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THE (n, N) POLICY OF THE $M^X/G/1/K$ QUEUE WITH MULTIPLE REMOVABLE SERVER

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Abstract : Every time the server completes service and finds ν customers in the system, the servers take a sequence of vacations. We consider an $M^x/G/C/K$ queue in which the removable server applied the following (ν, N) policy. At the end of vacations, the server inspects the length of the queue. If the queue length is greater than or equal to N, the server begins service until the number of customer's drops to ν. Under a classical cost structure, we characterize an optimal policy and develop an algorithm to find an optimal policy which minimize the expected cost per unit time. The (ν, N) policy has the following form "provide no service if the number of customers increases to N, turn the server on and continue serving until the customers drops again to ν. Teghem analyzed an optimal (ν, N) policy for the $M^x/G/C$ with finite capacity.

Keywords : $M^x/G/1/k$ queue, Multiple removable server, Queue length, Optimal policy.

1. Introduction

Generally, one of the objectives of the queueing control is to determine the time intervals at which the server is turned on or off in order to minimize the total operating cost of the system. At the end of the vacation, the server returns to the system to inspect the state of the system. Grabill et. al. [4] studies the classified bibliography of research on optimal design and control of queues. Hersh and Brosch [5] analysis the optimal strategy structure of an intermittently operated service channel. Teghem [14] considered the control of the service process in a queueing system. The control policies for $M^X/G/1$ queueing system made by Lee and Srinivasan [10]. Srinivasan and Lee [13] review production/inventory queueing systems with compound Poisson demands and arbitrary processing times. Lee [9] investigates the optimal control policy of the $M^x/G/1/K$ queue with multiple server vacations. Tian and Zhang [15] analyses the M/M/c queue with synchronous vacation of servers and its application to electronic commerce operations. Lewis [11] studied the load-capacity interference model for common mode failure in 1-out-of-2 : G system. Al-Seedy [1] introduced the queueing with fixed and variable channels considering balking and reneging concepts. Barron et. al. [3] analysis of R-out of-N repairable system, the case of phase-type distribution. Jain and Mishra [6] reliability analysis of unreliable server retrial queue with bulk arrivals. The optimal (d, c) vacation policy for a finite buffer M/M/c queue with unreliable servers and repairs was made by Ke et. al. [8]. If there are any waiting customers at this instant, the server begins service until the system becomes empty, otherwise, the server takes another vacation of length T and continues in this manner until there are customers awaiting service upon return from a vacation.

Jeyakumar and Senthilnathan [7] studies on the behavior of the server breakdown without interruption in a $M^x/G(a, b)/1$ queueing system with multiple vacations and closedown time. Banik [2] investigated the stationary distributions and optimal control of queues with batch Markovian arrival process under multiple adaptive vacations. The availability of k-out-of-n: F secondary subsystem with general repair time distribution was made by Mishra and Jain [12]. Yang and Ke [17] studied the cost optimization of a repairable M/G/1 queue with a randomized policy and single vacation. An unreliable Geo/G/1 queue with startup and closedown times under randomized finite vacations was investigated by Wang [16]. While much work has been done on control policies for queueing systems with infinite capacity only. In general these analysis consider queueing systems with infinite capacity. In many real queueing systems, however capacity of the queue is finite. For example, in manufacturing systems or in computer communication systems, buffer space is usually limited. Therefore in many cases, the finite capacity queueing model is a more realistic representation of the system.

In this paper, we extend these analyses to the more general $M^x/G/1/K$ queueing system with multiple server vacations. Consider an $M^x/G/1/K$ queue, where customers arrive to the system in batches. These batches arrive according to a Poisson process with rate λ. Each batch consists of a number, X, of customers where X is an independent and identically distributed random variable. That is, consecutive batch sizes are independent and have the common distribution with mean $\overline{X}$. Let x_K be the probability that the batch size X is K where K = 1, 2, …, we assume that the maximum number of customers allowed in the system at any time is given by K. If the waiting room available at the instant of a batch arrival is not sufficient to accommodate all customers of that batch, customers are partially accepted. The customers are served one at a time by a single server. The service time for an individual customers is independent and identically distributed with the common distribution function G(t) and the mean $1/\mu$. For this system, we consider the following control policy: every time the server completes service and finds ν customers in the system, the server takes a sequence of vacations. The durations of these vacations are independent and identically distributed and independent of the arrival process as well as the service times. Let $\nu(t)$ and $\overline{\nu}$ be the distribution function and the mean of these vacations. At the end of each vacation, the server inspects the length of queue. If the queue length at that time is below N ($N \le K$), the server immediately takes another vacation. On the other hand, if the queue length is equal to or greater than N, the server then begins to serve the queue until the number of customers drops to ν. We will call this policy the $(\nu, N)_K$ policy. We also consider the policy in which the server is never turned off as well as the policy in which the server is never turned on. Associated with this system, we impose the following cost structure.

(i) $C_1(C_2)$: The cost incurred per unit time when the server is off (on), with $C_1 \le C_2$ we set $C = C_1 - C_2$.

(ii) $A_1(A_2)$: fixed cost for turning the server on (off)

we set $A = A_1 + A_2$

(iii) C_1 : The cost for each last customers.

The objective of this paper is to determine the optimal policy with minimizes the expected cost per unit time in the long run under this cost structure. It is easy to note that our model has an analogy with the classical (s,S) inventory model with last sales [Srinivasan and Lee [13]]. Thus, the analytic results in this chapter might be used to obtain the optimal (s,S) policy in the production/inventory systems with lost sales. In the following sections, first an expression for the expected cost per unit time for a given policy will be derived and an efficient algorithm to determine the optimal will be presented next.

2. The Model

(i) The Approach

Note that the epoch marking the end of a busy period forms a regeneration point. Hence, we define the time interval from the end of one busy period to the end of the subsequent busy period as a cycle. To find an optimal policy we should first obtain an expression for the expected cost per unit time for given control values ν and N. To this end, we use the renewal reward theorem as follows.

Expected cost incurred unit time =

$$= \frac{\text{expected cost incurred per cycle}}{\text{expected length of a cycle}} \qquad .....(1)$$

Let the expected cost incurred per unit time for the $(\nu, N)_K$ policy in an $M^x/G/1/K$ system be denoted by $TC\,[\nu, N]_K$. Then, the following cost relationship between the $[\nu,N]_K$ policy and $[0, N-\nu]_{K-\nu}$ policy can be obtained.

Lemma 2:

$$TC[\nu, N]_K = TC\,[0, N-\nu]_{K-\nu}$$

Proof:

If we compare the queue length of the $[0, N-\nu]_{K-\nu}$ system with that of the $[\nu, N]_K$ system during a regeneration cycle, the queue length of the $[0, N-\nu]_{K-\nu}$ system has the same stochastic path as that of the $[\nu, N]_K$ system if ν customers are added to the queue length of the $[0, N-\nu]_{K-\nu}$ system. Thus, for these two policies, the expected length of a cycle, expected duration when the server is on (off) in a cycle and the expected numbers of lost customers per cycle are all the same. Consequently, under our cost structure, the expected costs incurred per unit time for these two policies are equivalent.

Since $TC[\nu, N]_K$ is equal to $TC\,[0, N-\nu]_{K-\nu}$.

We do not consider the policy with $\nu > 0$ any more. We can always find an equivalent policy with $\nu = 0$ by revising the capacity of the queueing system. Thus, if we can compute $TC\,[0, N]_K$ for any N and K, we can compute the expected cost incurred per unit time for any policy. For notational convenience, throughout the paper, we will use (N, K) instead of $[0, N]_K$. Besides, we will call the policy in which the server is never turned off the (0, K) policy and call the policy in which the server is never turned on the (K+1, K) policy.

(ii) Notation

We now introduce the following notations :

a_j	=	probability that j batches arrive during a vacation,
b_j	=	probability that j customers arrive during a vacation.
g_j	=	probability that j batches arrive during a service time.
h_j	=	Probability that j customers arrive during a service time.
μ	=	$m\mu$ = average number of customer
m	=	The number of servicing channels
ϕ_{ji}	=	$\Pr\{X_1+X_2+\text{------}+X_i = j\}$, the probability that there are j customers in i batches,
$L_1(N,K)$	=	Expected length of a cycle when the (N, K) policy is used.
$L_I\,(N, K)$	=	Expected length of an idle period in a cycle when the (N,K) policy is used.
$\pi(N, K)$	=	Expected penalty cost per unit time when the (N, K) policy is used.
TC (N, K)	=	Expected cost incurred per unit time when the (N, K) policy is used.

(iii) The Analysis:

Using the notations, from equation (4.1), TC (N,K) is expressed as

$$TC(N,K) = \frac{A + C_1 L_1(N,K) + C_2\{L(N,K) - L_1(N,K)\}}{L(N,K)} + \pi(N,K)$$

$$= \frac{A - c\ L_1(N,K)}{L(N,K)} + \pi(N,K) + C_2 \qquad .....(2)$$

In order to obtain TC(N,K) we need to compute the terms L_1(N,K), L(N,K) as well as the term π(N,K). To compute these terms, we first obtain the probabilities b_j and h_j as follows :

$$b_j = \sum_{i=0}^{j} a_i \phi_{ji}\ , \quad h_j = \sum_{i=0}^{j} g_i \phi_j \quad J \geq 0 \qquad .....(3)$$

where a_i and g_i are given by

$$a_i = \int_0^\infty \frac{(\lambda t)^i}{\underline{|i}} e^{-\lambda t} dv(t) \text{ and}$$

$$g_i = \int_0^\infty \frac{(\lambda t)^i}{\underline{|i}} e^{-\lambda t} dG(t) \qquad .....(4)$$

In equation (3), the term ϕ_{ji} can be quickly computed using the following recursive formula.

$$\phi_{ji} = \sum_{K=0}^{j-i+1} x_K \phi_{j-K,i-1} \qquad .....(5)$$

Based on these values, we can now compute L_1(N,K), L(N,K) as well as π(N,K) as follows :

Computing the Terms π(N,K).

Let the fraction of lost customers when the (N,K) policy is used in an $M^x/G/C/K$ queueing system be denoted by f(N,K) and let the offered load (traffic intensity) be denoted by

$$\rho = \frac{\lambda \bar{x}}{m\mu}$$

Then from the fact that the long-run fraction of time that the server is busy is $\rho\{1-f(N,K)\}$, the following equation is obtained

$$\rho(1 - f(N,K)) = 1 - \frac{L_I(N,K)}{L(N,K)} \qquad .....(6)$$

Equation (6) can be rewritten in term of f(N,K) as

$$f(N,K) = 1 - \frac{L(N,K) - L_I(N,K)}{\rho L(N,K)} \qquad .....(7)$$

Note that the expected number of lost customer per unit time is $\lambda \bar{X} f(N,K)$. Hence, the expected penalty cost per unit time owing to lost customers can be expressed as

$$\pi(N,K) = \lambda \bar{x} C_1 - \frac{m\mu\{L(N,K) - L_I(N,K)\}}{L(N,K)} C_1 \qquad .....(8)$$

Therefore, from equation (2) and (8), the expected cost per unit time is given by

$$TC(N,K) = \chi + \frac{A + \alpha L_I(N,K)}{L(N,K)} \qquad .....(9)$$

where

$$\chi = C_2 + \lambda \bar{x} C_2 - m\mu C_1 \text{ and } \alpha = m\mu C_1 - C$$

We now show how the terms L(N,K) and $L_I(N,K)$ can be computed.

Computing the terms L (N,K) and L_I (N,K) :

Let $T_j(K)$ denote the expected length of a busy period which is initiated with a queue length of j in an $M^x/G/1/K$ queueing system. Note that, in the (N,K) policy, for any value of N and K, the server takes at least one vacation. With probability b_j, j customers arrive during the first vacation – clearly if, $j \geq N$, then the idle period ends at the instant the server returns from the first vacation and the busy period begins, which continuous until the system becomes empty. In this case, the expected length of the busy period is $\tau_i(K)$ where i = min {j, K}. On the other hand, if $j < N$, then the idle period does not end at the instant the server returns from the first vacation. Consider, in this case, the interval starting from the instant the first vacation ends onwards until the instant the queue length drops to j for the first time during the busy period. Note that this interval is just a regeneration cycle of the $[j, N]_K$ system. As stated earlier, the cycle time of the $[j, N]_K$ system is equal, in distribution, to that of the $[0, N-j]_{K-j}$ system, which, in turn, can be represented by (N–j, K–j) system. Thus, the expected length of the interval is equal to L(N–j, K–j). From this fact, we can express L(N,K) as

$$L(N,K) = \bar{\nu} + \sum_{j=1}^{N-1} b_j \{L(N-j, K-j) + \tau_j(K)\} + \sum_{j=N}^{K} b_j \tau_j(K)$$

$$+ \sum_{j=K+1}^{\infty} b_j \tau_K(K) \quad .....(10)$$

Adding terms involving L(N,K) in equation (10), we obtain the following expression.

$$L(N,K) = \frac{\bar{\nu}}{1-b_0} + \sum_{j=1}^{N-1} \tilde{b}_j L(N-j, K-j) + \sum_{j=1}^{K} \bar{b}_j \tau_j(K)$$

$$+ \sum_{j=K+1}^{\infty} \tilde{b}_j \tau_K(K) \quad .....(11)$$

where $\bar{b}_j = (b_j / 1 - b_0)$ and an empty sum is, by definition, equal to 0.

By substituting N = 1 in equation (11), we obtain

$$L(1,K) = \frac{\bar{\nu}}{1-b_0} + \sum_{j=1}^{K} \tilde{b}_j(K)\tau_j(K) + \sum_{j=K+1}^{K} \tilde{b}_j \tau_K(K) + \sum_{J=K+1}^{\infty} \tilde{b}_j \tau_k(K)$$

$$.....(12)$$

Therefore, from equation (11) and (12), we obtain a recursive expression for L(N,K) as follows :

$$L(N,K) = \sum_{j=1}^{N-1} \tilde{b}_j L(N-j, K-j) + L(1,K) \quad ....(13)$$

If we let

$$B_j = \begin{cases} 1, & j = 0 \\ \sum_{i=1}^{j} \tilde{b}_i B_{j-i}, & j > 0 \end{cases}$$

L(N,K) is further simplified to

$$L(N,K) = \sum_{j=0}^{N-1} \beta_j \; L(1,K-j) \qquad .....(14)$$

Therefore L(N,K) can be obtained if we can compute L(1,K) for any value of K. We will now describe how the term L(1,K) can be computed.

From equation (12), we can obtain the following recursive expression for L(1,K);

$$L(1,K) \;=\; L(1,K-1) + \sum_{j=1}^{K-1} \tilde{b}_j \{\tau_j(K) - \tau_j(K-1)\}$$

$$+ \sum_{j=K}^{\infty} \tilde{b}_j \{\tau_K(K) - \tau_{K-1}(K-1)\} \qquad .....(15)$$

To analyze $\tau_j(K)$ in equation (15), we divide period $\tau_j(K)$ into j subpriods, where the nth subperiod (n=1, …j) is initiated when the queue length first reaches j+1–n and is terminated when the queue length first drops to j–n. The duration of this nth subperiod is equivalent is distribution to that of a busy period of an $M^x/G/1/(K-j+n)$ queue which is initiated with one customer.

Therefore, $\tau_j(K)$ can be expressed as

$$\tau_j(K) = \sum_{i=1}^{j} \tau_1(K-j+1) \qquad .....(16)$$

By substituting equation (16) into equation (15), we obtain

$$L(1,K) \;=\; L(1,K-1) + \sum_{j=1}^{K-1} \tilde{b}_j \{\tau_1(K) - \tau_1(K-j)\} + \sum_{j=K}^{\infty} \tilde{b}_j \tau_1(K)$$

$$= L(1,K-1) + \tau_1(K) - \sum_{j=1}^{K-1} \tilde{b}_j \tau_1(K-j). \qquad .....(17)$$

Hence, from equation (17), L(1,K) can be computed recursively using the initial value,

$$L(1,1) = \bar{\nu}/(1-b_0) + \tau_1(1) = \bar{\nu}/(1-b_0) + 1/m\mu$$

If the value of $\tau_1(n)$, $1 \le n \le K$ are available.

The expression of $\tau_1(n)$ can be obtained by conditioning on the number of customers that arrive during the service time of the first customer as follows.

$$\tau_1(n) = 1/m\mu + \sum_{j=1}^{n-1} h_j \tau_j(n) + \sum_{j=n}^{\infty} n_j \tau_{n-1}(n) \qquad .....(18)$$

From equation (18), we have

$$\tau_1(n+1) - \tau_1(n) = \sum_{j=1}^{n-1} n_j \{\tau_j(n+1) - \tau_j(n)\} + \sum_{j=1}^{\infty} n_j \{\tau_n(n+1) - \tau_{n-1}(n)\}$$

$$= \sum_{j=1}^{n-1} n_j \{\tau_1(n+1) - \tau_1(n+1-j)\} + \sum_{j=n}^{\infty} n_j \tau_1(n+1)$$

$$= \sum_{j=1}^{\infty} n_j \tau_1(n+1) - \sum_{j=1}^{n-1} n_j \tau_1(n+1-j) \qquad .....(19)$$

Adding terms involving τ_1 (n+1) in equation (19), we obtain the following recursive equation :

$$\tau_1(n+1) = \frac{1}{h_0}\left\{\tau_1(n) - \sum_{j=1}^{n-1} n_j \tau_1(n+1-j)\right\} \qquad .....(20)$$

As an initial value for equation (20), we use $\tau_1(1) = \dfrac{1}{m\mu}$ since we can compute $\tau_1(n)$ [hence, L(1,K)], we can now compute L(N,K) for any value of N and K $(1 \le N \le K)$.

The expected length of an idle period can be obtained in an analogous manner. By conditioning on the number of customers that arrive during the first vacation,

$L_I(N,K)$ is express as

$$L_I(N,K) = \bar{\nu} + \sum_{j=0}^{N-1} b_j L_I(N-j, K-j) \qquad .....(21)$$

By adding terms involving $L_I(N,K)$, we obtain

$$L_I(N,K) = \frac{\bar{\nu}}{1-b_0} + \sum_{j=0}^{N-1} \tilde{b}_j L_I(Nj, K-j) \text{ and} \qquad ...(22)$$

$$L_I(N,K) = \frac{\bar{\nu}}{1-b_0} \qquad .....(23)$$

Notice that the term $L_I(N,K)$ can be expressed analogous to equation (14), as

$$L_I(N,K) = \sum_{j=0}^{N-1} \beta_j L_I(1, K-j) = \frac{\bar{\nu}}{1-b_0} \sum_{j=0}^{N-1} \beta_j \qquad .....(24)$$

Now, by substituting equation (14) and (24) into equation (9), we obtain an expression for TC(N,K), which is stated as Lemma 3.

Lemma-4:

$$TC(N,K) = x + \frac{A + Y\sum_{j=0}^{N-1}\beta_j}{\sum_{j=0}^{N-1}\beta_j L(1, K-j)} \qquad .....(25)$$

where

$$\chi = C_2 + \lambda\bar{x}C_1 - m\mu C_1 \text{ and}$$

$$Y = (m\mu C_1 - C)\frac{\bar{\nu}}{1-b_0}$$

from Lemma 2, we can compute TC(N,K), for $O < N \le K$, by using equations (17) and (20). Analysis of the policies (K+1,K) and (0,K). We are now ready to describe how the expected casts of the policies (K+1,K) and (0,K) can be obtained, for the (K+1,K) policy in which the server is never turned on, the expected cost per unit time is easily obtained as

$$TC(K+1, K) = C_1 + \lambda\bar{x}C_1 \qquad .....(26)$$

The expected cost incurred per unit time for the (0,K) policy in which the server is never turned off, is expressed as

$$TC(0,K) = C_2 + \pi(N,K) = \chi + m\mu C_1 \frac{L_I(0,K)}{L(0,K)} \quad .....(27)$$

By conditioning on the number of customers in the first batch, the term L(0,K) in equation (27) is expressed as

$$L(0,K) = \frac{1}{\lambda} + \sum_{j=1}^{K-1} x_j \tau_j(K) + \sum_{j=K}^{\infty} x_j \tau_1(K) \quad .....(28)$$

using equation (28) together with equation (16), we can derive the following recursive equation for L(0,K)

$$L(0,K) = L(0,K-1) + \tau_1(K) - \sum_{j=1}^{K-1} x_j \tau_K(K - j) \quad .....(29)$$

With an initial value, $L(0,1) = (1/\lambda) + \tau_1(1)$

The terms $L_I(0,K)$ in equation (27) is easily obtained as $1/\lambda$.

3. The Optimal Control Policy

In this section, we find the optimal which minimizes the expected cost per unit time. To do so, we first present the following results which the function L(1,K) possesses. Since the proofs of these results are straight forward, we omit the proofs.

Lemma 3.1

L(1, K) is increasing in K for $K \geq 1$

From Lemma 3.1, we derive the following results, stated as Lemma 3.2

Lemma 3.2

(a) $TC(N, K-\nu) \geq TC(N,K)$ for $N \leq K$

(b) The policy $[\nu,N]_K$ with $\nu>0$ can not be an optimal policy.

From Lemma 3.2, in order to find an optimal policy in an $M^x/G/C/K$ queue with multi server vacations, we need only to consider policies (N,K) for $1 \leq N \leq K$, the policy (0,K) as well as the policy (K+1,K).

Now we present the main result of this section, which gives characteristics of the cost functions.

TC(N,K), TC(0,K) and TC(K+1,K).

Theorem 3.1: For a given value of K,

(a) If $m\mu C_1 > C$, then TC(N,K) is increasing or unimoded in N for $1 \le N \le K$.

(b) If $m\mu C_1 \le C$, then the policy (K+1,K) is an optimal policy.

Proof –

To prove (a), it is sufficient to show that if TC(N,K) < TC(N+1,K), then TC(N+1,K) < TC(N+2,K) for $N+2 \le K$.

Let

$$\theta_K = A + \gamma \sum_{j=0}^{K-1} \beta_j \quad \text{and} \quad \delta_K = \sum_{j=0}^{K-1} \beta_j L(1, K-j)$$

Then we need to show that

if

$$\frac{\theta_N}{\delta_N} < \frac{\theta_N + \beta_N \gamma}{\delta_n + \beta_N L(1, K-N)}$$

then

$$\frac{\theta_N + \beta_N \gamma}{\delta_n + \beta_N L(1, K-N)} < \frac{\theta_N + \beta_N \gamma + \beta_{N+1}\gamma}{\delta_n + \beta_N L(1, K-N) + \beta_{N+1} L(1, K-N-1)}$$

.....(30)

For positive values of a, b, c and d with a/b > c/d, it can be verified that if a> c and b > d, then

$$\frac{a-c}{b-d} > \frac{c}{d}$$

If we apply this algebric fact to the given condition

$$\frac{\theta_N}{\delta_N} < \frac{\theta_N + \beta_N \gamma}{\delta_n + \beta_N L(1, K-N)}$$

and using the fact that L(1, K-N-1) < L(1, K-n), we obtian,

$$\frac{\theta_N}{\delta_N} < \frac{\gamma}{L(1, K-N)} < \frac{\gamma}{L(1, K-N-1)} \qquad .....(31)$$

we now use another algebric fact that if a, b, c, d, e, f are all positive values, then

$$\frac{e'+e}{d+f} < \frac{a+c+e}{b+d+f}$$

holds if

$$\frac{e}{f} < \frac{c}{d} < \frac{a}{b}$$

applying the algebraic fact to (31), we finally obtain (30). To prove (b), Let us first show that if $m\mu C_1 \le C$, then TC(N,K) is decreasing in N for $1 \le N \le K$. To this end, it is sufficient to show that $TC(N,K) \ge TC(N+1,K)$ when $m\mu C_1 \le C$, i.e.

$$TC(N,K) = \chi\frac{\theta_N}{\delta_N} \ge \chi + \frac{\theta_N + \beta_N\gamma}{\delta_n + \beta_N L(1,K-N)} = TC(N+1,K)$$

.....(32)

Inequality (32) can be easily proved since δ_N, θ_N, 2(1, K-N) and β_N are all positive and γ is non-positive. From equation (26) and (27), we have

$$TC(0,K) - TC(K+1,K) = C - m\mu C_1 - m\mu C_1\frac{L_I(0,N)}{L(0,N)} \qquad .....(33)$$

From equation (4.33) if $m\mu C_1 \le C$,

We have $TC(K+1,K) \le + T_C\,(0,K)$

Since TC(N,K) is decreasing in N and $TC(K+1,K) \le TC(0,K)$ for $m\mu C_1 \le C$ to prob (b), we need only to show that $TC(K+1,K) \le TC(K,K)$ from equation (25) and (26),

$$TC(K,K) - TC(K+1,K) = c - m\mu C_1 + \frac{A + \gamma\sum_{j=0}^{K-1}\beta_j}{\sum_{j=0}^{K-1}\beta_j L(1,K-j)}$$

$$= C - m\mu C_1 + \frac{A - (C - m\mu C_1)\sum_{j=0}^{K-1}\beta_j\frac{\bar{\nu}}{1-b_0}}{\sum_{j=0}^{K-1}\beta_j L(1,K-j)}$$

$$= \left(C - m\mu C_1\right)\left\{\frac{\sum_{j=0}^{K-1} \beta_j \frac{\bar{\nu}}{1-b_0}}{\sum_{j=0}^{K-1} \beta_j L(1, K-j)}\right\} + \frac{A}{\sum_{j=0}^{K-1} \beta_j L(1, K-j)}$$

.....(34)

using the fact that

$$L(1, K-j) > \frac{\bar{\nu}}{1-b_0}$$

from equation (34), it is shown that TC(K,K) ≥ TC(K+1,K)

We are now in a position to describe the algorithm to find the optimal policy (N*,K). The algorithm makes use of Theorem 3.1 as well as the recursive nature of the functions L(1,n) and β_n from Theorem 4.1, if $m\mu C_1 \leq C$, then the policy is (K+1,K). However, if $m\mu C_1 \geq C$, then the obtain the optimal policy, we should first and policy which minimizes the function TC(N,K) for 1 ≤ N ≤ K. Let this local optimal policy be (N^0, K). To find the policy (N^0, K), the algorithm evaluates TC(N,K) sequentially starting with N=1 up to the point N= $\overline{N}$, at which the cost function TC(N,K) is first increased. Then owing to theorem 4.1 (a), the local optimal policy. (N^0,K) is ($\overline{N}$ -1,K). Recall that TC(N,K) for 1 ≤ N ≤ K is obtained directly using equation (25) if L (1,n) and β_n are computed for 1 ≤ n ≤ N-1. Note also that L(1, n) and β_n are computed recursively starting from n=1 and n=0 respectively. Owing to this recursive nature, TC(N,K) is computed very quickly once TC(N-1,K) has been computed. Therefore, there is very little computational effort in our sequential evaluation. After we find TC(N^0,K), we can obtain the optimal policy (N*,K), easily by comparing TC(N^0,K) with TC(0,K) and TC(K+1,K).

Algorithm to find the optimal policy

Case-1. $m\mu C_1 \leq C$; the optimal policy is (K+1,K).

Case-2. $m\mu C_1 \geq C$,

(a) Find the policy (N^0,K) :

0. Set N=1. Compute TC(N,K) using equation (17) (20) and (25).
1. Set N=N+1. Compute TC(N,K) using equations (17), (20) and (25)
2. If TC (N,K) > TC(N-1,K), then N^0=N-1. go to step (b) otherwise, go to step 1.

(b) Compute TC(0,K) using equations (27) and (29) compute TC(K+1,K) using equation (26)

(c) Find the optimal policy (N*,K) by TC(N*,K) = Min_N {TC(N,K) $N=0,N^0,K+1$}

4. Conclusion

We have introduced control policies for the $M^x/G/1/K$ queueing system with multiple server vacations. Under a classical cost structure, we obtained an expression for the expected cost per unit time for a given policy. We derived a set of properties of the cost functions and based on these properties, we develop an efficient method to find an optimal policy.

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