Journal of Rajasthan Academy of Physical Sciences
ISSN : 0972-6306; URL : http://raops.org.in
Vol.14, No.1, March, 2015, 101-113

THEORY OF FINSLER SPACES WITH SPECIAL 4-th ROOT METRIC CONVEXITY OF INDICATRIX, GEODESIC SPRAY, V-CURVATURE AND V(h)-TORSION TENSORS

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Abstract : In the present paper, we have considered the metric of a n-dimensional Finsler space with 4^{th} root metric $L^4 = \alpha^4 + \beta^4$ and examined the convexity of the indicatrix, S_3- likeness of v-curvature tensor, the co-efficient G^i [of the geodesic spray of the Finsler Space] and (v) h-torsion tensor has been working out in terms of generalised 4-th order Christoffel symbols.

Keywords & Phrases : Finsler spaces, 4-th Root metric.

2010 AMS classification No. : 5C60, 53B40

1. Introduction

The theory of m-th root metrics has been first developed by Shimada [7] as an interesting example of Finsler Metrics, immediately following M. Motsumoto and S. Numata's theory of cubic metrics[6]. By introducing the regularity of metric various fundamental quantities as a Finsler metric could be found. In particular, the Cartan connection of a Finsler space with m-th root metric could be discuss from a theoretical stand point.

Recently the m-th root metric have begun to be applied to theoretical Physics but the result of our investigations are not yet ready for acceding to the demands of various applications.

In the present paper we are considering 4-th root metric of the form $L^4 = \alpha^4 + \beta^4$ where $\alpha^2 = a_{ij}(x)y^iy^j$ and $\beta^2 = a_i(x)y^i$.

2. The general theory of n-dimensional Finsler spaces with m-th root metric

We consider the n-dimensional Finsler space (M,L) equipped with the m-th ($m \geq 3$) root metric.

$L^m = a_{i1}i_2...i_m(x^1,x^2,...x^n)y^{i1}y^{12}...y^{im}$ ...(1)

where the suffixes run from 1 to n and the $a_{i1i2}...i_m(x^1,x^2,...,x^4)$ are the components of a symmetric tensor field covariant of order m and are the functions of position alone.

We denote this Finsler space (M,L) by F_m^n.

We define the symmetric tensors $\hat{a}_i$, $\hat{a}_{ij}$ and $\hat{a}_{ijk}$ as follows ([3],[4],[5]);

$L^{m-1}\hat{a}_i = a_{ij1j2j3...jm-1}y^{j1}y^{j2}...y^{jm-1}$ ...(2)

$L^{m-2}\hat{a}_{ij} = a_{ijk1k2...km-2}y^{k1}y^{k2}...y^{km-2}$ ...(3)

$L^{m-3}\hat{a}_{ijk} = a_{ijks_1s_2...s_{m-3}}y^{s1}y^{s2}...y^{sm-3}$

Differentiating the expression (1) by y^i and using the definition (2), we get ([3],[4],[5]):

$l_i = \hat{a}_i$ ...(4)

Next differentiating (2) w.r.t. y^i, we get

$g_{ij} = (m-1)\hat{a}_{ij} - (m-2)\hat{a}_i\hat{a}_j$ ...(5)

and $h_{ij} = (m-1)(\hat{a}_{ij} - \hat{a}_i\hat{a}_j)$

Throughout this paper, we suppose that $\det(g_{ij}) \neq 0$.

Using (5), we get the results given by Motsumoto [4,5].

Differentiating (3) and (5) by y^k, we get

$2LC_{ijk} = (m-1)(m-2)(\hat{a}_{ijk} - \hat{a}_{ij}\hat{a}_k - \hat{a}_{jk}\hat{a}_i - \hat{a}_{ik}\hat{a}_j + 2\hat{a}_i\hat{a}_j\hat{a}_k)$ ...(6)

Now we define the symmetric tensor $\hat{a}^{ij}$ and $\hat{a}^i$ by

$(\hat{a}^{ij}) := (\hat{a}_{ij})^{-1}$ and $\hat{a}^i := \hat{a}^{ij}\hat{a}_j$, respectively.

Transvecting (3) and (2) by $\hat{a}^i a^j$ and $\hat{a}^i$ respectively and using the formula (4), we get $\hat{a}_i\hat{a}^i = l_i l^i = 1$.

We denote the inverse matrix of (g_{ij}) by (g^{ij}). Then we get the results given by Matsumoto [3,4,5]

$g^{ij} = \{\hat{a}^{ij} + (m-2)\hat{a}^i\hat{a}^j\}/(m-1)$ ...(7)

using (7), we get $l^i = \hat{a}^i$.

Finally we will calculate $C_i = C_{ijk}g^{jk}$, using (6) and (7), we get

$2LC_i = (m-2)(\hat{a}_{ijk}a^{jk} - n\hat{a}_i)$.

It is well-known that the formula

$C_i = \partial_k(\log\sqrt{|g|})$, where $g = \det(g_{ij})$, holds.

Proposition 2.1. [7] In the Finsler space F_m^n, the formula $C_i = 0$ holds goods if and only if $\det(\hat{a}_{ij})$ is a function of the position (x^i) alone.

Using (1) and $G_i = \left(\frac{1}{4}\right)\left\{\left(\frac{\partial^2 L^2}{\partial x^r \partial y^r}\right) y^r - \frac{\partial L^2}{\partial x^i}\right\}$, wet get

$$G_i = \left(\frac{1}{2}\right)\left\{\left(\frac{2}{m} - 1\right) L^{m-1} \hat{a} \frac{\partial a_{i_1 i_2 \dots i_m}}{\partial x^r} y^{i_1} y^{i_2} \dots \dots . y^{i_m} y^r + \right\}$$

$$L^{2-m} \frac{\partial a_{i_1 i_2 \dots i_m}}{\partial x^r} y^{i_1} y^{i_2} \dots \dots . y^{i_m} y^r - \frac{1}{m} L^{2-m} \frac{\partial a_{i_1 i_2 \dots i_m}}{\partial x^r} y^{i_1} y^{i_2} \dots \dots . y^{i_m} .$$

Definition 2.1. The symbols $\{i_1, i_2, \dots\dots i_m; i\}$ are defined by

$$\{i_1, i_2, \dots \dots . i_m ; i\} = \frac{1}{2(m-1)}\{\frac{\partial a_{ii_2 \dots i_m}}{\partial x^{i_1}} + \frac{\partial a_{ii_3 \dots i_m i_1}}{\partial x^{i_2}} + \frac{\partial a_{ii_4 \dots i_m i_1 i_2}}{\partial x^{i_3}} +$$

$$\frac{\partial a_{ii_1 \dots i_{m-1}}}{\partial x^{i_m}} - \frac{\partial a_{ii_1 i_2 \dots i_m}}{\partial x^i}\} \quad \text{...(8)}$$

We call $\{i_1, i_2, \dots\dots i_m; i)$ the Christoffel symbols of m-th order.

Then, we get the results given by Matsumoto [3,4]

$$mL^{m-2}\hat{a}_{ir}G^r = \{00\dots\dots0;i), \quad \text{...(9)}$$

where

$$\{00\dots\dots0;i) = \{i_1 i_2 i_3 \dots\dots i_m;i\} y^{i1} y^{i2} \dots\dots y^{im}$$

From (8), we get

$$\frac{\partial a_{i_1 i_2 \dots i_m}}{\partial x^i} = \{ii_2 i_3 \dots\dots i_m;i_1\} + \{ii_3 \dots\dots i_m i_1;i_2\} + \dots\dots +$$

$$\{ii_1 i_3 \dots\dots i_{m-1};i_m\} - (m-2)\{i_1 i_2 \dots i_m;i\} \quad \text{...(10)}$$

Using (9) and (10), we get

$$\frac{\partial G^i}{\partial x^k} = \frac{1}{m} L^{-(m-2)} \hat{a}^{ir} \left\{\frac{\partial(00\dots\dots0;r)}{\partial x^k} - m \frac{\partial(L^{m-2} a_{rs})}{\partial x^k} G^s\right\} \quad \text{...(11)}$$

Now from (9), we have

$$G^i = \frac{1}{m} L^{-(m-2)} \hat{a}^{ir} \{00 \dots . .0; r\}, \quad \text{...(12)}$$

Differentiating (9) by y^j and using $\frac{\partial(L^{m-2} a_{ijk})}{\partial y^1} = (m-2)L^{m-3}\hat{a}_{ijk}$, we get

$$(m-2)L^{m-3} \hat{a}_{irj} G^r + L^{m-2} \hat{a}_{ir} G^r_j = \{00 \dots . .0j; i\} \quad \text{...(13)}$$

Differentiating (13) by x^k, we have

$$(m-2) \frac{\partial L^{m-2} \hat{a}_{irj}}{\partial x^k} + (m-2)L^{m-3} a_{irj} \frac{\partial G^r}{\partial x^k} + \frac{\partial(L^{m-2} \hat{a}_{ir})}{\partial x^k} G^r_j +$$

$$L^{m-2} a_{ir} \frac{\partial G^r}{\partial x^k} = \frac{\partial\{00 \dots . . 0_{j;i}\}}{\partial x^k}$$

From the above equation, we get

$$L^{m-2} a_{ir} \frac{\partial G_j^r}{\partial x^k} = \frac{\partial\{00.....0_{j;i}\}}{\partial x^k} - (m-2)\frac{\partial L^{m-3}\hat{a}_{irj}}{\partial x^k} G^r -$$
$$(m-2)L^{m-3} a_{irj} \frac{\partial G^r}{\partial x^k} - \frac{\partial(L^{m-2}\hat{a}_{ir})}{\partial x^k} G_j^r \qquad ...(14)$$

Solving the equation (13) for G_j^r, we get

$$G_j^r = L^{-(m-2)}\hat{a}^{ir}\{(00.....0j;r) - (m-2)L^{m-3}\hat{a}_{rsj} G^s. \qquad ...(15)$$

Differentiating (13) by y^k and using $\frac{\partial(L^{m-3}\hat{a}_{ijk})}{\partial y^l} = (m-3)L^{m-4}\hat{a}_{ijkl}$ we have,

$$(m-2)(m-3)L^{m-4}\hat{a}_{irjk} G^r + (m-2)L^{m-3}\hat{a}_{irj} G_k^r$$
$$+(m-2)L^{m-3}\hat{a}_{irk} G_j^r + L^{m-2}\hat{a}_{ir} G_{jk}^r = (m-1)\{00......0jk;i\}$$

Solving the equation for $L^{m-2}\hat{a}_{ir} G_{jk}^r$, we get

$$L^{m-2}\hat{a}_{ir} G_{jk}^r = (m-1)\{00......0jk;i\} - (m-2)(m-3)L^{m-4}\hat{a}_{irjk} G^r -$$
$$(m-2)L^{m-3}\hat{a}_{irj} G_k^r - (m-2)\, L^{m-3}\hat{a}_{irk} G_j^r \qquad ...(16)$$

Now we shall show that the (v)h - torsion tensor R_{jk}^r can be represented by the Christoffel symbols of m-th order.

The (v)h - torsion tensor R_{jk}^r is defined by, $R_{jk}^r = \pi_{\{jk\}}\{\frac{\partial G^r}{\partial x^k} - G_k^s G_{sj}^r$, where the symbol $\pi_{\{jk\}}$stands for the exchange of the suffices j and k, and subtraction of them. Transvecting the above equation by $L^{m-2}\hat{a}_{ir}$, we get

$$L^{m-2}\hat{a}_{ir} R_{jk}^r = \pi_{\{jk\}}\{L^{m-2}\hat{a}_{ir} \frac{\partial G_j^r}{\partial x^k} - G_k^s L^{m-2}\hat{a}_{ir} G_{sj}^r \qquad ...(17)$$

Substituting (14) and (16) in (17), we get

$$L^{m-2}\hat{a}_{ir} R_{jk}^r = \pi_{\{jk\}}\{\frac{\partial\{00...0j;i\}}{\partial x^k} - (m-2)\frac{\partial L^{m-3}\hat{a}_{irj}}{\partial x^k} G^r -$$
$$(m-2)L^{m-3}\hat{a}_{irj} \frac{\partial G^r}{\partial x^k} - \frac{\partial(L^{m-2}\hat{a}_{ir})}{\partial x^k} G_j^r - G_k^s(m-1)\{00......0sj;i\} -$$
$$(m-2)(m-3)L^{m-4}\hat{a}_{irsj} G^r - (m-2)L^{m-3}\hat{a}_{irs} G_j^r - (m-2)L^{m-3}\hat{a}_{irj} G_s^r$$

Summarizing the above discussion, we get

Theorem 2.1. [7] The (v)h - torsion tensor $R_{jk}^r$of the n-dimensional Finsler space F_m^n with the m($\geq$3)-th root metric is represented using the Christoffel symbols of m-th order as follows :

$$L^{m-2}\hat{a}_{ir} R_{jk}^r = \pi_{\{jk\}}\{\frac{\partial\{00...0j;i\}}{\partial x^k} - (m-2)\frac{\partial L^{m-3}\hat{a}_{irj}}{\partial x^k} G^r -$$

$$(m-2)L^{m-3}\hat{a}_{irj}\frac{\partial G^r}{\partial x^k}-\frac{\partial(L^{m-2}\hat{a}_{ir})}{\partial x^k}G^r_j-G^s_k(m-1)\{00\ldots..0sj;i\}-$$

$$(m-2)(m-3)L^{m-4}\hat{a}_{irsj}G^r-(m-2)L^{m-3}\hat{a}_{irs}G^r_j-(m-2)L^{m-3}\hat{a}_{irj}G^r_s$$

where $\frac{\partial G^r}{\partial x^k}$, G^r and G^r_j are as (11), (12) and (15) respectively.

Further more, using the derivation $\frac{\delta}{\delta x^k}-G^s_k(\frac{\partial}{\partial y^s})$, we have

$$L^{m-2}\hat{a}_{ir}R^r_{jk}=\pi_{\{jk\}}\{\frac{\partial\{00\ldots0j;i\}}{\partial x^k}-(m-2)\frac{\partial L^{m-3}\hat{a}_{irj}}{\partial x^k}G^r-$$

$$(m-2)L^{m-3}\hat{a}_{irj}\frac{\partial G^r}{\partial x^k}-\frac{1}{m-1}\frac{\delta}{\delta x^k}(\frac{\partial(L^{m-1}\hat{a}_i)}{\partial x^j})\}$$

$$\pi_{\{jk\}}\{\frac{\delta}{\delta x^k}(\{00\ldots..0j;i\}-(m-2)L^{m-3}\hat{a}_{irj}G^r-\frac{1}{m-1}\frac{\partial(L^{m-1}\hat{a}_i)}{\partial x^j}\}.$$

using

$$\frac{\partial L^{m-1}\hat{a}_i}{\partial x^j}=\{j000\ldots..0;i\}+(m-1)\{ji000\ldots..0;0\}-(m-2)\{i000\ldots..0;j\}$$

we get

$$L^{m-2}\hat{a}_{ir}R^r_{jk}=\pi_{\{jk\}}\{\frac{\delta}{\delta x^k}(\frac{m-2}{m-1}\{j000\ldots..0;i)+\frac{m-2}{m-1}\{j000\ldots..0;j\}$$

$$-\{ji000\ldots..0;0\}-(m-2)L^{m-3}\hat{a}_{irj}G^r)\}$$

Therefore, we can rewrite theorem 2.1 as

Theorem 2.2. [7] The (v)h - torsion tensor R^r_{jk} of the n-dimensional Finsler space F^n_m with the m($\geq$3)-th root metric is represented using the Christoffel symbols of m-th order as the equation (18).

3. The Condition for the Space to be Positive definite

We consider the Finsler space (M,L), where $L=\sqrt[4]{\alpha^4+\beta^4}$, $\alpha^2=a_{ij}(x)y^iy^j$ and $\beta=b_i(x)y^i$. Since L = α in case of β = 0, we mainly consider the case of β ≠ 0 in this paper.

putting $s=\frac{\beta}{\alpha}$, we get

$$L=\alpha\phi(s),\phi(s)=\sqrt[4]{1+s^4} \qquad ...(19)$$

We know that the fundamental tensor $g_{ij}=\frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\,\partial y^3}$. We shall check whether the matrix (g_{ij}) of the Finsler space (M,L) is positive definite or not. We have examined for the +ve definiteness of the metric (g_{ij}). It is the following Chern and Shen's lemma.

Lemma 3.1. [1] $L = \alpha\phi\left(\frac{\beta}{\alpha}\right)$ is a Minkowski norm for any Riemannian metric α and 1-form β with $||\beta||_\alpha < b_0$ if any only if $\phi = \phi(s)$ satisfies the following conditions:

$$\phi(s) > 0,\ (\phi(s) - s\phi'(s)) + (b^2 - s^2)\phi''(s) > 0 \qquad ...(20)$$

where s and b are arbitrary numbers with $|s| \leq b < b_0$.

After that, in [2], B.L. Lovas defined a Finsler-Minkowski norm as follows:

Definition 3.1. A Finsler-Minkowski norm on a vector space V is a function $F:V \to R$ that satisfies the following axioms:

(F_1)$F(\upsilon) > 0$ if $\upsilon \neq 0$ (Positively);

(F_2) if $\lambda \in R$ is positive then $F(\lambda\nu)=\lambda F(\nu)$ for all $\nu \in V$ (Positive homogeneity)

(F_3) F is class of C^∞ over $V \setminus \{0\}$;

(F_4)$E = ½F^2$, then for all $p \in V \setminus \{0\}$ the symmetric bilinear form

$g_p = E''(p) : V \times V \to R$

is non-degenerate and he proved.

Theorem 3.1. The metric tensor g of a Finsler - Minkowski norm is positive definite.

Lemma (3.1) follows from theorem (3.1)

For the metric (19), we can take b_0 as ∞ and have only to check the two conditions in (20).

The first condition in (20) trivially satisfies, since we have

$$\phi'(s) = \frac{s^3}{(1+s^4)^{\frac{3}{4}}} \ and\ \phi''(s) = \frac{3s^3}{(1+s^4)^{\frac{7}{4}}}$$

we get

$$\left(\phi(s) - s\phi'(s)\right) + (b^2 - s^2)\phi''(s) = \sqrt[4]{1+s^4} - \frac{s^4}{\sqrt[4]{(1+s^4)^3}} +$$

$$(b^2 - s^2)\frac{3s^4}{\sqrt[4]{(1+s^4)^7}} = \frac{1}{\sqrt[4]{(1+s^4)^7}}\{-2s^4 + 3s^2b^2 + 1\}$$

Since, we have

$$-2s^4 + 3s^2b^2 + 1 = -2\{(s^2 - \frac{3}{4}b^2)^2 - \frac{9b^4+8}{16}\}$$

therefore $-2s^4 + 3s^2b^2 + 1 > 0$ if $s^2 < \frac{3}{4}b^2 + \frac{1}{2}\sqrt{\frac{9b^4}{4} + 2}$ which is always true for $s^2 \leq b^2$

Theorem 3.2. The matrix g_{ij} of the Finsler space (M,L), where L = $L = \sqrt[4]{\alpha^4 + \beta^4}$ is always positive definite.

4. The Christoffel symbol of fourth order

For $L^4 = a_{ijkl}y^iy^jy^ky^l$, the symmetric tensors $\hat{a}_{ijk}$, $\hat{a}_{ij}$ and $\hat{a}_i$ are defined as follows :

$$L\hat{a}_{ijk} = a_{ijks_1}y^{s_1} \quad \text{... (21)}$$

$$L^2\hat{a}_{ij} = a_{ijs_1s_2}y^{s_1}y^{s_2} \quad \text{... (22)}$$

$$L^3\hat{a}_i = a_{is_1s_2s_3}y^{s_1}y^{s_2}y^{s_3} \quad \text{... (23)}$$

Since $\alpha^2 = a_{ij}y^iy^j$, $\beta = b_iy^i$ and $L^4 = \alpha^4 + \beta^4$, we have

$$L^4 = (a_{ij}a_{kl} + b_ib_jb_kb_l)y^iy^jy^ky^l, \quad \text{... (24)}$$

Comparing with,

$L^4 = a_{ijkl}y^iy^jy^ky^l$, we get

$3a_{ijkl} = a_{ij}a_{kl} + a_{ik}a_{jl} + a_{il}a_{jk} + 3b_ib_jb_kb_l$

So, we get

Lemma 4.1. For the quadratic metric $L = 4\sqrt{\alpha^4+\beta^4}$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$, the following equations holds :

$$a_{ijkl} = 1/3(a_{ij}a_{kl} + a_{ik}a_{jl} + a_{il}a_{jk}) + b_ib_jb_kb_l \quad \text{...(25)}$$

where a_{ijkl} are the components of a symmetric tensor field covariant of order 4 defined by $L^4 = a_{ijkl}y^iy^jy^ky^l$.

$L^3\hat{a}_i = a_{ijkl}y^iy^ky^l$

$L^3\hat{a}_i = a_{i0}\alpha^2 + b_i\beta^3$

From the definition (22) and (25), we get

$L^2\hat{a}_{ij} = a_{ijkl}y^ky^l$

$L^2\hat{a}_i = 1/3(a_{ij}\alpha^2 + 2a_{i0}a_{j0}) + b_ib_j\beta^2$

Now from the definition (1) and (5), we get

$L\hat{a}_{ijk} = a_{ijkl}y^1$

$L\hat{a}_{ijk} = 1/3(a_{ij}a_{k0} + a_{ik}a_{j0} + a_{i0}a_{jk}) + b_ib_jb_k\beta$

Hence, we get

Lemma 4.2. For the quartic metric $L = \sqrt[4]{\alpha^4 + \beta^4}$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y_i$, the following equations hold:

$L^3\hat{a}_i = a_{i0}\alpha^2 + b_i\beta^3$

$L^2 \hat{a}_{ij} = 1/3(a_{ij}\alpha^2 + 2a_{i0}a_{j0})b_i b_j \beta^2$

and

$L\hat{a}_{ijk} = 1/3(a_{ij}a_{k0} + a_{ik}a_{j0} + a_{i0}a_{jk}) + b_i b_j b_k \beta$

where $\hat{a}_1$, $\hat{a}_{ij}$ and $\hat{a}_{ijk}$ are the symmetrical tensor defined by (21), (22) and (23) respectively.

From (8) the Christoffel symbols of the fourth order is defined by

$$\{ijkl;p\} = \frac{1}{6}\{\frac{\partial a_{jklp}}{\partial x^i} + \frac{\partial a_{klpi}}{\partial x^j} + \frac{\partial a_{lpij}}{\partial x^k} + \frac{\partial a_{pijk}}{\partial x^l} - \frac{\partial a_{ijkl}}{\partial x^p}\} \qquad ...(26)$$

and from (12) the coefficient G^i of the geodesic spray is given by

$$G^i = \tfrac{1}{4} L^{-2}\hat{a}^{ip}\{0000;p\} \qquad ...(27)$$

where $(L^{-2}\hat{a}^{ij})$ is the inverse of the matrix $(L^2\hat{a}_{ij})$.

Calculation of $L^{-2}\hat{a}^{ij}$

Since $L^2\hat{a}_{ij} = 1/3(a_{ij}\alpha^2 + 2a_{i0}a_{j0}) + b_i b_j \beta^2$

$\Rightarrow \hat{a}_{ij} = 1/3L^2(a_{ij}\alpha^2 + 2a_{i0}a_{j0}) + 1/L^2 b_i b_j \beta^2$

since $\hat{a}_{jk}a^{ik} = \delta_j^i$

So,

$$\frac{1}{3}\left[\frac{1}{L^2}\left(a_{jk}\,\alpha^2 \hat{a}^{ik}\right) + 2\hat{a}^{ik} a_{jp}\, a_{kq}\, l^p\, l^q\right] + \frac{1}{l^2}\hat{a}^{ik} b_j\, b_k\, \beta^2 = \delta_j^i \qquad ...(28)$$

Contracting (4.8) with a^{jh} we get

$$\frac{1}{3}\left[L^{-2}\hat{a}^{ih}\,\alpha^2 + 2l^h\left(\hat{a}^{ik} a_k\right)\right] + \frac{1}{L^2}\left(\hat{a}^{ik} b_k\right) b^h \beta^2 = a^{ih} \qquad ...(29)$$

Contracting (28) with b^j we get

$$\frac{1}{3}\left[\frac{1}{L^2} b_k \hat{a}^{ik}\,\alpha^2 + 2\left(\hat{a}^{ik} a_k\right) a_j\, b^j\right] + \frac{1}{L^2}\left(\hat{a}^{ik} b_k\right) b^h \beta^2 = b^i \qquad ...(30)$$

Again contracting (28) with l^j then we get

$$\hat{a}^{ik} a_k = \frac{3L^3 l^i - \hat{a}^{ik} b_k \beta^3}{L(\alpha^2 + 2L^2)} \qquad ...(31)$$

using equation (31) in (30) we get

$$\hat{a}^{ik} b_k = \frac{3L^2 b^i(\alpha^2+2L^2) - 6a_j b^j L^4 l^i}{(\alpha^2+2L^2)(\alpha^2+b^2\beta^2) - 6La_3 b^j \beta^3} \qquad ...(32)$$

using (32) in (31)

$$\hat{a}^{ik} a_k = \frac{3L^2 l^i}{(\alpha^2 + 2L^2)} - \frac{3\beta^3}{L(\alpha^2 + 2L^2)}\left[\frac{3L^2 b^i(\alpha^2 + 2L^2) - 6a_j b^j L^4 l^i}{(\alpha^2 + 2L^2)(\alpha^2 + 3b^2\beta^2) - 6La_3 b^j \beta^3}\right] \qquad ...(33)$$

After using equation (32) and (33) in equation (29) and putting $a_j b^j = \beta/L$ then we get

$$L^{-2}\hat{a}^{ij} = \frac{3}{\alpha^2}\Big[a^{ij} - \frac{1}{\{(\alpha^2+2L^2)(\alpha^2+3b^2\beta^2)-6\beta^4} x\{2(\alpha^2 + 3b^2\beta^2)y^i y^j - 6\beta^3(y^i b^j + y^j b^i) + 3\beta^2(\alpha^2 + 3L^2)b^i b^j\}\Big]$$

Now we shall calculate {0000;p}. Using (25) and (26), we get

$$\{ijk;l\} = \frac{1}{6}\{\frac{2}{3}(a_{ij}\gamma_{klp} + a_{ik}\gamma_{jlp} + a_{il}\gamma_{jkp} + a_{jk}\gamma_{ilp} + a_{jl}\gamma_{ikp} + a_{kl}\gamma_{ijp} + \frac{1}{3}\sum_{\{ijkl\}} a_{ip}\left(\frac{\partial a_{kl}}{\partial x^j} + \frac{\partial a_{lj}}{\partial x^k} + \frac{\partial a_{jk}}{\partial x^l}\right) + 2(b_j b_k b_l s_{pi} + b_i b_l b_k s_{pj} + b_i b_j b_l s_{pk} + b_i b_j b_k s_{pl}) + 2(b_j b_k b_p r_{il} + b_j b_l b_p r_{ki} + b_k b_l b_p r_{ij} + b_i b_p b_k r_{jl} + b_i b_p b_l r_{kj} + b_i b_j b_p r_{lk}$$

where,

$$\gamma^i_{jk} = \frac{a^{ih}}{2}(\frac{\partial a_{hj}}{\partial x^k} + \frac{\partial a_{hk}}{\partial x^j} - \frac{\partial a_{jk}}{\partial x^h})$$

$$b_{i;j}\ \frac{\partial b_i}{\partial x^j} - b_h\gamma^h_{ij}\, and\, b_{j;i} = \frac{\partial b_j}{\partial x^i} - b_h\gamma^h_{ji}$$

$$s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}) = \frac{1}{2}(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i})$$

$$r_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}) = \frac{1}{2}\left(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\right) - b_h\gamma^h_{ij} \text{ and}$$

$$\gamma_{jkl} = \frac{1}{2}[\frac{\partial a_{ij}}{\partial x^k} + \frac{\partial a_{lk}}{\partial x^j} - \frac{\partial a_{jk}}{\partial x^l}]$$

there we get

Lemma 4.3. For the quartic metric $L = \sqrt[4]{\alpha^4 + \beta^4}$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$, the following equations holds:

$$\{ijkl;p\} = \frac{1}{18}\{2(a_{ij}\gamma_{klp} + a_{ik}\gamma_{jlp} + a_{il}\gamma_{jkp} + a_{jk}\gamma_{ilp} + a_{jl}\gamma_{ikp} + a_{kl}\gamma_{ijp}) + \sum_{ijkl} a_{ip}\left(\frac{\partial a_{kl}}{\partial x^j} + \frac{\partial a_{lj}}{\partial x^k} + \frac{\partial a_{jk}}{\partial x^l}\right) + 6(b_j b_k b_l s_{pi} + b_i b_l b_k s_{pj} + b_i b_j b_l s_{pk} + b_i b_j b_k s_{pl}) + 6(b_j b_k b_p r_{il} + b_j b_l b_p r_{ki} + b_k b_l b_p r_{ij} + b_i b_p b_k r_{jl} + b_i b_p b_l r_{kj} + b_i b_j b_p r_{lk}$$

using equation (24) we get

$$\{0000;p\} = \frac{2}{3}[\alpha^2\gamma_{00p} + 2a_{0p}\gamma_{000} + 2\beta^2 s_{p0} + 3\beta^2 b_p r_{00}] \qquad ...(34)$$

using the value of $L^{-2}\hat{a}^{ij}$ and equation (34) in (33) we get

$G^i = \frac{3}{4\alpha^2}\left[a^{ip} - \frac{1}{\{(\alpha^2+2L^2)(\alpha^2+3b^2\beta^2)-6\beta^4\}} x\{2(\alpha^2+3b^2\beta^2)y^i y^p - 6\beta^3(y^i b^p + y^p b^i) + 3\beta^2(\alpha^2+3L^2)b^i b^p\}\right] x\ [\frac{2}{3}(\alpha^2\gamma_{00p} + 2a_{0p}\gamma_{000} + 2\beta^3 s_{p0} + 3\beta^2 b_p r_{00})]$

$$= \frac{1}{2\alpha^2}[\alpha^2\gamma^i_{00} + 2\beta^3 s^i_0 + 3b^i\beta^2 r_{00} - \frac{1}{\{(\alpha^2+2L^2)(\alpha^2+3b^2\beta^2)-6\beta^4\}}$$

$$x\{6\beta^3((\alpha^2+3b^2\beta^2)r_{00} - \alpha^2\gamma_{00*} - \beta^3 s_{*0} - 3\beta^2 b^2 r_{00})y^i + 3\beta^2(\alpha^2(\alpha^2+2L^2)\gamma_{00*} + 2\beta^3(\alpha^2+2L^2)s_{*0} + 3\beta^2(\alpha^2+2L^2)b^2 r_{00} - 6\beta^4 r_{00})b^i\}]$$

where we put $r_{00p}b^p = \gamma_{00*}$ and $s_{p0}b^p = s_{*0}$

summarizing the above discussion we get

Theorem 4.1. Let (M,L) be a Finsler space with an (α,β)-metric L = $L = \sqrt[4]{\alpha^4+\beta^4}$. The coefficients G^i of the geodesic spray of (M,L) are given by

$$G^i = \frac{1}{2}\gamma^i_{00} + \frac{1}{2\alpha^2}(2\beta^3 s^i_0 + 3b^i\beta^2 r_{00}) - \frac{1}{2\alpha^2\{(\alpha^2+2L^2)(\alpha^2+3b^2\beta^2)-6\beta^4\}}\ x$$

$$\{6\beta^3(\alpha^2 r_{00} - \alpha^2\gamma_{00*} - \beta^3 s_{*0})y^i + 3\beta^2(3\{b^2\beta^2(\alpha^2+2L^2)\gamma_{00*} - 2\beta^4\}r_{00} + \alpha^2(\alpha^2+2L^2)\gamma_{00*} + 2\beta^3(\alpha^2+2L^2)s_{*0})b^i)\}$$

5. A Finsler space (M,L), where $L = \sqrt[4]{\alpha^4+\beta^4}$, which is S_3-like space

In[6], we consider S_3-like Finsler spaces (M, αϕ(β/α)) and obtained

Theorem 5.1. [6] Suppose that an (n ≥ 4)-dimensional Finsler space (M, αϕ(β/α)) satisfy $C_i \neq 0$. It is S_3-like if and only if the function w(s) satisfy the differential equation.

$$w - sw' + (b^2 - s^2)w'' = 0 \qquad ...(35)$$

where s = β/α and $w = \frac{\phi'}{\phi - s\phi'}$

For a Finsler space (M,L), where $L = \sqrt[4]{\alpha^4+\beta^4}$ where we have

$$\phi(s) = \sqrt[4]{1+s^4}\ and\ \phi'(s) = \frac{s^3}{(1+s^4)^{3/4}}$$

then we have $w = s^3$

$$w - sw' + (b^2 - s^2)w'' = 2s(3s^2 - 4s^2) \neq \qquad \text{for } n > 1$$

thus from theorem (35) we obtain

Theorem 5.2. An n(≥4) – dimensional Finsler space (M,L), where L = $L = \sqrt[4]{\alpha^4+\beta^4}$, is not S_3-like.

6. The Curvature tensor

Putting m = 4 in (18), we get

$$L^2 \hat{a}_{ir} R^r_{jk} = \Pi_{\{jk\}}\{\frac{\delta}{\delta x^k}\left(\frac{2}{3}\{i000;j\} + \frac{2}{3}\{j000;i\} - \{ji00;0\} - 2a_{irj0}G^r\right)\}$$

From (25) we have

$$\{i000;p\} = \frac{1}{6}\{2\left(a_{i0}\gamma_{00p} + \alpha^2\gamma_{i0p}\right) + \left(2a_{ip}\gamma_{000} + 4a_{0p}\gamma_{0i0} + 2a_{0p}\gamma_{00i}\right) + 2\left(\beta^3 s_{pi} + 3b_i\beta^2 s_{p0}\right) + 6\left(\beta^2 b_p r_{i0} + b_i b_p \beta r_{00}\right)\}$$

and

$$\{ij00;0\} = \frac{1}{6}\{\frac{2}{3}\left[a_{ij}\gamma_{000} + 2a_{i0}\gamma_{00j} + 2a_{j0}\gamma_{i00} + \alpha^2\gamma_{ij0}\right] + \frac{1}{3}\left[a_{i0}\left(2\gamma_{00j} + 4\gamma_{0j0}\right) + a_{j0}(4\gamma_{0i0} + 2\gamma_{00i}) + \alpha^2\left(4\gamma_{ij0} + 4\gamma_{0ji} + 4\gamma_{0ij}\right)\right] + 2\left[\beta^2 b_j s_{0i} + \beta^2 b_i s_{0j}\right] + 2\left[2b_j\beta^2 r_{i0} + 2\beta^2 b_i r_{j0} + \beta^3 r_{ij} + b_i b_j \beta r_{00}\right]\}$$

Therefore we have

$$\frac{2}{3}\{i000;j\} + \frac{2}{3}\{j000;i\} - \{ji00;0\}$$

$$= \frac{1}{9}[2\{a_{i0}\gamma_{00j} + \alpha^2\gamma_{i0j} + a_{j0}\gamma_{00i} + \alpha^2\gamma_{j0i}\} + 2\{2a_{ij}\gamma_{000} + 2a_{0j}\gamma_{0i0} + 2\gamma_{0i}\gamma_{0j0} + a_{0j}\gamma_{00i} + a_{0i}\gamma_{00j}\} + 6\{b_i\beta^2 s_{j0} + b_j\beta^2 s_{i0}\} + 6\{\beta^2 b_j r_{i0} + \beta^2 b_i r_{j0} + 2b_i b_j \beta r_{00}\}]$$

$$-\frac{1}{18}[2\{a_{ji}\gamma_{000} + 2a_{ji}\gamma_{00i} + 2a_{i0}\gamma_{j00} + \alpha^2\gamma_{ji0}\} + \{a_{j0}(2\gamma_{00i} + 4\gamma_{0i0}) + a_{i0}\left(4\gamma_{0j0} + 2\gamma_{00j}\right) + \alpha^2\left(4\gamma_{ji0} + 4\gamma_{0ij} + 4\gamma_{0ji}\right)\} + 6\{\beta^2 b_i s_{0j} + \beta^2 b_j s_{0i}\} + 6\{2b_i\beta^2 r_{j0} + 2\beta^2 b_j r_{i0} + \beta^3 r_{ji} + b_i b_j \beta r_{00}\}]$$

$$= \frac{1}{9}[3a_{i0}\gamma_{00j} + 3a_{j0}\gamma_{00i} + 3a_{ij}\gamma_{000} + 2a_{0j}\gamma_{0i0} - 2a_{ij}\gamma_{00i} - 3\alpha^2\gamma_{ij0} + 3b_i\beta^2 s_{0j} + 3b_j\beta^2 s_{0i} + 9b_i b_j \beta r_{00} - 3\beta^3 r_{ij}]$$

Next we shall calculate $2a_{irj0}G^r$.

From (25) we have

$$a_{irj0} = \frac{1}{3}\left[a_{ir}a_{j0} + a_{ij}a_{r0} + a_{i0}a_{rj}\right] + b_i b_r b_j \beta$$

Now putting

$$G(s) = \frac{\beta^3}{\alpha^2} \qquad ...(36)$$

$$G(y) = \frac{6\beta^3(\alpha^2 r_{00} - \alpha^2\gamma_{00*} - \beta^3 s_{*0})}{2\alpha^2\{(\alpha^2+2L^2)(\alpha^2+3b^2\beta^2)-6\beta^4\}} \quad ...(37)$$

$$G(b) = \frac{(\alpha^2+2L^2)(3\beta^3 r_{00}\alpha^2 - \alpha^2\gamma_{00*} - 2\beta^3 s_{*0})}{2\alpha^2\{(\alpha^2+2L^2)(\alpha^2+3b^2\beta^2)-6\beta^4\}} \quad ...(38)$$

then we have

$$G^r = \frac{1}{2}\gamma^r_{00} + G_{(s)}s^r_0 + G_{(y)}y^r + G_{(b)}b^r$$

therefore

$$2a_{irj\,0}G^r = \left[\frac{2}{3}\left(a_{ir}a_{j0} + a_{ij}a_{r0} + a_{i0}a_{rj}\right) + 2b_i b_r b_j \beta\right] x\, [\frac{1}{2}\gamma^r_{00} + G_{(s)}s^r_0 + G_{(y)}y^r + G_{(b)}b^r]$$

$$= \frac{1}{3}[\left(a_{j0}\gamma_{00i} + a_{i0}\gamma_{00j} + a_{ij}\gamma_{000}\right) + 3b_i b_r b_j \beta\gamma^r_{00}]$$

$$+ \frac{2}{3}G_{(s)}[\left(s_{i0}a_{j0} + a_{i0}s_{j0}\right) + 3b_i b_j \beta s_o]$$

$$+ \frac{2}{3}G_{(y)}[\left(a_{i0}a_{j0} + a_{ij}\alpha^2 + a_{i0}a_{0j}\right) + 3b_i b_j \beta^2]$$

$$+ \frac{2G_b}{3}[\left(b_i a_{j0} + a_{ij}\beta + a_{i0}b_j\right) + 3b_i b_j b^2\beta]$$

Hence we get

$$\begin{aligned}
&\tfrac{2}{3}\{i000;j\} + \tfrac{2}{3}\{j000;i\} - \{ji00;0\} - 2a_{irj0}G^r \\
&= \tfrac{1}{9}[3a_{i0}\gamma_{00j} + 3a_{j0}\gamma_{00i} + 3a_{ij}\gamma_{000} + 2a_{j0}\gamma_{0i0} - a_{ij}\gamma_{00i} - 3\alpha^2\gamma_{ij0} + 3b_i\beta^2 s_{0j} \\
&\quad + 3b_j\beta^2 s_{0i} + 9b_i b_j \beta r_{00} - 3\beta^3 r_{ij}] - \tfrac{1}{3}[(a_{j0}\gamma_{00i} + a_{i0}\gamma_{00j} + a_{ij}\gamma_{000})] \\
&\quad - b_i b_j b_r \beta\gamma^r_{00} - \tfrac{2G_{(s)}}{3}[(s_{i0}a_{j0} + a_{i0}s_{j0}) + 3b_i b_j \beta s_0] \\
&\quad - \tfrac{2G_{(y)}}{3}[(a_{i0}a_{j0} + a_{ij}\alpha^2 + a_{i0}a_{0j}) + 3b_i b_j \beta^2] \\
&\quad - \tfrac{2G_{(b)}}{3}[(b_i a_{j0} + a_{ij}\beta + a_{i0}b_j) + 3b_i b_j b^2\beta] \\
&= \tfrac{1}{9}[2a_{j0}\gamma_{0i0} - 2a_{ij}\gamma_{00i} - 3\alpha^2\gamma_{ij0} + 3b_i\beta^2 s_{0j} + 3b_j\beta^2 s_{0i} + 9b_i b_j \beta r_{00} \\
&\quad - 9b_i b_j b_r \beta\gamma^r_{00} - 3\beta^3 r_{ij}] \\
&\quad - \tfrac{2G_{(s)}}{3}[(s_{i0}a_{j0} + a_{i0}s_{j0}) + 3b_i b_j \beta s_0] \\
&\quad - \tfrac{2G_{(y)}}{3}[(a_{i0}a_{j0} + a_{ij}\alpha^2 + a_{i0}a_{0j}) + 3b_i b_j \beta^2] \\
&\quad - \tfrac{2G_{(b)}}{3}[(b_i a_{j0} + a_{ij}\beta + a_{i0}b_j) + 3b_i b_j b^2\beta]
\end{aligned}$$

Summarizing the about discussion, we get

Theorem 6.1. Denoting the (v)h-torsion tensor of the n-dimensional Finsler space (M,L), where $L = \sqrt[4]{\alpha^4 + \beta^4}$, by R^r_{jk} , we get

$$L^2\hat{a}_{ir}R^r_{jk} = \Pi_{\{j,k\}}\{\frac{\delta}{\delta x^k}[\frac{2}{3}(i000;j) + \frac{2}{3}(j000;i) - \{ji00;0\} - 2a_{irj0}G^r]\}$$

where

$$\begin{aligned}&\frac{2}{3}\{i000;j\} + \frac{2}{3}\{j000;i\} - \{ji00;i\} - 2a_{irj0}G^r \\ &= \frac{1}{9}[2a_{j0}\gamma_{0i0} - 2a_{ij}\gamma_{00i} - 3\alpha^2\gamma_{ij0} + 3b_i\beta^2 s_{0j} + 3b_j\beta^2 s_{0i} + 9b_ib_j\beta r_{00} - \\ &\qquad 9b_ib_j\beta\gamma^r_{00} - 3\beta^3 r_{ij}] \\ &\quad - \frac{2G_{(s)}}{3}[(s_{i0}a_{j0} + a_{i0}s_{j0}) + 3b_ib_j\beta s_0] \\ &\quad - \frac{2G_{(y)}}{3}[(2a_{i0}a_{j0} + a_{ij}\alpha^2) + 3b_ib_j\beta^2] \\ &\quad - \frac{2G_{(b)}}{3}[(b_ia_{j0} + a_{ij}\beta + a_{i0}b_j) + 3b_ib_jb^2\beta]\end{aligned}$$

where $G_{(s)}$, $G_{(y)}$ and $G_{(b)}$ are defined in (36), (37) and (38) respectively.

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