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GENERALIZED MATSUMOTO CHANGE OF FINSLER METRIC BY H-VECTOR

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Abstract : The purpose of the present paper is to find necessary and sufficient conditions under which a generalized Matsumoto change of Finsler metric by h-vector becomes a projective change. We have also found the condition under which the above change of metric of Douglas space gives a Douglas space.

Key Words : Finsler metric, Matsumoto change, Generalized Matsumoto change, h-vector and Projective change.

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1. Introduction

Let $F^n = (M^n, L)$ be n-dimensional Finsler space on a differentiable manifold M^n, equipped with the fundamental function L(x,y). Various changes of Finsler metric have been studied recently in papers Shukla et al. [6,7,8,9,10,11]

The necessary and sufficient conditions for these changes to be projective have been obtained. The condition for the space with the changed metric to be Douglas space has been found out.

The generalized Matsumoto change of Finsler metric is given by

$$\overline{L}(x, y) = \frac{L^{m+1}}{(L-\beta)^m}, \text{ where } \beta = b_i(x) y^i. \qquad \ldots(1)$$

In the present paper we have considered the transformation (1) in which $b_i(x)$ in β has been replaced by h-vector $b_i(x,y)$ so that $\frac{\partial b_i}{\partial y^j}$ is proportional to the angular metric tensor h_{ij}.

Let $\frac{\partial b_i}{\partial y^j} = \rho h_{ij}$, where ρ is any scalar function of x,y and $h_{ij} = g_{ij} - \ell_i \ell_j$. …(2)

It has been shown by Shukla, Pandey and Joshi in [12] that

$$\dot{\partial}_k \rho = -\frac{\rho}{L}\ell_k, \text{ for } n > 2, \text{ where } \dot{\partial}_k = \frac{\partial}{\partial y^k} \quad \text{…(3)}$$

We shall use the equation (3) without quoting it in the present paper.

Let $\beta = B_i(x, y)y^i$ be defined on the manifold M^n. Then $L \to \frac{L^{m+1}}{(L-\beta)^m}$ is called generalized Matsumoto change of Finsler metric by h-vector. If we write $\overline{L} = \frac{L^{m+1}}{(L-\beta)^m}$ and $\overline{F}^n = (M^n, \overline{L})$ then the Finsler space $\overline{F}^n$ is said to be obtained from F^n by a generalized Matsumoto change of Finsler metric by h-vector.

If m = 1, then the generalized Matsumoto change of Finsler metric by h-vector reduces to Matsumoto change of Finsler metric by h-vector. The quantities corresponding to $\overline{F}^n$ will be noted by putting bar over those quantities.

The fundamental quantities of F^n are given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \ell_i = \frac{\partial L}{\partial y^i} \text{ and } h_{ij} = L\frac{\partial^2 L}{\partial y^i \partial y^j} = g_{ij} - \ell_i \ell_j.$$

We shall denote the partial derivatives with respect to x^i, y^i by $\partial_i, \dot{\partial}_i$ respectively and write

$$L_i = \dot{\partial}_i L, \; L_{ij} = \dot{\partial}_i \dot{\partial}_j L, \; L_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L.$$

Then $L_i = l_i, \; L^{-1} h_{ij} = L_{ij}$.

The geodesic of F^n are given by the system of differential equations

$$\frac{d^2 x^i}{ds^2} + 2G^i\left(x, \frac{dx}{ds}\right) = 0,$$

where $G^i(x,y)$ are positively homogeneous of degree two in y^i:

$$2G^{i} = g^{ij}(y^{r}\dot{\partial}_{j}\partial_{r}F - \partial_{j}F)\; F = \frac{L^{2}}{2}$$

and g^{ij} is the inverse of g_{ij}.

Berwald connection $B\Gamma = (G^{i}_{jk}, G^{i}_{j}, 0)$ of Finsler space is given by (Matsumoto [2]).

$$G^{i}_{j} = \frac{\partial G^{i}}{\partial y^{j}}, G^{i}_{jk} = \frac{\partial G^{i}_{j}}{\partial y^{k}}.$$

The Cartan connection $C\Gamma = (F^{i}_{jk}, G^{i}_{j}, G^{i}_{jk})$ is constructed from L with the help of following axioms given by Matsumoto [2]

1. Cartan connection CΓ is v-material;
2. Cartan connection CΓ is h-material;
3. The (v) v torsion tensor field S of Cartan connection vanishes;
4. The (h) h torsion tensor field T of Cartan connection vanishes;
5. The deflection tensor field D of cartan connection vanishes;

The h-and v-covariant derivatives with respect to Cartan connection are denoted by $|_k$ and $|_k$ respectively. It is clear that the h-covariant derivative of L with respect to BΓ and CΓ is the same and vanishes identically. Furthermore, the h-covariant derivatives of L_i, L_{ij} with respect to CΓ are zero. We shall write

$$2r_{ij} = b_{i/j} + b_{j/i}, 2s_{ij} = b_{i/j} - b_{j/i}.$$

2. The difference tensor D^i

The generalized Matsumoto change of Finsler metric by h-vector is given by

$$\overline{L} = \frac{L^{m+1}}{(L-\beta)^{m}}, \qquad \ldots(4)$$

where $\beta = b_i(x, y)y^i$ and $b_i(x, y)$ is an h-vector.

We may put

$$\overline{G}^{i} = G^{i} + D^{i} \qquad \ldots(5)$$

Then $\overline{G}^i_j = G^i_j + D^i_j$ and $\overline{G}^i_{jk} = G^i_{jk} + D^i_{jk}$,

where $D^i_j = \partial_j D^i$ and $D^i_{jk} = \partial_k D^i_j$

The tensors D^i, D^i_j and D^i_{jk} are positively homogeneous in y^i of degree two, one and zero respectively.

To find D^i we deal with equation $L_{ij/k} = 0$, (Matsumoto [1]):

$$\partial_k L_{ij} - L_{ijr} G^r_k - L_{rj} F^r_{ik} - L_{ir} F^r_{jk} = 0 \qquad \ldots(6)$$

Since $\partial_i \beta = b_i$, from (4), we have

(a) $$\overline{L}_i = pL_i + qb_i, \qquad \ldots(7)$$

(b) $$\overline{L}_{ij} = \mu L_{ij} + \eta[\beta^2 L^{-1} L_i L_j - \beta(L_i b_j + L_j b_i) + Lb_i b_j],$$

(c) $$\partial_j \overline{L}_i = p\partial_j L_i + \psi(\beta L_i - Lb_i)\partial_j L + \eta(Lb_i - \beta L_i)\partial_j \beta + q\partial_j b_i,$$

(d) $$\partial_k \overline{L}_{ij} = \mu\partial_k L_{ij} - \phi\left\{\nu L_{ij} - \frac{\beta^2}{L^2}\right\}(\beta - m\beta - 3L)L_i L_j$$

$$-\frac{\beta}{L}(m\beta + 2L)(L_i b_j + L_j b_i) - (\beta + m\beta + L)b_i b_j\}\partial_k L$$

$$+\phi\{(\rho L^2 - \beta)(L - \beta)L_{ij} + 2\frac{\beta}{L}(L + \beta)L_i L_j)$$

$$-(\beta + m\beta + L)(L_i b_j + L_j b_i) + (m + 2)Lb_i b_j\}\partial_k \beta$$

$$+\psi(\beta L_j - Lb_j)\partial_k l_i + \psi(\beta L_i - Lb_i)\partial_k l_j$$

$$+\eta(Lb_j - \beta L_j)\partial_K b_i + \eta(Lb_i - \beta L_i)\partial_K b_j,$$

(e) $$\bar{L}_{ijk} = \mu L_{ijk} + \eta[\left(\frac{\beta^2}{L} - \rho L^2\beta\right)(L_i L_{jk} + L_j L_{ik} + L_k L_{ij})$$

$$+(\rho L^2 - \beta)(b_i L_{jk} + b_j L_{ik} + b_k L_{ij})] + \phi[\left(2\beta + \frac{m\beta^2}{L}\right)$$

$$(L_i L_j b_k + L_k L_i b_j + L_k b_i) + \left\{\frac{\beta^3(1-m)}{L^2} - \frac{3\beta^2}{L}\right\} L_i L_j L_k$$

$$-(L + \beta + m\beta)(L_i b_j b_k + L_j b_k b_i + L_k b_i b_j) + (m+2)Lb_i b_j b_k],$$

where we have used:

$$p = \frac{L^m}{(L-\beta)^m}[L - \beta(m+1)],\ q = \frac{mL^{m+1}}{(L-\beta)^{m+1}},\ \eta = \frac{m(m+1)L^m}{(l-\beta)^{m+2}},$$

$$\mu = \frac{L^m}{(L-\beta)^{m+1}}[L - \beta(m+1) + \rho m L^2],\ \phi = \frac{m(m+1)\beta L^m}{(L-\beta)^{m+3}}$$

$$\nu = \left[\rho\beta L(L-\beta) + \frac{\beta^2(L-\beta)}{L} - \frac{\rho L(L-\beta)^2}{(m+1)}\right],\ \psi = \frac{m(m+1)\beta L^{m-1}}{(L-\beta)^{m+2}},$$

Since $\bar{L}_{ij/k} = 0 \text{ in } \bar{F}^n$, after using (5), we have

$$\partial_k \bar{L}_{ij} = \bar{L}_{ijr}(G^r_k + D^r_k) - \bar{L}_{rj}(F^r_{ik} + {}^c D^r_{ik}) - (F^r_{jk} + {}^c D^r_{jk}) = 0,$$

where $\bar{F}^i_{jk} - F^i_{jk} = {}^c D^r_{jk}$. (Park [5]).

Substituting in the above equation the values of $\partial_k \bar{L}_{ij}, \bar{L}_{ir} \text{ and } \bar{L}_{ijr}$ from (7) and using (6), and then contracting the equation thus obtained with y^k, we get

$$2\bar{L}_{ijr}D^r + \bar{L}_{jr}D^r_i + \bar{L}_{ir}D^r_j - [\phi\{(L-\beta)(\rho L^2 - \beta)L_{ij}$$

$$+\left(2\beta + \frac{\beta^2}{L}\right)L_i L_j - (L + \beta + m\beta)(L_i b_j + L_j b_i)$$

$$+(m+2)\,Lb_i b_j\}r_{00}]+\eta[(Lb_i-\beta L_i)(r_{j0}-S_{j0})$$

$$-(Lb_j-\beta L_j)(r_{i0}+S_{i0})]+q\rho_0 LL_{ij}+2\rho q L_r L_{ij} G^r=0, \qquad \text{...(8)}$$

where '0' stands for contraction with y^k viz $r_{j0}=r_{jk}t^k, r_{00}=r_{ij}y^i y^j$

and we have used the fact that $D^i_{jk}y^k={}^c D^i_{jk}y^k=D^i_j$. (Matsumoto [2]):

Next, we deal with $\overline{L}_{i/j}=0,$ that is

$$\partial_j\overline{L}_I-\overline{L}_{ir}\overline{G}^r_j-\overline{L}_r\overline{F}^r_{ij}=0.$$

Then, we have

$$\partial_j\overline{L}_i-\overline{L}_{ir}(G^r_j+D^r_j)-\overline{L}_r(F^r_{ij}+{}^c D^r_{ij})=0. \qquad \text{...(9)}$$

Putting the value of $\partial_j\overline{L}_i, \overline{L}_{ir}$ and $\overline{L}_r$ from (7) in (9) and using the equation $L_{i/j}=\partial_j L_i-L_{ir}G^r_j-\overline{L}_r F^r_{ij}=0,$ and rearranging the terms, we get

$$q\,b_{i/j}=[\mu L_{ir}+\eta\{\beta^2 L^{-1}L_i L_r-\beta(L_i b_r+L_r b_i)+Lb_i b_r\}]D^r_j$$

$$+[pL_r+qb_r]\,{}^c D^r_{ij}+\eta(\beta L_i-Lb_i)(r_{0j}+S_{0j}),$$

which after using $2r_{ij}=b_{i/j}+b_{j/i}, 2S_{ij}=b_{i/j}-b_{j/i},$ we have

$$2q\,r_{ij}=[\mu L_{ir}+\eta\{\beta^2 L^{-1}L_i L_r-\beta(L_i b_r+L_r b_i)+b_i b_r L\}]\,D^r_j$$

$$+[\mu L_{jr}+\eta\{\beta^2 L^{-1}L_j L_r-\beta(L_j b_r-b_r b_j)+b\,b_j b_r\}]D^r_i$$

$$+2[pL_r+qb_r]\,{}^c D^r_{ij}+\eta[(\beta L_i-Lb_i)(r_{0j}+S_{0j})$$

$$-(\beta L_j-Lb_j)(r_{i0}+S_{i0})], \qquad \text{...(10)}$$

$$2qS_{ij} = [\mu L_{ir} + \eta\{\beta^2 L^{-1} L_i L_r - \beta(L_i b_r + L_r b_i) + Lb_i b_r\}]D_j^r$$

$$-[\mu L_{jr} + \eta\{\beta^2 L^{-1} L_j L_r - \beta(L_j b_r - b_r b_j)) + b\, b_j b_r\}]D_i^r$$

$$+\eta[(\beta L_i - Lb_i)(r_{0j} + S_{0j}) - (\beta L_j - Lb_j)(r_{i0} + S_{i0})], \qquad \ldots(11)$$

Subtracting (10) from (8) and contracting the resulting equation with y^i, we obtain

$$[-\mu L_{jr} - \eta\{\beta^2 L^{-1} L_j L_r - \beta(L_j b_r + L_r b_j) + Lb_j b_r\}]D^r$$

$$+\frac{1}{2}\eta(Lb_j - \beta L_j)r_{00} + q\, r_{0j} = [pL_r + qb_r]D_j^r, \qquad \ldots(12)$$

Contracting (12) with y^j, we get

$$2[\{L - \beta(m+1)\}L_r + mLb_r]D^r = m\, Lr_{00} \qquad \ldots(13)$$

Subtracting (11) from (8) and contracting the resulting equation with y^i, we get

$$[\mu L_{ir} + \eta\{\beta^2 L^{-1} L_i L_r - \beta(L_i b_r + L_r b_i) + Lb_i b_r\}]D^r$$

$$= \eta\left[\frac{L(L-\beta)}{(m+1)} S_{i0} + \frac{1}{2}(Lb_i - \beta L_i)r_{00}\right]. \qquad \ldots(14)$$

In view of $LL_{ir} = g_{ir} - L_i L_r$, the equation (2.11) can be written as

$$\lambda g_{ir} D^r - [\chi\{m)m+1)\beta^2 - (L-\beta)(L-\beta-m\beta)$$

$$+\rho m L^2)L-\beta)\}L_i + m(m+1)\beta b_i]L_r D^r + \eta(Lb_i - \beta L_i)b_r D^r$$

$$= qS_{i0} + \frac{1}{2}\eta(Lb_i - \beta L_i)r_{00}, \qquad \ldots(15)$$

where $\lambda = \dfrac{L^{m-1}}{(L-\beta)^{m+1}}[L - \beta(m+1) + \rho m L^2], \chi = \dfrac{mL^{m-1}}{(L-\beta)^{m+2}},$

Contracting (15) Contracting with $b_i = g^{ij} b_j$, we get

$$\left(-\frac{\beta}{L}\right)u\,L_rD^r + ub_rD^r = mL^2(L-\beta)S_0 + \frac{1}{2}m(m+1)(b^2L^2-\beta^2)r_{00}, \qquad \ldots(16)$$

where we have written S_0 for $S_{r_0} b^r$. The equation (13) and (16) constitute the system of algebraic equation in L_rD^r and b_rD^r whose solution is given by

$$b_rD^r = \frac{2mL^2\{L-\beta(m+1)\}S_0 + \{m(m+1)(b^2L^2-\beta^2)+m\beta V\}r_{00}}{2U} \qquad \ldots(17)$$

and

$$L_rD^r = \frac{mL[Ur_{00}-2mL^2S_0]}{2U}, \qquad \ldots(18)$$

where,

$$U = [m(m+1)(b^2L^2-\beta^2)+(L-\beta)(L-\beta-m\beta)+\rho L^2(L-\beta)$$

$$V = [L-\beta(m+1)+\rho mL^2].$$

Contracting (15) by g^{ij} and putting the value of L_rD^r and b_rD^r from (17) and (18) respectively, we get

$$D^i = \frac{(L-\beta)^{m+2}(Vr_{00}-2mL^2S_0]}{2UVL^m}\left[\eta L^2b^i + \frac{1}{L}\{\mu q - \eta\beta\bar{L}\}y^i\right] + \frac{mL^2}{V}S^i_0, \quad \ldots(19)$$

where $l^i = \dfrac{y^i}{L}$.

Proposition 2.1. The difference tensor $D^i = \bar{G}^i - G^i$ of generalized Matsumoto change of Finsler metric by h-vector is given by (19).

3. Conditions to be projective change

The Finsler space $\bar{F}^n$ is said to be projective to Finsler space F^n if every geodesic of F^n is transformed to a geodesic of $\bar{F}^n$. It is well known that the change

$L \to \bar{L}$ is projective if $\bar{G}^i = G^i + P(x, y)y^i$, where P(x,y) is a homogeneous scalar function of degree one in y^i, called projective factor (Matsumoto [3]).

Thus from (5) it follows that $L \to \bar{L}$ is projective iff $D^i = Py^i$.

Now we consider that the Matsumoto change $L \to \bar{L} = \dfrac{L^{m+1}}{(L-\beta)^m}$ is projective. Then from (19), we have

$$Py^i = \frac{(L-\beta)^{m+2}(Vr_{00} - 2mL^2S_0)}{2UVL^m}\left[\eta L^2 b^i + \frac{1}{L}\{\mu q - \eta\beta\bar{L}\}y^i\right] + \frac{mL^2}{V}S^i_0, \quad \ldots(20)$$

Contracting (20) with $y_i (= g_{ij}y^j)$ and using the fact that $S^i_0 y_i = 0$ and $y_i y^i = L^2$, we get

$$P = \frac{m[Vr_{00} - 2L^2S_0]}{2U}. \quad \ldots(21)$$

Putting the value of P from (21) in (20), we get

$$m(m+1)(Vr_{00} - 2mL^2S_0)(\beta y^i - L^2b^i) = 2mL^2US^i_0. \quad \ldots(22)$$

Transecting (22) by b_i, we get

$$r_{00} = \frac{2(L-\beta)S_0}{(m+1+\Delta}, \text{where } \Delta = \left(\frac{\beta}{L}\right)^2 - b^2 \neq 0 \quad \ldots(23)$$

Substituting the value of r_{00} from (3.4) in (3.2), we get

$$P = \frac{mS_0}{(m+1)\Delta}. \quad \ldots(24)$$

Eliminating P and r_{00} from (24), (23) and (20), we get

$$S^i_0 = \left(b^i - \frac{\beta}{L^2}y^i\right)\frac{S_0}{\Delta}. \quad \ldots(25)$$

The equation (23) and (25) gives the necessary conditions under which the generalized Matsumoto change becomes a projective change.

Conversely, if conditions (23) and (25) are satisfied, then putting these conditions in (19), we get

$$D^i = \frac{mS_0}{(m+1)\Delta} y^i, \text{i.e.} D^i = Py^i, \text{where } P = \frac{mS_0}{(m+1)\Delta},$$

Thus $\overline{F}^n$ is projective to F^n.

Theorem (3.1): The generalized Matsumoto change of a Finsler metric by h-vector of a projective if and only if (23) and (25) hold good.

4. Douglas space transforming to a Douglas space

The Finsler space F^n is called a Douglas space iff $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three in y^i (Matsumoto [4]). We shall write hp(r) to denote a homogeneous polynomial in y^i of degree r.

If we put $B^{ij} = D^i y^j - D^j y^i$, then from (2.16), we get

$$B^{ij} = (b^i y^j - b^j y^i) \frac{\eta(L-\beta)^{m+2} L^2 \{Vr_{00} - 2mL^2 S_0\}}{2UVL^m} + \frac{mL^2}{U}(S^i_0 y^i - S^i_0 y^i). \quad \ldots(26)$$

If a Douglas space is transformed to a Douglas space by generalized Matsumoto change (4) then B^{ij} must be hp(3) and vice-versa.

Theorem (4.1). The generalized Matsumoto change of Finsler metric by h-vector of a Douglas space gives rise to a is a Douglas if and only if B^{ij} given byk (26) hp(3).

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