{\boldsymbol{i}}^t)_{i=0}^{\infty}$ and $t = 1,2, \dots, n-1$.

Now in view of **(12)**, for all $\boldsymbol{r} \in \mathbb{N}$ we get

$$\boldsymbol{a}.\boldsymbol{x^r} = \boldsymbol{a^1}.\boldsymbol{a^2} \dots \boldsymbol{a^{n-1}}.\boldsymbol{x^r} = (0, 0, \dots, a_{\boldsymbol{r}}, 0, \dots);$$
$$\|\boldsymbol{ax^r}\|_{\boldsymbol{p}} = |a_r|$$

and for all $\boldsymbol{r}, \boldsymbol{s} \in \mathbb{N}$

$$\|\boldsymbol{ax^r} - \boldsymbol{ax^s}\|_{\boldsymbol{p}} = \begin{cases} (|a_r|^p + |a_s|^p)^{1/p} & ; \text{ if } r \neq s \\ 0 & ; \text{ if } r = s \end{cases} \tag{13}$$

Now, $\boldsymbol{a} \in \boldsymbol{l^p}$ because $\|\boldsymbol{a}\|_p \leq \left\|\boldsymbol{a^1}\right\|_{\boldsymbol{p}} . \left\|\boldsymbol{a^2}\right\|_{\boldsymbol{p}} \dots \left\|\boldsymbol{a^{n-1}}\right\|_{\boldsymbol{p}} < \infty$ **,** therefore for every given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that

$$\sum_{i=N}^{\infty} |a_i|^p < \varepsilon^p$$

which implies that

$(|a_r|^p + |a_s|^p) \leq \sum_{i=N}^{\infty} |a_i|^p < \varepsilon^p$; for all $r, s \geq N$ (14)

therefore

$\|\boldsymbol{ax^r} - \boldsymbol{ax^s}\|_{\boldsymbol{p}} < \varepsilon$; for all $r, s \geq N$ (15)

Thus, $(\boldsymbol{ax^r})_{r=0}^{\infty} = \left(\boldsymbol{a^1}.\boldsymbol{a^2} \dots \boldsymbol{a^{n-1}}.\boldsymbol{x^r}\right)_{r=0}^{\infty}$ is a Cauchy sequence in $\left(\boldsymbol{l^p}, \| \,.\, \|_p\right)$. But $\boldsymbol{a^1}, \boldsymbol{a^2}, \dots, \boldsymbol{a^{n-1}} \in \boldsymbol{l^p}$ were taken arbitrarily therefore, $\left(\boldsymbol{a^1}.\boldsymbol{a^2} \dots \boldsymbol{a^{n-1}}.\boldsymbol{x^r}\right)_{r=0}^{\infty}$ is a Cauchy sequence in $\left(\boldsymbol{l^p}, \| \,.\, \|_p\right)$ and hence Cauchy sequence in the n-normed space $\left(\boldsymbol{l^p}, \overline{\overline{\| \,.\,, .\,, \dots, .\, \|}}_{\boldsymbol{p}}\right)$ also, for all $\boldsymbol{a^1}, \boldsymbol{a^2}, \dots, \boldsymbol{a^{n-1}} \in \boldsymbol{l^p}$.

Clearly by definition of the sequence $(\boldsymbol{x^r})_{\boldsymbol{r=0}}^{\infty}$ we see that

$\|\boldsymbol{x}^r - \boldsymbol{x}^s\|_{\boldsymbol{p}} = 2^{1/p}$; for $r \neq s$

This shows that $(\boldsymbol{x^r})_{\boldsymbol{r=0}}^{\infty}$ is not a Cauchy sequence in $\left(\boldsymbol{l^p}, \| \,.\, \|_p\right)$.

Next, for $t = 1,2,\dots,n-1$ let us define $\boldsymbol{b^t} = \left(b_i^t\right)_{i=0}^{\infty}$ such that

$$b_i^t = \begin{cases} 1 \ ; & \text{if } i = t \\ 0 \ ; & \text{otherwise} \end{cases} .$$

Let $\varepsilon = \frac{1}{2}$ is given then for all $N \geq \boldsymbol{n}$, taking a rearrangement $\overline{\overline{\mathbb{N}}} = (N, 1,2,\dots,n-1,0,n,n+1,\dots,N-1,N+1,\dots)$ We have;

$$\left|\overline{\overline{\boldsymbol{x}}}^N - \overline{\overline{\boldsymbol{x}}}^{N+1}, \overline{\overline{\boldsymbol{b}}}^{\boldsymbol{1}}, \dots, \overline{\overline{\boldsymbol{b}}}^{\boldsymbol{n-1}}\right| = 1 \tag{16}$$

Now from **(16)** and in view of **(3),** for all $N \geq \boldsymbol{n}$ we have:

$$\overline{\overline{\| \boldsymbol{x}^N - \boldsymbol{x}^{N+1}, \boldsymbol{b^1}, \boldsymbol{b^2}, \dots, \boldsymbol{b^{n-1}} \|_{\boldsymbol{p}}}} > \frac{1}{2}$$

This shows that $(\boldsymbol{x^r})_{\boldsymbol{r=0}}^{\infty}$ is not a Cauchy sequence in the n-normed space $\left(\boldsymbol{l^p}, \overline{\overline{\| .,.,\dots,. \|_{\boldsymbol{p}}}}\right)$also.

Thus in view of above example, we have the following remark.

Remark 2.6: In general if $(\boldsymbol{x^r})_{\boldsymbol{r=0}}^{\infty}$ is a sequence in $\left(\boldsymbol{l^p}, \| . \|_p\right)$ such that for all $\boldsymbol{a^1}, \boldsymbol{a^2} \dots, \boldsymbol{a^{n-1}} \in \boldsymbol{l^p}$, $\left(\boldsymbol{a^1}.\boldsymbol{a^2} \dots \boldsymbol{a^{n-1}}.\boldsymbol{x^r}\right)_{r=0}^{\infty}$ is a Cauchy sequence in $\left(\boldsymbol{l^p}, \| . \|_p\right)$ and hence Cauchy sequence in the n-normed space $\left(\boldsymbol{l^p}, \overline{\overline{\| .,.,\dots,. \|_{\boldsymbol{p}}}}\right)$also then;

- (i) $(\boldsymbol{x^r})_{\boldsymbol{r=0}}^{\infty}$ need not be a Cauchy sequence in$\left(\boldsymbol{l^p}, \| . \|_p\right)$;
- (ii) $(\boldsymbol{x^r})_{\boldsymbol{r=0}}^{\infty}$ need not be a Cauchy sequence in the n-normed space $\left(\boldsymbol{l^p}, \overline{\overline{\| .,.,\dots,. \|_{\boldsymbol{p}}}}\right)$also.

In [9], we found that every Cauchy sequence need not converge in the n-normed space$\left(\boldsymbol{l^p}, \overline{\overline{\| .,.,\dots,. \|_{\boldsymbol{p}}}}\right)$. Here, our effort is to investigate some sufficient condition for the convergence of a Cauchy sequence.

Theorem 2.7: Let $(\boldsymbol{x^r})_{r=0}^{\infty}$ is a Cauchy sequence in the n-normed space$\left(\boldsymbol{l^p}, \overline{\overline{\| .,.,\dots,. \|_{\boldsymbol{p}}}}\right)$, if there exists $M > 0$ such that for all $r \in \mathbb{N}$; $\| x^r \|_p \leq M$, then $(\boldsymbol{x^r})_{r=0}^{\infty}$ converges in the n-normed space$\left(\boldsymbol{l^p}, \overline{\overline{\| .,.,\dots,. \|_{\boldsymbol{p}}}}\right)$.

Proof: let $(\boldsymbol{x^r})_{r=0}^{\infty}$ is a Cauchy sequence in the n-normed space$\left(\boldsymbol{l^p}, \overline{\overline{\| .,.,\dots,. \|_{\boldsymbol{p}}}}\right)$, therefore by **(7)** of theorem 2.1**;** for a given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $r, s \geq N$ and for all $i \in \mathbb{N}$ we have:

$$|x_i^r - x_i^s| < \varepsilon \ . \tag{17}$$

That is, for each fixed $\boldsymbol{i} \in \mathbb{N}$**,** the sequence $(x_i^r)_{r=0}^{\infty}$ is Cauchy in $\mathbb{K}$, let $x_i^r \to x_i$ in $\mathbb{K}$. Letting $s \to \infty$, in **(17)**; for all $r \geq N$ and for all $i \in \mathbb{N}$ we get:

$$|x_i^r - x_i| \leq \varepsilon \ . \tag{18}$$

Define $\boldsymbol{x} = (x_i)_{i=0}^{\infty}$. We shall first show that $\boldsymbol{x} \in \boldsymbol{l^p}$. Assume that $\boldsymbol{x} \notin \boldsymbol{l^p} \Longrightarrow$ for every given $K > 0$, $\exists\, T \in \mathbb{N}$ such that:

$$\left(\sum_{i=0}^{T} |x_i|^p\right)^{1/p} > K$$

Taking, $K = M + 1$, we have:

$$\left(\sum_{i=0}^{T} |x_i|^p\right)^{1/p} > M + 1 \tag{19}$$

Now taking $\varepsilon = \frac{1}{2.(T+1)^{1/p}}$ and taking $r = N$ in **(18),** for all $i \in \mathbb{N}$ we get:

$$\left|x_i^N - x_i\right| \leq \frac{1}{2.(T+1)^{1/p}} \tag{20}$$

Now, from **(19)** and **(20)**, and then using Minkowski Inequality (finite form), we get:

$$\left(\sum_{i=0}^{T} |x_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=0}^{T} \left|\left(x_i - x_i^N\right) + x_i^N\right|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=0}^{T} \left|\left(x_i - x_i^N\right)\right|^p\right)^{\frac{1}{p}} + \left(\sum_{i=0}^{T} \left|x_i^N\right|^p\right)^{\frac{1}{p}}$$
$$\leq \frac{1}{2} + M$$

Which contradicts **(19)**, therefore our assumption is wrong and hence $\boldsymbol{x} \in \boldsymbol{l^p}$.

Claim: $\boldsymbol{x^r} \to \boldsymbol{x}$ in $\left(\boldsymbol{l^p}, \overline{\overline{\|\cdot,\cdot,\ldots,\cdot\|_{\boldsymbol{p}}}}\right)$:

Let $\varepsilon > 0$ is given, then from **(18)**; $\exists N \in \mathbb{N}$ such that for all $r \geq N$,

$$\|\boldsymbol{x^r} - \boldsymbol{x}\|_{\infty} \leq \varepsilon \ . \tag{21}$$

Hence, in view of **(5)** and **(21);** $\boldsymbol{x^r} \to \boldsymbol{x}$ in $\left(\boldsymbol{l^p}, \overline{\overline{\|\cdot,\cdot,\ldots,\cdot\|_{\boldsymbol{p}}}}\right)$.

Theorem 2.8: Let $(\boldsymbol{x^r})_{r=0}^{\infty}$ is a Cauchy sequence in the n-normed space $\left(\boldsymbol{l^p}, \overline{\overline{\|\cdot,\cdot,\ldots,\cdot\|_{\boldsymbol{p}}}}\right)$, if there exists $M > 0$ such that for all $r \in \mathbb{N}$; $\| x^r \|_p \leq M$, then $(\boldsymbol{x^r})_{\boldsymbol{r=0}}^{\infty}$ need not be Cauchy sequence in $\left(\boldsymbol{l^p}, \|\cdot\|_p\right)$.

Proof: let us take a sequence $(\boldsymbol{x^r})_{\boldsymbol{r=0}}^{\infty}$ in $\boldsymbol{l^p}$, where:

$\boldsymbol{x^0} = (0,0,\ldots)$ and $\boldsymbol{x^1} = (0,0,\ldots)$

For $r \geq 2$; $\boldsymbol{x^r} = (x_i^r)_{i=0}^{\infty}$ is defined as follows:

If $r = 2m$, then $x_i^r = \begin{cases} \frac{1}{m^{1/p}} & ; \text{ for } i \leq m-1 \\ 0 & ; \text{ for } i \geq m \end{cases}$

And

for, $r = 2m+1$; $x_i^r = \begin{cases} \frac{-1}{m^{1/p}} & ; \text{ for } i \le m-1 \\ 0 & ; \text{ for } i \ge m \end{cases}$.

Obviously, by the definition of the sequence $(\boldsymbol{x^r})_{\boldsymbol{r=0}}^{\infty}$, $\|\boldsymbol{x^r} - \boldsymbol{x^s}\|_\infty \to 0$ as $r, s \to \infty$ therefore in view of **(5)**, $(\boldsymbol{x^r})_{r=0}^{\infty}$ is a Cauchy sequence in the n-normed space$\left(\boldsymbol{l^p}, \overline{\overline{\|\cdot,\cdot,\ldots,\cdot\|}}_{\boldsymbol{p}}\right)$. Again, for all $r \in \mathbb{N}$; $\|\boldsymbol{x^r}\|_{\boldsymbol{p}} \le \boldsymbol{1}$ while for every $N \in \mathbb{N}$, we have:

$$\left\|\boldsymbol{x^{2N}} - \boldsymbol{x^{2N+1}}\right\|_{\boldsymbol{p}} = 2$$

which shows that $(\boldsymbol{x^r})_{\boldsymbol{r=0}}^{\infty}$ fails to be a Cauchy sequence in $\left(\boldsymbol{l^p}, \|\cdot\|_p\right)$.

Theorem 2.9: Let $(\boldsymbol{x^r})_{r=0}^{\infty}$ is a Cauchy sequence in the n-normed space$\left(\boldsymbol{l^p}, \overline{\overline{\|\cdot,\cdot,\ldots,\cdot\|}}_{\boldsymbol{p}}\right)$ then $(\boldsymbol{x^r})_{r=0}^{\infty}$ need not be bounded in $\left(\boldsymbol{l^p}, \|\cdot\|_p\right)$.

Proof: let us define the sequence $(\boldsymbol{x^r})_{\boldsymbol{r=0}}^{\infty}$ in$\boldsymbol{l^p}$ such that $\boldsymbol{x^r} = (x_{nl}^r, x_{nl+1}^r, \ldots, x_{nl+(n-1)}^r)_{\boldsymbol{l=0}}^{\infty}$

Where,

$$x_i^r = \begin{cases} 1 & ; \text{ for } i = 0,1 \\ \frac{1}{i^{1/p}} & ; \text{ for } 1 \le i \le r \\ 0 & ; \text{ for } i \ge r+1 \end{cases}$$

Here, we obtain $\boldsymbol{x^r} - \boldsymbol{x^s} = (0, \ldots, 0, \frac{1}{(r+1)^{1/p}}, \ldots, \frac{1}{(s)^{1/p}}, 0, 0, \ldots)$, for $r \le s$. Now, as we studied in above theorem, we can easily find that above defined sequence is Cauchy sequence in$\left(\boldsymbol{l^p}, \overline{\overline{\|\cdot,\cdot,\ldots,\cdot\|}}_{\boldsymbol{p}}\right)$, but not bounded in $\left(\boldsymbol{l^p}, \|\cdot\|_p\right)$.

Discussion and Conclusion

In this paper, we have investigated characteristics, behaviors of Cauchy sequences in $\boldsymbol{l^p}$ with respect to normed space and n-normed space defined by us in [9]. We have presented counter examples for the proof of converse part of some theorems. These results differ in many aspects to those who find in [6].

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