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THE LOMAX-FRECHET DISTRIBUTION

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Abstract : The Frechet probability distribution is the power transformation of an exponentially distributed variable and is widely applied for problems in engineering. The Lomax (Pareto II) probability distribution has wide application in a variety of fields. In this paper, we introduce a new statistical distribution constructed by composition of the cumulative density functions (cdf) of Frechet probability distribution and Lomax probability distribution and is named as the Lomax-Frechet distribution. It may have wider applicability in engineering, order statistics and other fields as it has more number of parameters. We derive expressions for its moments, characteristic function, hazard rate function and survivor function. We plot some graphs for its probability density function (pdf) using the software 'Mathematica'. We also investigate the variation of the skewness and kurtosis and discuss estimation by the method of maximum-likelihood.

Keywords : Lomax-Frechet distribution, composition of cumulative density functions, characteristic function, estimation, hazard rate function, moments, survivor function.

MSC Subject Classification : 60E, 62N.

1. Introduction

The Frechet distribution has been of great use to the adequate modelling of market-returns which are often heavy tailed. Applications of the Frechet distribution discussed in Harlow [12] indicates that it is an important distribution for modelling the statistical behaviour of materials properties for a variety of engineering applications. Nadarajah et al. [17] discussed the sociological models based on Frechet random variables. Later, Zaharim et al. [18] applied Frechet distribution for analysing the wind speed data.

Recent developments focus on new techniques for building meaningful distributions. These include the two-piece approach introduced by Hansen [11], the perturbation approach of Azzalini et al.[2], and the generator approach pioneered by Eugene et al. [8]. The generator technique and composition of cdf's to create new distributions are almost similar. Many researchers have done work on using generator technique, a few to mention are Gupta et al. [10] on exponential family, Nadarajah et al.[16] on beta family, Akinsete et al. [1] on beta-pareto, Zografos et al. [20] on beta and gamma, Codeiro et al.[7] on Kumaraswami-Weibull, Barreto-Souza et al. [3] on beta-exponential, Zhu et al. [19] on student t, Cordeiro et al. [4,5,6] on beta, exponential and Kumaraswami distribution. Nadarajah and Gupta [15] introduced the beta Frechet distribution, derived the analytical shapes of the probability density function (pdf) and the hazard rate function and calculated the asymptotic distribution of the extreme order statistics. In the present paper, a new statistical distribution is discussed by considering the composition of cdf's of the Frechet and Lomax distributions and name it as Lomax-Frechet distribution.

The rest of the paper is organized as follows. In Section 2, Lomax and Frechet distribution are introduced. In Section 3, the Lomax-Frechet distribution is introduced and the graph of the pdfs of Lomax, Frechet and Lomax-Frechet distributions are drawn. The moments, characteristic function, hazard rate function, survivor function and the reliability for the newly defined distribution are also obtained. In Section 4, we discuss estimation of the four parameters α, β, μ, σ by the method of maximum-likelihood. In Section 5, we provide conclusions.

2. Preliminaries

2.1 Frechet Distribution: The Frechet probability distribution is the power transformation of an exponentially distributed variable and is widely applied for problems in engineering. The pdf of Frechet distribution is given by

$$g(x) = \frac{\mu}{\sigma}\left(\frac{x}{\sigma}\right)^{-\mu-1} e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}, \; x > 0,\; \mu,\; \sigma > 0 \qquad ...(1)$$

The cumulative distribution function (cdf) is

$$G(x) = e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}, \; x > 0,\; \mu,\; \sigma > 0 \qquad ...(2)$$

2.2 Lomax Distribution: The Lomax (Pareto II) distribution is a heavy tail probability distribution, often used in business, economics and actuarial modelling. It has wide application in a variety of fields. Its applications in modelling and analysing the lifetime data in medical and biological sciences and engineering are widely studied by Lomax [13] and Moghadam et al. [14].

The pdf of Lomax distribution is given by

$$\mathrm{f}(x;\alpha,\beta) = \frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-(1+\alpha)}, \quad \alpha,\beta{>}0,\, x \geq 0 \qquad ...(3)$$

$$= 0\,, \qquad x < 0$$

The cdf of Lomax distribution is given by

$$\mathrm{F}(x;\alpha,\beta) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}, \quad \alpha, \beta > 0,\, x \geq 0 \qquad ...(4)$$

$$= 0\,, \qquad x < 0$$

2.3 The Composition of cdfs: We define new distribution with cdf as

$$F_G(x) = \frac{F[G(x)]}{F(1)} \qquad ...(5)$$

and the corresponding pdf as

$$f_G(x) = \frac{F'[G(x)]}{F(1)} g(x) \qquad ...(6)$$

3 The Lomax - Frechet distribution

Substituting $G(x)$ from equation (2.2) in equation (2.5), the cdf of Lomax - Frechet distribution is defined as

$$F_{LF}(x;\alpha,\beta,\mu,\sigma) = K\left[1 - \left(1 + \frac{1}{\beta} e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right)^{-\alpha}\right], \qquad ...(7)$$

and the pdf of Lomax- Frechet distribution is defined as

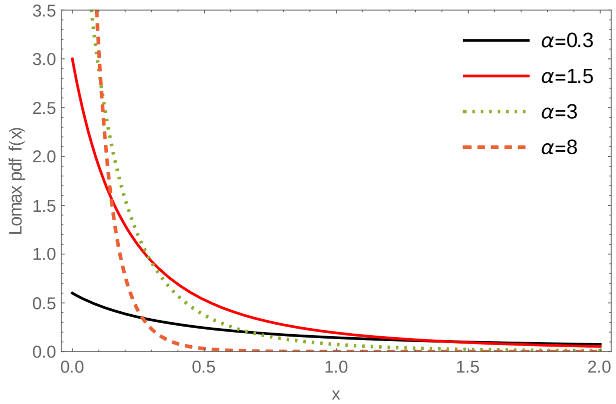
$$f_{LF}(x;\alpha,\beta,\mu,\sigma) = \frac{K\alpha\mu}{\beta\sigma}\left(\frac{x}{\sigma}\right)^{-(1+\mu)} e^{-\left(\frac{x}{\sigma}\right)^{-\mu}} \left[1 + \frac{1}{\beta} e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right]^{-(1+\alpha)}, \qquad ...(8)$$

where

$$K^{-1} = 1 - \left(1 + \frac{1}{\beta}\right)^{-\alpha};\ \alpha,\beta,\sigma,\mu > 0,\ x > 0 \qquad ...(9)$$

3.1 Graphs: The four set of graphs are drawn using the software ‘Mathematica’ to illustrate the effects of various parameters on the shape of the pdf in support to the study. In all the sets graphs of pdf of Lomax, Frechet and Lomax- Frechet distributions are drawn for varying values of one parameter and fixed values of the remaining parameters.

Set 1: β=0.5, μ=3, σ =1 and α is varying (Fig 1.1-1.3)



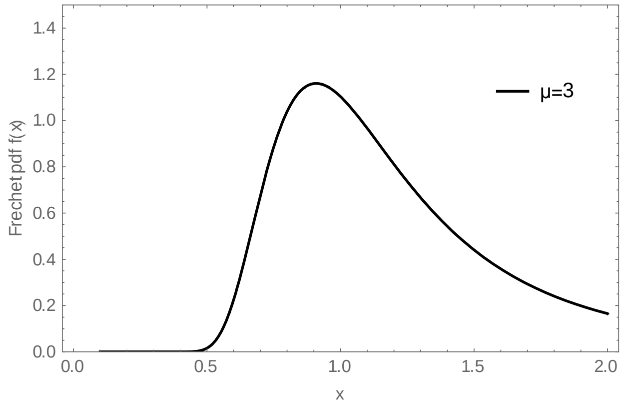


Fig.1.1 The Lomax pdf for β = 0.5

Fig.1.2 The Frechet pdf for μ = 3, σ = 1

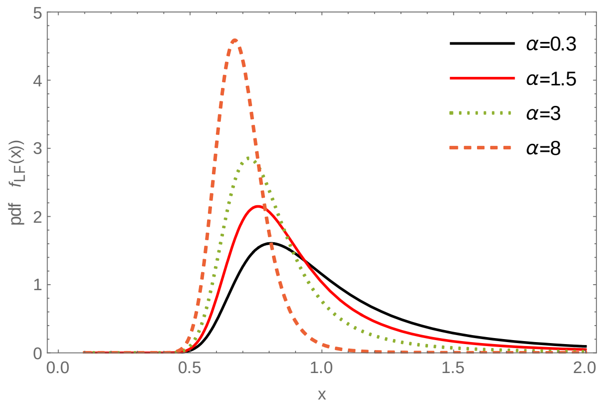
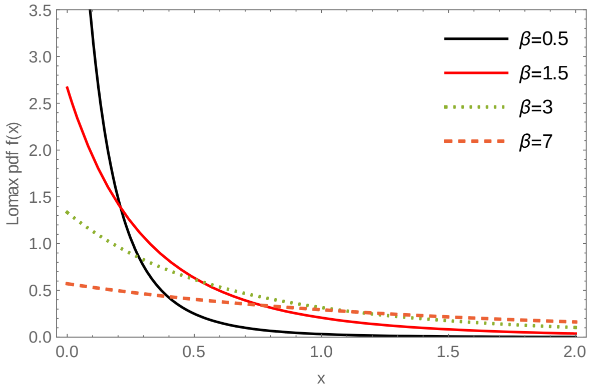


Fig.1.3 The Lomax-Frechet pdf for β = 0.5, μ= 3, σ=1

Set 2: α =4, μ=3, σ =1 and β is varying (Fig 2.1-2.3)



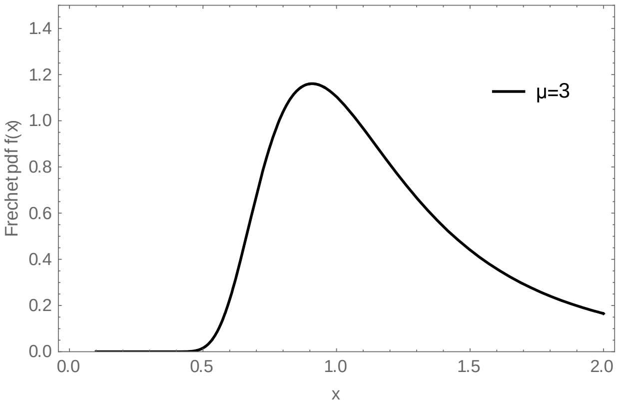


Fig.2.1 The Lomax pdf for $\alpha = 4$ Fig.2.2 The Frechet pdf for $\mu = 3$, $\sigma = 1$

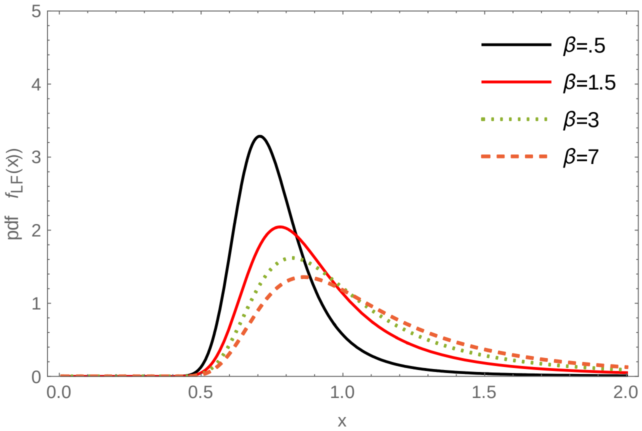
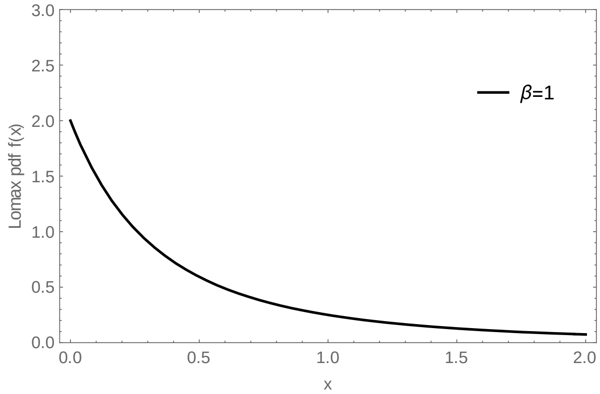


Fig.2.3 The Lomax-Frechet pdf for $\alpha = 4$, μ= 3, σ=1

Set 3: $\alpha = 2$, $\beta = 1$, $\sigma = 2.5$ and μ is varying (Fig 3.1–3.3)



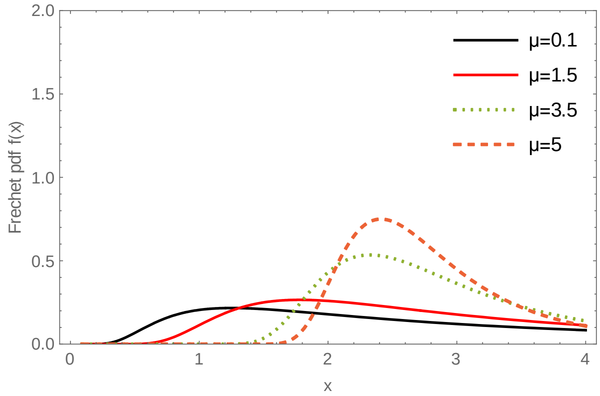


Fig.3.1 The Lomax pdf for $\alpha = 2$, $\beta = 1$ Fig.3.2 The Frechet pdf for σ = 2.5

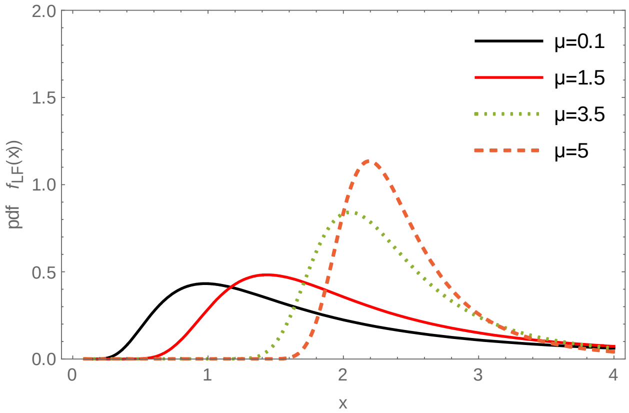
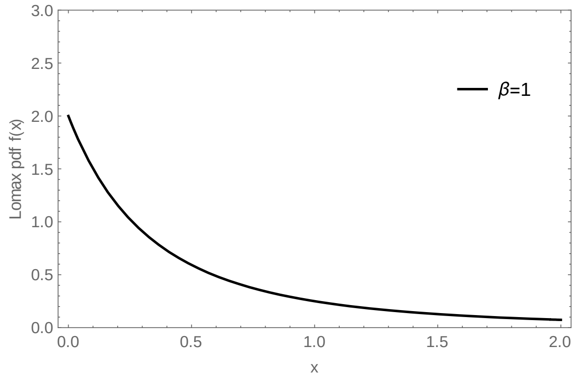


Fig.3.3 The Lomax-Frechet pdf for $\alpha = 2$, $\beta = 1$, σ=2.5

Set 4: α = 2, β = 1, μ= 3 and σ is varying (Fig 4.1–4.3)



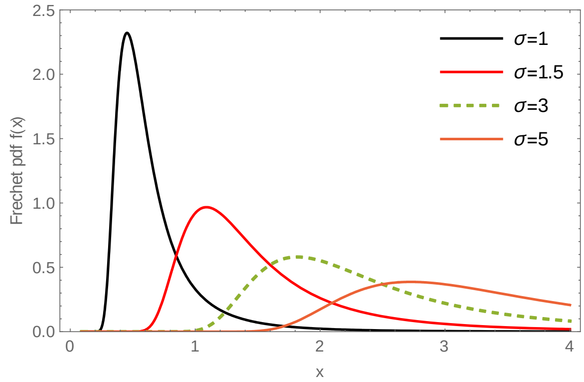


Fig.4.1 The Lomax pdf for α = 2, β = 1

Fig.4.2 The Frechet pdf for α = 2, β = 1, μ= 3

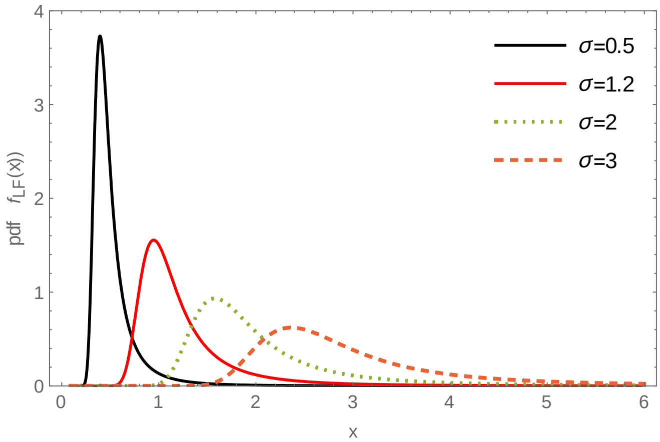


Fig.4.3 The Lomax-Frechet pdf for α = 2, β = 1, μ= 3

3.2 Characteristics Function: The Characteristics Function of $f_{LF}(x)$ is

$$\phi(t) = E\left[\exp\ (itx)\right] = \int_{-\infty}^{\infty} e^{itx} f_{LF}(x)\, dx$$

$$= \int_0^\infty e^{itx} \frac{K\alpha\mu}{\beta\sigma}\left(\frac{x}{\sigma}\right)^{-(1+\mu)} e^{-\left(\frac{x}{\sigma}\right)^{-\mu}} \left[1+\frac{1}{\beta} e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right]^{-(1+\alpha)} dx \tag{10}$$

To expand $\left[1+\frac{1}{\beta} e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right]^{-(1+\alpha)}$, let $\frac{1}{\beta} e^{-\left(\frac{x}{\sigma}\right)^{-\mu}} = A$

$$\left[1+A\right]^{-(1+\alpha)} = 1-(1+\alpha)A + \frac{(1+\alpha)(\alpha+2)}{2!} A^2 + \frac{(1+\alpha)(\alpha+2)(\alpha+3)}{3!} A^3 + ...$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(\alpha+1)(\alpha+2)(\alpha+3).....(\alpha+n)}{n!} A^n$$

$$\left[1+\frac{1}{\beta} e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right]^{-(1+\alpha)} = \sum_{n=0}^{\infty} (-1)^n \frac{(\alpha+1)_n}{n!} \left[\frac{e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}}{\beta}\right]^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(\alpha+1)_n}{n!\beta^n} e^{-n\left(\frac{x}{\sigma}\right)^{-\mu}} \qquad ...(11)$$

Also, $$e^{itx} = 1 + (itx) + \frac{(itx)^2}{2!} + ...$$

$$= \sum_{m=0}^{\infty} \frac{(itx)^m}{m!} \qquad ...(12)$$

Substituting equations (11) and (12) in equation (10),

$$\phi(t) = \frac{K\alpha\mu}{\beta\sigma} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha+1)_n}{n!\beta^n} \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \int_0^\infty x^m \left(\frac{x}{\sigma}\right)^{-(1+\mu)} e^{-\left(\frac{x}{\sigma}\right)^{-\mu}} e^{-n\left(\frac{x}{\sigma}\right)^{-\mu}} dx$$

$$= \frac{K\alpha\mu}{\beta\sigma} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha+1)_n}{n!\beta^n} \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \int_0^\infty x^m e^{-(n+1)\left(\frac{x}{\sigma}\right)^{-\mu}} \left(\frac{x}{\sigma}\right)^{-(1+\mu)} dx \qquad ...(13)$$

Let $\left(\frac{x}{\sigma}\right)^{-\mu} = y \Rightarrow -\mu\left(\frac{x}{\sigma}\right)^{-\mu-1} . \frac{1}{\sigma} dx = dy$

$$\Rightarrow \left(\frac{x}{\sigma}\right)^{-(1+\mu)} dx = \frac{\sigma}{\mu} dy$$

Also, $x=\sigma(y)^{-\frac{1}{\mu}}$

When $x = 0, y = \infty$ and when $x = \infty, y = 0$.

$$\therefore\ \phi(t)=\frac{K\alpha\mu}{\beta\sigma}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha+1)_n}{n!\beta^n}\sum_{m=0}^{\infty}\frac{(it)^m}{m!}\int_{\infty}^{0}\left(\sigma y^{-\frac{1}{\mu}}\right)^m e^{-(n+1)y}\left(-\frac{\sigma}{\mu}\right)dy$$

$$=\frac{K\alpha}{\beta}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha+1)_n}{n!\beta^n}\sum_{m=0}^{\infty}\frac{(it)^m\sigma^m}{m!}\int_{0}^{\infty}e^{-(n+1)y}y^{-\frac{m}{\mu}}dy$$

$$=\frac{K\alpha}{\beta}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha+1)_n}{n!\beta^n}\left[\lim_{\mu\to\infty}\sum_{m=0}^{\mu}\frac{(it)^m\sigma^m}{m!}\frac{\left\lceil 1-\frac{m}{\mu}\right.}{(n+1)^{1-\frac{m}{\mu}}}\right]\qquad\left[\because\ \int_0^{\infty}e^{-kx}x^n dx=\frac{\lceil n+1}{k^{n+1}}\right]$$

$$=\frac{K\alpha}{\beta}\left[\lim_{\mu\to\infty}\sum_{m=0}^{\mu}\frac{(it\sigma)^m}{m!}\left\lceil 1-\frac{m}{\mu}\right.\sum_{n=0}^{\infty}\frac{(\alpha+1)_n}{n!(n+1)^{1-\frac{m}{\mu}}}\left(-\frac{1}{\beta}\right)^n\right]$$

$$=\frac{K\alpha}{\beta}\sum_{m=0}^{\mu}\frac{(it\sigma)^m}{m!}\left\lceil 1-\frac{m}{\mu}\right.\ \phi^*_{\alpha+1}\left(-\frac{1}{\beta},1-\frac{m}{\mu},1\right)\quad,\quad|\beta|>1 \qquad (14)$$

where $\phi^*_\alpha(z,s,a)$ is the generalized Riemann Zeta function defined by Goyal and Laddha [9] as

$$\phi^*_\alpha(z,s,a)=\sum_{n=0}^{\infty}\frac{(\alpha)_n z^n}{n!\,(n+a)^s}\ ,\ a\neq\{-1,-2,....\},\ \alpha\geq 1 \text{ and either } |z|<1,\ \mathrm{Re}(s)>0 \text{ or } z=1 \text{ and } \mathrm{Re}(s)>\mu$$

3.3 The moments: The r[th] moment of $f(x)$ about the origin is

$$\mu_r^{'}=E(x^r)=\int_{-\infty}^{\infty}x^r f(x)\,dx,\quad r=0,1,2,......$$

$$=\int_0^{\infty}x^r f_{LF}(x)\,dx$$

$$=\int_0^{\infty}x^r\frac{K\alpha\mu}{\beta\sigma}\left(\frac{x}{\sigma}\right)^{-(1+\mu)}e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\left[1+\frac{1}{\beta}e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right]^{-(1+\alpha)}dx$$

Expanding $\left[1+\frac{1}{\beta}e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right]^{-(1+\alpha)}$ using binomial expansion,

$$\mu_r' = \int_0^\infty x^r \frac{K\alpha\mu}{\beta\sigma}\left(\frac{x}{\sigma}\right)^{-(1+\mu)} e^{-\left(\frac{x}{\sigma}\right)^{-\mu}} \sum_{n=0}^{\infty}\frac{(-1)^n(\alpha+1)_n}{n!}\frac{1}{\beta^n} e^{-n\left(\frac{x}{\sigma}\right)^{-\mu}} dx$$

$$= \frac{K\alpha\mu}{\beta\sigma}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha+1)_n}{n!\beta^n}\int_0^\infty x^r\left(\frac{x}{\sigma}\right)^{-(1+\mu)} e^{-(n+1)\left(\frac{x}{\sigma}\right)^{-\mu}} dx$$

Putting $\left(\frac{x}{\sigma}\right)^{-\mu} = y$,

$$\mu_r' = \frac{K\alpha\mu}{\beta\sigma}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha+1)_n}{n!\beta^n}\int_\infty^0\left(\sigma y^{-\frac{1}{\mu}}\right)^r e^{-(n+1)y}\left(-\frac{\sigma}{\mu}\right)dy$$

$$= \frac{K\alpha\sigma^r}{\beta}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha+1)_n}{n!\beta^n}\int_0^\infty e^{-(n+1)y} y^{-\frac{r}{\mu}}\, dy$$

$$= \frac{K\alpha\sigma^r}{\beta}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha+1)_n}{n!\beta^n}\frac{\left|\overline{1-\frac{r}{\mu}}\right.}{(n+1)^{1-\frac{r}{\mu}}}$$

$$= \frac{K\alpha\sigma^r}{\beta}\left|\overline{1-\frac{r}{\mu}}\right.\sum_{n=0}^{\infty}\frac{(\alpha+1)_n}{n!(n+1)^{-\frac{r}{\mu}+1}}\left(-\frac{1}{\beta}\right)^n$$

$$= \frac{K\alpha\sigma^r}{\beta}\left|\overline{1-\frac{r}{\mu}}\right.\ \phi^*_{\alpha+1}\left(-\frac{1}{\beta},\ 1-\frac{r}{\mu},\ 1\right), \quad |\beta|>1,\ r<\mu \tag{15}$$

where $\phi^*_\alpha(z, s, a)$ is the generalized Riemann Zeta function defined by Goyal and Laddha [9].

The first four cumulants if μ >4 are

$$\mu_1' = \frac{K\alpha\sigma}{\beta}\left|\overline{1-\frac{1}{\mu}}\right.\ \phi^*_{\alpha+1}\left(-\frac{1}{\beta},\ 1-\frac{1}{\mu},\ 1\right) \tag{16}$$

$$\mu_2' = \frac{K\alpha\sigma^2}{\beta}\left|\overline{1-\frac{2}{\mu}}\right.\ \phi^*_{\alpha+1}\left(-\frac{1}{\beta},\ 1-\frac{2}{\mu},\ 1\right) \tag{17}$$

$$\mu_3' = \frac{K\alpha\sigma^3}{\beta}\left|\overline{1-\frac{3}{\mu}}\right.\ \phi^*_{\alpha+1}\left(-\frac{1}{\beta},\ 1-\frac{3}{\mu},\ 1\right) \tag{18}$$

$$\mu_4' = \frac{K\alpha\sigma^4}{\beta}\left|\overline{1-\frac{4}{\mu}}\right.\ \phi^*_{\alpha+1}\left(-\frac{1}{\beta},\ 1-\frac{4}{\mu},\ 1\right) \tag{19}$$

The variance, skewness, and kurtosis measures can now be calculated using the following relations

$$Var(x) = \mu_2' - \mu_1'^2 \tag{20}$$

$$\text{Skewness}(x) = \frac{\mu_3' - 3\mu_1'\mu_2' + 2\,\mu_1'^3}{Var^{3/2}(x)} \tag{21}$$

$$\text{Kurtosis}(x) = \frac{\mu_4' - 4\mu_1'\mu_3' + 6\mu_2'\mu_1'^2 - 3\,\mu_1'^4}{Var^2(x)} \tag{22}$$

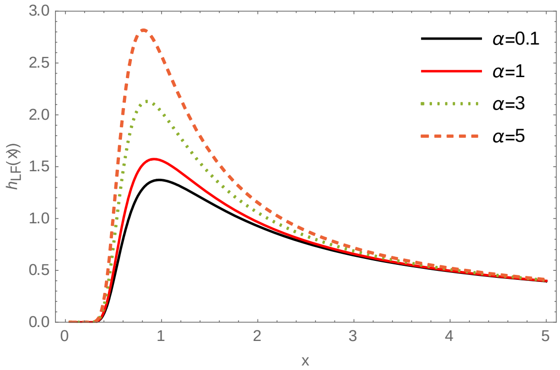
3.4 Hazard rate function: The hazard rate function *h(x)* is given by

$$h(x) = \frac{f_{LF}(x)}{1 - F_{LF}(x)}$$

$$= \frac{\dfrac{K\alpha\mu}{\beta\sigma}\left(\dfrac{x}{\sigma}\right)^{-(1+\mu)} e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\left[1 + \dfrac{1}{\beta}e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right]^{-(1+\alpha)}}{1 - K\left[1 - \left\{1 + \dfrac{1}{\beta}e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right\}^{-\alpha}\right]}$$

$$= \frac{\alpha\mu}{\beta\sigma}\left(\frac{x}{\sigma}\right)^{-(1+\mu)} \frac{e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\left[1 + \dfrac{1}{\beta}e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right]^{-(1+\alpha)}}{\left\{1 + \dfrac{1}{\beta}e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right\}^{-\alpha} - \left(1 + \dfrac{1}{\beta}\right)^{-\alpha}} \quad ; \; \alpha, \beta, \sigma, \mu > 0 \tag{23}$$

Fig.3.4.1 and Fig.3.4.2 illustrates some possible shapes of $h(x)$ for different values of parameter α and σ.



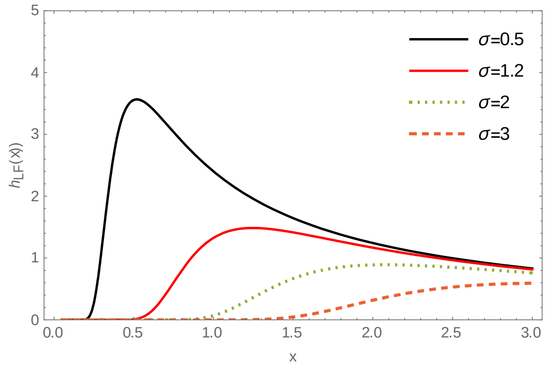


Fig.3.4.1 The Hazard rate function for $\beta = 1.5$, $\mu = 2$, $\sigma = 1$

Fig.3.4.2 The Hazard rate function for $\alpha = 2$, $\beta = 1$, $\sigma = 3$

3.5 Survivor function: The survivor function *S(x)* is given by

$$S(x) = P(X > x) = \int_x^\infty f_{LF}(u)\, du$$

$$= 1 - F_{LF}(x)$$

$$= 1 - K\left[1 - \left\{1 + \frac{1}{\beta} e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right\}^{-\alpha}\right],$$

$$= 1 - \frac{1}{1 - \left(1 + \frac{1}{\beta}\right)^{-\alpha}}\left[1 - \left\{1 + \frac{1}{\beta} e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right\}^{-\alpha}\right]$$

$$=1-\frac{\left(1+\frac{1}{\beta}\right)^{\alpha}}{\left(1+\frac{1}{\beta}\right)^{\alpha}-1}+\frac{\left(1+\frac{1}{\beta}\right)^{\alpha}}{\left(1+\frac{1}{\beta}\right)^{\alpha}-1}\left\{1+\frac{1}{\beta}e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right\}^{-\alpha}$$

$$=-\frac{1}{\left(1+\frac{1}{\beta}\right)^{\alpha}-1}+\frac{\left(1+\frac{1}{\beta}\right)^{\alpha}}{\left(1+\frac{1}{\beta}\right)^{\alpha}-1}\left\{1+\frac{1}{\beta}e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}\right\}^{-\alpha}$$

$$=-\frac{1}{\left(1+\frac{1}{\beta}\right)^{\alpha}-1}\left[\left\{\frac{1+\frac{1}{\beta}e^{-\left(\frac{x}{\sigma}\right)^{-\mu}}}{1+\frac{1}{\beta}}\right\}^{-\alpha}-1\right] \qquad (24)$$

3.6 Reliability: The reliability R is given by

$$R=\mathrm{P_r}(X_2<X_1)$$

$$=\int_0^{\infty} f_1(x)\,F_2(x)\,dx$$

where X_1 and X_2 are independent random variables with positive support. $f_1(x)$ is the pdf of X_1 and $F_2(x)$ is the cdf of X_2.

The reliability of Lomax-Frechet distribution is

$$R=\int_0^{\infty}\frac{K_1\alpha_1\mu_1}{\beta_1\sigma_1}\left(\frac{x}{\sigma_1}\right)^{-(1+\mu_1)}e^{-\left(\frac{x}{\sigma_1}\right)^{-\mu_1}}\left[1+\frac{1}{\beta_1}e^{-\left(\frac{x}{\sigma_1}\right)^{-\mu_1}}\right]^{-(1+\alpha_1)}.K_2\left[1-\left(1+\frac{1}{\beta_2}e^{-\left(\frac{x}{\sigma_2}\right)^{-\mu_2}}\right)\right]^{-\alpha_2}dx$$

$$=\frac{K_1\alpha_1\mu_1K_2}{\beta_1\sigma_1}\left[\int_0^{\infty}\left\{1+\frac{1}{\beta_1}e^{-\left(\frac{x}{\sigma_1}\right)^{-\mu_1}}\right\}^{-(1+\alpha_1)}\left(\frac{x}{\sigma_1}\right)^{-(1+\mu_1)}e^{-\left(\frac{x}{\sigma_1}\right)^{-\mu_1}}dx\right.$$

$$\left.-\int_0^{\infty}\left\{1+\frac{1}{\beta_1}e^{-\left(\frac{x}{\sigma_1}\right)^{-\mu_1}}\right\}^{-(1+\alpha_1)}\left\{1+\frac{1}{\beta_2}e^{-\left(\frac{x}{\sigma_2}\right)^{-\mu_2}}\right\}^{-\alpha_2}\left(\frac{x}{\sigma_1}\right)^{-(1+\mu_1)}.\,e^{-\left(\frac{x}{\sigma_1}\right)^{-\mu_1}}dx\right] \qquad ...(25)$$

Expanding using binomial expansion,

$$R=\frac{K_1\alpha_1\mu_1K_2}{\beta_1\sigma_1}\left[\int_0^\infty\left\{\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}(\alpha_1+1)_{n_1}}{\beta_1^{\,n_1}}\,e^{-n_1\left(\frac{x}{\sigma_1}\right)^{-\mu_1}}\right\}\left(\frac{x}{\sigma_1}\right)^{-(1+\mu_1)}e^{-\left(\frac{x}{\sigma_1}\right)^{-\mu_1}}dx\right.$$

$$\left.-\int_0^\infty\left\{\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}(\alpha_1+1)_{n_1}}{\beta_1^{\,n_1}}\,e^{-n_1\left(\frac{x}{\sigma_1}\right)^{-\mu_1}}\cdot\sum_{n_2=0}^{\infty}\frac{(-1)^{n_2}(\alpha_2)_{n_2}}{\beta_2^{\,n_2}}\,e^{-n_2\left(\frac{x}{\sigma_2}\right)^{-\mu_2}}\left(\frac{x}{\sigma_1}\right)^{-(1+\mu_1)}\cdot e^{-\left(\frac{x}{\sigma_1}\right)^{-\mu_1}}\right\}dx\right]$$

$$=\frac{K_1K_2\alpha_1\mu_1}{\beta_1\sigma_1}\left[\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}(\alpha_1+1)_{n_1}}{\beta_1^{\,n_1}}\int_0^\infty e^{-(n_1+1)\left(\frac{x}{\sigma_1}\right)^{-\mu_1}}\left(\frac{x}{\sigma_1}\right)^{-(1+\mu_1)}dx\right.$$

$$\left.-\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}(\alpha_1+1)_{n_1}}{\beta_1^{\,n_1}}\sum_{n_2=0}^{\infty}\frac{(-1)^{n_2}(\alpha_2)_{n_2}}{\beta_2^{\,n_2}}\int_0^\infty e^{-(n_1+1)\left(\frac{x}{\sigma_1}\right)^{-\mu_1}}e^{-n_2\left(\frac{x}{\sigma_2}\right)^{-\mu_2}}\left(\frac{x}{\sigma_1}\right)^{-(1+\mu_1)}dx\right]\qquad ...(26)$$

Putting $\left(\frac{x}{\sigma_1}\right)^{-\mu_1}=y$,

$$-\mu_1\left(\frac{x}{\sigma_1}\right)^{-\mu_1-1}.\frac{1}{\sigma_1}dx=dy\Rightarrow\left(\frac{x}{\sigma_1}\right)^{-(1+\mu_1)}dx=-\frac{\sigma_1}{\mu_1}dy$$

Also, $x=\sigma_1(y)^{-\frac{1}{\mu_1}}$

When $x = 0$, $y = \infty$ and when $x = \infty$, $y = 0$.

$$R=\frac{K_1K_2\alpha_1\mu_1}{\beta_1\sigma_1}\left[\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}(\alpha_1+1)_{n_1}}{\beta_1^{\,n_1}}\int_\infty^0 e^{-(n_1+1)y}\left(-\frac{\sigma_1}{\mu_1}\right)dy\right.$$

$$\left.-\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}(\alpha_1+1)_{n_1}}{\beta_1^{\,n_1}}\sum_{n_2=0}^{\infty}\frac{(-1)^{n_2}(\alpha_2)_{n_2}}{\beta_2^{\,n_2}}\int_\infty^0 e^{-n_2\left(\frac{\sigma_1 y^{-\frac{1}{\mu_1}}}{\sigma_2}\right)^{-\mu_2}}e^{-(n_1+1)y}\left(-\frac{\sigma_1}{\mu_1}\right)dy\right]$$

$$=\frac{K_1K_2\alpha_1}{\beta_1}\left[\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}(\alpha_1+1)_{n_1}}{\beta_1^{\,n_1}}\int_0^\infty e^{-(n_1+1)y}\,dy\right.$$

$$\left.-\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(-1)^{n_1+n_2}(\alpha_1+1)_{n_1}(\alpha_2)_{n_2}}{\beta_1^{\,n_1}\beta_2^{\,n_2}}\int_0^\infty e^{-(n_1+1)y}\,e^{-n_2\left(\frac{\sigma_1}{\sigma_2}\right)^{-\mu_2}y^{\frac{\mu_2}{\mu_1}}}dy\right]$$

$$=\frac{K_1K_2\alpha_1}{\beta_1}\left[\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}(\alpha_1+1)_{n_1}}{\beta_1^{n_1}}\left\{\frac{1}{n_1+1}\right\}\right.$$

$$\left.-\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(-1)^{n_1+n_2}(\alpha_1+1)_{n_1}(\alpha_2)_{n_2}}{\beta_1^{n_1}\beta_2^{n_2}}\int_0^{\infty}e^{-(n_1+1)y}\sum_{m=0}^{\infty}\frac{(-1)^m\left\{n_2\left(\frac{\sigma_1}{\sigma_2}\right)^{-\mu_2}y^{\frac{\mu_2}{\mu_1}}\right\}^m}{m!}dy\right]$$

$$=\frac{K_1K_2\alpha_1}{\beta_1}\left[\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}(\alpha_1+1)_{n_1}}{\beta_1^{n_1}(n_1+1)}\right.$$

$$\left.-\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(-1)^{n_1+n_2}(\alpha_1+1)_{n_1}(\alpha_2)_{n_2}}{\beta_1^{n_1}\beta_2^{n_2}}\sum_{m=0}^{\infty}\frac{(-1)^m n_2^{\,m}\left(\frac{\sigma_1}{\sigma_2}\right)^{-\mu_2 m}}{m!}\int_0^{\infty}e^{-(n_1+1)y}\,y^{\frac{m\mu_2}{\mu_1}}dy\right]$$

$$=\frac{K_1K_2\alpha_1}{\beta_1}\left[\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}(\alpha_1+1)_{n_1}}{\beta_1^{n_1}(n_1+1)}\right.$$

$$\left.-\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(-1)^{n_1+n_2}(\alpha_1+1)_{n_1}(\alpha_2)_{n_2}}{\beta_1^{n_1}\beta_2^{n_2}}\sum_{m=0}^{\infty}\frac{(-1)^m n_2^{\,m}\left(\frac{\sigma_1}{\sigma_2}\right)^{-\mu_2 m}}{m!}\frac{\Gamma\left(\frac{m\mu_2}{\mu_1}+1\right)}{(n_1+1)^{\frac{m\mu_2}{\mu_1}+1}}\right] \quad ...(27)$$

4 Estimation

The estimation of the parameters is done by the method of maximum likelihood. The likelihood function is a product of the pdf functions with one element of each data point in the data set.

$$L=\prod_{i=1}^{n} f(x_i\ ;\theta_1,\theta_2,.......,\theta_k)$$

$$\log L=\sum_{i=1}^{n}\log f(x_i;\theta_1,\theta_2,.......,\theta_k)$$

$$=\sum_{i=1}^{n}\log\left[\frac{K\alpha\mu}{\beta\sigma}\left(\frac{x_i}{\sigma}\right)^{-(1+\mu)}e^{-\left(\frac{x_i}{\sigma}\right)^{-\mu}}\left\{1+\frac{1}{\beta}e^{-\left(\frac{x_i}{\sigma}\right)^{-\mu}}\right\}^{-(1+\alpha)}\right]$$

$$=\sum_{i=1}^{n}\left[\log\frac{K\alpha\mu}{\beta\sigma}-(1+\mu)\{\log x_i-\log\sigma\}-\left(\frac{x_i}{\sigma}\right)^{-\mu}-(1+\alpha)\log\left\{1+\frac{1}{\beta}e^{-\left(\frac{x_i}{\sigma}\right)^{-\mu}}\right\}\right]$$

$$=n\log K+n\log\alpha+n\log\mu-n\log\beta-n\log\sigma-(1+\mu)\sum_{i=1}^{n}\log x_i$$

$$+n(1+\mu)\log\sigma-\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{-\mu}-(1+\alpha)\sum_{i=1}^{n}\log\left\{1+\frac{1}{\beta}e^{-\left(\frac{x_i}{\sigma}\right)^{-\mu}}\right\}$$

$$=-n\log\left[1-\left(1+\frac{1}{\beta}\right)^{-\alpha}\right]+n\log\alpha+n\log\mu-n\log\beta+n\mu\log\sigma$$

$$-(1+\mu)\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{-\mu}-(1+\alpha)\sum_{i=1}^{n}\log\left\{1+\frac{1}{\beta}e^{-\left(\frac{x_i}{\sigma}\right)^{-\mu}}\right\} \quad ...(28)$$

Differentiating log L w.r.t. α, β and σ partially,

$$\frac{\partial\log L}{\partial\alpha}=-\frac{n}{\left[1-\left(1+\frac{1}{\beta}\right)^{-\alpha}\right]}\left[\left(1+\frac{1}{\beta}\right)^{-\alpha}\log\left(1+\frac{1}{\beta}\right)\right]+\frac{n}{\alpha}-\sum_{i=1}^{n}\log\left[1+\frac{1}{\beta}e^{-\left(\frac{x_i}{\sigma}\right)^{-\mu}}\right]$$

$$=\frac{n}{\alpha}-\frac{n\log\left(1+\beta^{-1}\right)}{\left(1+\beta^{-1}\right)^{\alpha}-1}-\sum_{i=1}^{n}\log\left[1+\frac{1}{\beta}e^{-\left(\frac{x_i}{\sigma}\right)^{-\mu}}\right] \quad ...(29)$$

$$\frac{\partial\log L}{\partial\beta}=-\frac{n}{\left[1-\left(1+\frac{1}{\beta}\right)^{-\alpha}\right]}\left[\alpha\left(1+\frac{1}{\beta}\right)^{-\alpha-1}\left(-\frac{1}{\beta^2}\right)\right]-\frac{n}{\beta}-(1+\alpha)\sum_{i=1}^{n}\frac{1}{1+\frac{1}{\beta}e^{-\left(\frac{x_i}{\sigma}\right)^{-\mu}}}\left(-\frac{1}{\beta^2}\right)e^{-\left(\frac{x_i}{\sigma}\right)^{-\mu}}$$

$$=-\frac{n}{\beta}+\frac{n\alpha}{\beta(\beta+1)\left[\left(1+\beta^{-1}\right)^{\alpha}-1\right]}+\left(\frac{(1+\alpha)}{\beta^2}\right)\sum_{i=1}^{n}\frac{1}{e^{\left(\frac{x_i}{\sigma}\right)^{-\mu}}+\frac{1}{\beta}} \quad ...(30)$$

$$\frac{\partial \log L}{\partial \mu}=\frac{n}{\mu}+n\log\sigma-\sum_{i=1}^{n}\log x_i+\mu\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{-\mu-1}-(1+\alpha)\sum_{i=1}^{n}\frac{1}{1+\frac{1}{\beta}e^{-\left(\frac{x_i}{\sigma}\right)^{-\mu}}}\left(\frac{1}{\beta}\right)e^{-\left(\frac{x_i}{\sigma}\right)^{-\mu}}\left(\frac{x_i}{\sigma}\right)^{-\mu}\log\left(\frac{x_i}{\sigma}\right)$$

$$=\frac{n}{\mu}+n\log\sigma-\sum_{i=1}^{n}\log x_i+\mu\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{-\mu-1}-\frac{(1+\alpha)}{\beta}\sum_{i=1}^{n}\frac{1}{e^{\left(\frac{x_i}{\sigma}\right)^{-\mu}}+\frac{1}{\beta}}e^{-\left(\frac{x_i}{\sigma}\right)^{-\mu}}\left(\frac{x_i}{\sigma}\right)^{-\mu}\log\left(\frac{x_i}{\sigma}\right) \quad ...(31)$$

$$\frac{\partial \log L}{\partial \sigma}=\frac{n\mu}{\sigma}-\sum_{i=1}^{n}(x_i)^{-\mu}(\mu\sigma^{\mu+1})-(1+\alpha)\sum_{i=1}^{n}\frac{1}{1+\frac{1}{\beta}e^{-\left(\frac{x_i}{\sigma}\right)^{-\mu}}}\left(\frac{1}{\beta}\right)e^{-\left(\frac{x_i}{\sigma}\right)^{-\mu}(x_i)^{-\mu}(\mu\sigma^{\mu+1})}$$

$$=\frac{n\mu}{\sigma}-\mu\sigma\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{-\mu}-\frac{(1+\alpha)\mu\sigma}{\beta}\sum_{i=1}^{n}\frac{1}{e^{\left(\frac{x_i}{\sigma}\right)^{-\mu}}+\frac{1}{\beta}}\left(\frac{x_i}{\sigma}\right)^{-\mu} \quad ...(32)$$

Setting these expressions to zero and solving them simultaneously, the maximum likelihood estimates of the four parameters is obtained.

5 Conclusions

A new class of distribution referred to as Lomax-Frechet distribution is developed by taking Lomax distribution as the parent distribution and the Frechet distribution as the generator distribution by using generator technique. The explicit expressions for the moments, characteristic function, hazard rate function, survivor function and reliability for the Lomax-Frechet distribution are derived. Some graphs for the pdf of these distributions are drawn using Mathematica to show the effect of variation of different parameters occurring in the definition of these distributions. The results derived in this paper are likely to find certain applications in the theory of special functions.

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