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CONVEXITY CONDITION FOR CERTAIN INTEGRAL OPERATOR

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Abstract : In the present paper, a sufficient condition is established for which the image of a normalized analytic function under a well-known integral operator given by

$$I(z) = \frac{\beta+1}{z^{\beta}} \int_0^z \left[f(t)e^{g(t)}\right]^{\beta} dt \qquad (z \in \mathbb{U})$$

is convex univalent function.

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1. Introduction

Let $\mathbb{U}$ be the open unit disk of the complex plane:

$$\mathbb{U} = \{z : z \in \mathbb{C} \ \ and \ \ |z| < 1\}.$$

Let $\mathcal{H}[\mathbb{U}]$ denote the class of holomorphic functions in $\mathbb{U}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} := \{1,2,3, \dots\}$,

let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}[\mathbb{U}], \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \qquad z \in \mathbb{U}\}, \qquad \dots (1)$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[\mathbb{U}], \qquad f(z) = z + a_{n+1} z^{n+1} + \cdots, \qquad z \in \mathbb{U}\}. \qquad \dots (2)$$

In particular, for n=1, we write $\mathcal{A}_1 = \mathcal{A}$.

A function $f(z)$ in $\mathcal{A}$ is said to be univalent in $\mathbb{U}$ if $f(z)$ is one to one in $\mathbb{U}$. Let S denote the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$ (see [4]).

Let

$$\mathcal{K} = \left\{ f \in \mathcal{A}:\ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0;\ z \in \mathbb{U} \right\} \quad \text{... (3)}$$

and

$$\mathcal{S}^* = \left\{ f \in \mathcal{A}:\ \Re\, \frac{zf'(z)}{f(z)} > 0;\ z \in \mathbb{U} \right\}, \quad \text{... (4)}$$

respectively denote the class of convex and starlike functions in $\mathbb{U}$.

Finding sufficient conditions of univalence, starlikeness, convexity of integral, derivative and other operators is an important topic of research in Geometric Function Theory. In recent years, several authors have investigated sufficient conditions for the univalence, starlikeness and convexity of various linear and non-linear integral operators. For example, Ularu [13] studied the following integral operator:

$$\mathrm{K(z)} = \int_0^z \pi_{i=1}^n \left(\frac{f_i(z)}{z}\right)^{\gamma_i} \left(g_i'(t)\right)^{\eta_i} dt\ , \quad \text{... (5)}$$

where the functions $f_i,\ g_i \in \mathcal{A}$ and the parameters $\gamma_i,\ \eta_i\ (i = 1,2,3, \dots, n)$ are so constrained that the integral (5) exists. Taking n=1, $\gamma_1 = \gamma,\ f_1 = f\ and\ \eta_1 = 0$ in (5) i-e the operator

$$F_\gamma(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\gamma dt \quad \text{... (6)}$$

has been studied in [1]. On the other hand, Oros [11] (also, see [12]) obtained sufficient conditions for convexity and starlikeness of operator given by

$$L_\gamma(f)(z) = F(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z f(t) t^{\gamma - 1} dt \quad (\gamma \geq 1; z \in \mathbb{U}). \quad \text{... (7)}$$

Recent expository work on this topic can also be found in ([1, 2, 3, 8, 9, 10]). Motivated by aforementioned work, we introduce the following integral operator.

Definition 1.

$L\ et\ \beta > 0\ be\ a\ real\ number\ and\ f\ be\ a\ function\ in\ \mathcal{A}\ such\ that\ \frac{f(z)}{z} \neq 0\ in\ \mathbb{U}.$ $Furthermore,\ suppose\ that\ h\ is\ a\ function\ in\ \mathcal{H}[1, 1]\ such\ that\ h(z) \neq 0\ in\ \mathbb{U}.$

$Define\ the\ integral\ operator\ I_h(z) :\ \mathcal{A} \to \mathcal{A}\ by$

$$I_h(z) = \frac{\beta + 1}{z^\beta} \int_0^z [f(t)h(t)]^\beta dt \quad (f \in \mathcal{A};\ z \in \mathbb{U}). \quad \text{... (8)}$$

We note that the totality of the function $I_h(z)$ is single-valued analytic and belong to the class $\mathcal{A}$.

For the choice of the functions $h(z)$, we can take $\frac{sinz}{z}$ or any function satisfying $h(0) = 1$ and $\Re\{h(z)\} > 0 \ or\ h(0) = 1\ and\ \Im\{h(z)\} > 0$. As a consequent of Cauchy's theorem, every such function h, we can write $h(z) = e^{g(z)}\ where\ g \in \mathcal{A}$. Therefore, throughout this paper, we shall prefer to write

$$I(z) = \frac{\beta+1}{z^\beta}\int_0^z \left[f(t)e^{g(t)}\right]^\beta dt \qquad (f \in \mathcal{A};\ z \in \mathbb{U}). \qquad ...(9)$$

2. Preliminary

To prove our main result, we have to recall the following lemma.

Lemma 2.1. (see [5, 6, 7]) let $\psi :\ \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ satisfying the condition

$$\Re\psi(is, t; z) \leq 0, \ \ (z \in \mathbb{U})$$

for $s, t \in \mathbb{R},\ t \leq \frac{-(1+s^2)}{2}$.

If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \ \ satisfies$

$\Re\{p(z),\ zp^{'}(z); z\} > 0$,

then

$$\Re\{p(z)\} > 0 \ \ (z \in \mathbb{U}) .$$

More general forms of this lemma can be found in [7].

3. Main Result

We determine the sufficient condition such that, for a function $f \in \mathcal{A}$, the image under the new integral operator $I(z)$ is convex.

Theorem 1. *Let $f \in \mathcal{A}$ and β be a real number such that $\beta \geq 1$. If*

$$\Re\left[1 + \frac{z\phi^{''}(z)}{\phi^{'}(z)}\right] > \ -\frac{1}{2\beta} \quad (z \in \mathbb{U}), \qquad ...\ (10)$$

where

$$\phi(z) = \frac{\left(f(z)e^{g(z)}\right)^\beta}{z^{\beta-1}}, \qquad ...\ (11)$$

then the integral operator $I(z)$ defined by (9) is convex.

Proof. *Let $f, g \in \mathcal{A}$ so that*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ (z \in \mathbb{U}),$$

and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \ (z \in \mathbb{U}).$$

Then from (9), we have

$$\mathrm{I}(z) = \frac{\beta+1}{z^\beta}\int_0^z \left[(t + a_2t^2 + a_3t^3 + \cdots)e^{\left(t+b_2t^2+b_3t^3+\cdots\right)}\right]^\beta dt$$

$$= \frac{\beta+1}{z^\beta}\int_0^z [(t + a_2t^2 + a_3t^3 + \cdots)(1 + (t + b_2t^2 + b_3t^3 + \cdots) + \cdots)]^\beta dt$$

$$= \frac{\beta+1}{z^\beta}\int_0^z \left[t + (a_2+1)t^2 + \left(a_2 + a_3 + b_2 + \frac{1}{2!}\right)t^3 + \cdots\right]^\beta dt$$

$$= \frac{\beta+1}{z^\beta}\int_0^z (t^\beta + (a_2+1)\beta t^{\beta+1} + \cdots)dt$$

$$= \frac{\beta+1}{z^\beta}\left[\frac{z^{\beta+1}}{\beta+1} + \frac{(a_2+1)\beta z^{\beta+2}}{\beta+2} + \cdots\right]$$

$$= z + c_2z^2 + c_3z^3 + \cdots \qquad \ldots (12)$$

Thus, $\mathrm{I}(z) \in \mathcal{A}$.

Differentiating both sides of (9) with respect to z, we have

$$\beta\mathrm{I}(z) + z\mathrm{I}'(z) = (\beta+1)\phi(z) \qquad \ldots (13)$$

where $\phi(z)$ is given by (11). Differentiating both sides of (13) with respect to 'z' and simplifying, we obtain

$$\mathrm{I}'(z)\left(1 + \frac{z\mathrm{I}''(z)}{\mathrm{I}'(z)} + \beta\right) = (\beta+1)\phi'(z) \qquad \ldots (14)$$

Let

$$p(z) = 1 + \frac{z\mathrm{I}''(z)}{\mathrm{I}'(z)} \qquad (z \in \mathbb{U}) . \qquad \ldots (15)$$

Clearly, $p(0) = 1,\ p(z) = 1 + p_nz^n + \cdots$.

Making use of (15) in (14), we have

$$(p(z) + \beta)\mathrm{I}'(z) = (\beta+1)\phi'(z). \qquad \ldots (16)$$

Since $\mathrm{I}'(z) \neq 0,\ p'(z) \neq 0,\ p(z) + \beta \neq 0,$ so taking logarithmic differentiation on both sides of (16) and multiplying resulting equation by z give

$$\frac{z\mathrm{I}''(z)}{\mathrm{I}'(z)} + \frac{z\mathrm{p}'(z)}{p(z)+\beta} = \frac{z\phi''(z)}{\phi'(z)}$$

which implies

$$p(z) + \frac{z\mathrm{p}'(z)}{p(z)+\beta} = 1 + \frac{z\phi''(z)}{\phi'(z)} \quad (z \in \mathbb{U}) . \qquad \ldots (17)$$

Using the given condition (10) in (17), we have

$$\Re\left[p(z)+\frac{z\mathrm{p}'(z)}{p(z)+\beta}\right]> -\frac{1}{2\beta} \quad (\beta \geq 1; \ z\in\mathbb{U}),$$

which is equivalent to

$$\Re\left[p(z)+\frac{z\mathrm{p}'(z)}{p(z)+\beta}+\frac{1}{2\beta}\right]>0 \quad (z\in\mathbb{U}). \qquad \text{... (18)}$$

Let $\psi: \mathbb{C}\times\mathbb{U}\to\mathbb{C}$ given by

$$\psi(p(z), zp'(z); z) = p(z)+\frac{zp'(z)}{p(z)+\beta}+\frac{1}{2\beta}\,. \qquad \text{... (19)}$$

Then (19) is equivalent to

$$\Re\psi(p(z), zp'(z); z)>0 \quad (z\in\mathbb{U}).$$

To prove our result, we can make use of Lemma 2.1.

Now we calculate

$$\Re\psi(is,t;z)=\Re\left[is+\frac{t}{is+\beta}+\frac{1}{2\beta}\right]$$

$$=\Re\left[is+\frac{1}{2\beta}+\frac{t(\beta-is)}{s^2+\beta^2}\right]$$

$$=\frac{1}{2\beta}+\frac{t\beta}{s^2+\beta^2}$$

$$\leq \frac{1}{2\beta}-\frac{\beta\left(\frac{1+s^2}{2}\right)}{s^2+\beta^2}$$

$$=\frac{(1-\beta^2)s^2}{2\beta(s^2+\beta^2)} \quad \leq 0 \quad (\beta\geq 1).$$

Therefore, it follows from Lemma 2.1 that

$$\Re p(z)>0 \quad (z\in\mathbb{U}),$$

which implies

$$\Re\left[1+\frac{z\mathrm{I}''(z)}{\mathrm{I}'(z)}\right]>0 \quad (z\in\mathbb{U}).$$

Hence the result follows. Thus, the proof of Theorem 1 is completed.

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