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SOME CURVATURE PROPERTIES OF SP-SASAKIAN MANIFOLD WITH QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract : In this paper, we study quarter-symmetric metric connection in an SP-Sasakian manifold. Some results related to this connection are studied and obtained. Also some curvature properties of an SP-Sasakian with quarter-symmetric metric connection are studied.

Key words and Phrases : SP-sasakian manifold, quarter-symmetric metric connection, conharmonic, projective, quasi-conformal, M-projective curvature tensor.

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1. INTRODUCTION

In 1975, Golab initiated the study of quarter-symmetric linear connection on a differentiable manifold M^n. A linear connection ∇ in an n-dimensional differentiable manifold M^n is said to be aquarter-symmetric connection if torsion tensor T is of the form

$$T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y]$$

$$= \eta(Y)\,\phi X - \eta(X)\,\phi Y$$

where η is a 1-form and ϕ is a tensor of type (1,1).

In addition, if a quarter-symmetric linear connection $\overline{\nabla}$ satisfies the condition $\overline{\nabla}_{xg} = 0$ for all X,Y,Z $\in$ TM, where TM is a Lie algebra of vectors fields of the differentiable manifold M^n, then $\overline{\nabla}$ is said to be quarter-symmetric metric connection. Quarter-symmetric metric connection is also studied by Biswas and De [3], De and Mondal [6], Singh and Pandey [16], Mishra and Pandey [10], Rastogi [12], Yano and Imai [22], Sular, et al. [20] and many others.

In particular, if $\phi X = X$ and $\phi Y = Y$, the the quarter-symmetric connection reduces to a semi-symmetric connection which is the generalized case of quarter-symmetric metric connection.

The semi-symmetric metric connection in an SP-Sasakian manifold have been studied by Sinha and Kalpna [14]. In the present paper, the quarter-symmetric metric connection in an SP-Sasakian manifold has been studied. The different type of curvatures with a quarter-symmetric metric connection in an SP-Sasakian manifold also have been studied. This paper is organized as follows after preliminaries. In section 3, we have studied some Riemannian curvature properties of SP-Sasakian manifold with respect to quarter-symmetric connection. In section 4, we have studied projective curvature tensor.

The quasi-conformal curvature tensor, M-projective curvature tensor, Conhormonic curvature tensor and projective Ricci tensor have been studied in section 5, section 6, section 7 and section 8 respectively.

2. PRELIMINARIES

An n-dimensional differential manifold M^n is called an almost para-contact manifold [15], if it admits an almost para-contact structure (ϕ,ξ,η) consisting of a (1,1) tensor field ϕ, vector field ξ, and 1-form η satisfying

$$\phi^2 X = X - \eta(X)\xi \quad \ldots(1)$$

$$\eta(\xi) = 1 \quad \ldots(2)$$

$$\phi_0 \xi = 0 \quad \ldots(3)$$

$$\eta_0 \phi = 0 \quad \ldots(4)$$

$$\text{Rank}\,(\phi) = n - 1 \quad \ldots(5)$$

for all $X \in TM$.

Let g be Riemannian metric satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad \ldots(6)$$

$$g(\phi X, Y) = g(X, \phi Y) \quad \ldots(7)$$

for all $X,Y \in TM$.

Then M^n becomes an almost para-contact Riemannian manifold equipped with the almost para-contact Riemannian structure (ϕ,ξ,η,g) [13].

If we define

$$\Phi(X, Y) = g(\phi X, Y) \quad \ldots(8)$$

Then in addition to the above equation, we have

$$\Phi(X, Y) = \Phi(Y, X) \quad \ldots(9)$$

$$\Phi(\phi X, \phi Y) = \Phi(X, Y) \quad \ldots(10)$$

An almost para-contact Riemannian manifold is called a P-Sasakian manifold [2],if it satisfies

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \quad \ldots(11)$$

for all X,Y $\in$ TM.

where ∇ denotes he covariant differentiation with respect to Riemannian metric g. From the above equation, it follows that

$$\nabla_X \xi = \phi X \quad \ldots(12)$$

$$(\nabla_X \eta)(Y) = g(X, \phi Y) = (\nabla_Y \eta)(X) \quad \ldots(13)$$

Especially, a P-Sasakian manifold M^n is called an special para – Sasakian manifold or briefly an SP-Sasakian manifold [11], if M^n admits a 1-form η satisfying

$$(\nabla_X \eta)(Y) - g(X, Y)\xi + \eta(X)\eta(Y) \quad \ldots(14)$$

For an SP-Sasakian manifold, we have [2],

$$\Phi(X, Y) = -g(X, Y)\xi + \eta(X)\eta(Y) \quad \ldots(15)$$

In an n-dimensional P-Sasakian manifold M^n, the curvature tensor R, the Ricci tensor S satisfy the following properties [2], [1] and [21],

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X \quad \ldots(16)$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi \quad \ldots(17)$$

$$R(\xi, X)\xi = X - \eta(X)\xi \quad \ldots(18)$$

$$\eta(R(X, Y)U) = g(X, U)\eta(Y) - g(Y, U)\eta(X) \quad \ldots(19)$$

$$\eta(R(X, Y)\xi) = 0 \quad \ldots(20)$$

$$\eta(R(\xi, X)Y) = \eta(X)\eta(Y) - g(X, Y) \quad \ldots(21)$$

$$S(X, \xi) = -(n-1)\eta(X) \quad \ldots(22)$$

$$R\xi = -(n-1)\xi \quad \ldots(23)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y) \quad \ldots(24)$$

$$S(X, \phi Y) = S(\phi X, Y) \quad \ldots(25)$$

A quarter-symmetric metric connection $\overline{\nabla}$ in an SP-Sasakian Manifold can be defined as

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi \quad \ldots(26)$$

and the curvature tensor $\overline{R}$ of M^n be

$$\overline{R}(X, Y)Z = R(X, Y)Z + 3g(X, Z)Y - 3g(Y, Z)X - 2g(X, Z)\eta(Y)\xi$$

$$- 2\eta(X)\eta(Z)T + 2g(Y, Z)\eta(X)\xi + 2\eta(Y)\eta(Z)X \quad \ldots(27)$$

The Ricci tensor $\overline{S}$ and scalar curvature r of M^n with respect to quarter-symmetric metric connection $\overline{\nabla}$ is defined as

$$\overline{S}(Y, Z) = S(Y, Z) - (3n-5)g(Y, Z) + 2(n-2)\eta(Y)\eta(Z) \quad \ldots(28)$$

$$\bar{r} = r - (n-1)(3n-4) \quad \ldots(29)$$

where $\overline{S}$ and S are the Ricci tensor of the connection $\overline{\nabla}$ and ∇ respectively.

Similarly $\bar{r}$ and r are the scalar curvature of the connection $\overline{\nabla}$ and ∇ respectively.

Definition 1. The projective curvature tensor is defined as [19],

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}\{S(Y, Z)X - S(X, Z)Y\} \quad \ldots(30)$$

Definition 2. The quasi-conformal curvature tensor is defined as [18],

$$C(X, Y, Z, U) = aR(X, Y, Z, U) + b[S(Y, Z) g(X, U) - S(X, Z)S(Y.U)]$$

$$+ b[g(Y, Z)S(X, U) - g(X, Z)S(Y, U)] \frac{r}{(2n+1)}\left[\frac{a}{2n} + b\right]\left[\begin{array}{c} g(Y, Z)g(X, U) \\ - g(X, Z) g(Y, U) \end{array}\right]$$

…(31)

Definition 3. The M-projective curvature tensor (w*) is defined as [18],

$$w^*(X, Y, Z, U) = R(X, Y, Z, U) - \frac{1}{2(n-1)}[S(Y, Z)g(X, U) - S(X, Z)g(Y, U)]$$

$$- \frac{1}{2(n-1)}[g(Y, Z)S(X, U) - g(X, Z)S(Y, U)] \quad \ldots(32)$$

Definition 4. The conhormonic curvature tensor (C*) is defined as [19],

$$C^*(X, Y, Z, U) = R(X, Y, Z, U) - \frac{1}{(n-2)}[S(Y, Z)g(X, U) - S(X, Z)g(Y, U)]$$

$$- \frac{1}{(n-2)}[g(Y, Z)S(X, U) - g(X, Z)S(Y, U)]$$

Definition 5. The projective Ricci curvature tensor is defined as

$$P^*(X, Y)Z = \frac{(2n+1)}{2n}[S(X, Z)Y] - \frac{r}{2n}[g(X, Y)Z] \quad \ldots(33)$$

3. SOME RIEMANNIAN CURVATURE PROPERTIES OF SP-SASAKIAN MANIFOLD WITH RESPECT TO QUARTER-SYMMETRIC METRIC CONNECTION

Let K and $\overline{K}$ be the curvature tensor of type (0,4) given by

$$K(X, Y, Z, U) = g(R(X, Y)Z, U) \quad \ldots(34)$$

$$\overline{K}(X, Y, Z, U) = g(\overline{R}(X, Y)Z, U) \quad \ldots(35)$$

Therefore we can state that

Theroem 1. In an SP-Sasakian manifold with quarter-symmetric metric connection $\overline{\nabla}$, we have

$$\bar{K}(X,Y,Z,U)+\bar{K}(Y,X,Z,U)=0 \quad \ldots(36)$$

Proof. From (27), we have

$$\bar{K}(X,Y,Z,U)=K(X,Y,Z,U)+3g(X,Z)g(Y,U)-3g(Y,Z)g(X,U)-2g(X,Z)\eta(Y)\eta(U)$$

$$-2\eta(X)\eta(Z)g(Y,U)+2g(X,Z)\eta(X)\eta(U)+2\eta(Y)\eta(Z)g(X,U) \quad \ldots(37)$$

and

$$\bar{K}(Y,X,Z,U)=K(Y,X,Z,U)+3g(Y,Z)g(X,U)-3g(X,Z)g(Y,U)-2g(Y,Z)\eta(X)\eta(U)$$

$$-2\eta(Y)\eta(Z)g(X,U)+2g(X,Z)\eta(Y)\eta(U)+2\eta(X)\eta(Z)g(Y,U) \quad \ldots(38)$$

Adding (34) and (35), we have

$$\bar{K}(X,Y,Z,U)+\bar{K}(Y,X,Z,U)=K(X,Y,Z,U)+K(Y,X,Z,U) \quad \ldots(39)$$

$$K(X,Y,Z,U)=-K(Y,X,Z,U) \quad \ldots(40)$$

by using (36) and (37), we obtain (33).

4. PROJECTIVE CURVATURE TENSOR

Let M^n be an n-dimensional SP-Sasakian manifold. The projective curvature tensor of M^n with respect to quarter-symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{P}(X,Y)Z=\bar{R}(X,Y)Z-\frac{1}{(n-1)}\{\bar{S}(Y,Z)X-\bar{S}(X,Z)Y\} \quad \ldots(41)$$

From (27), (28) and (41), we obtain

$$\bar{P}(X,Y)Z=P(X,Y)Z+\frac{2}{(n-1)}\{g(X,Z)Y-g(Y,Z)X\}$$

$$-\frac{2}{(n-1)}(\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\}-2\{g(X,Z)\eta(Y)-g(Y,Z)X\}\xi \quad \ldots(42)$$

From (42), we get

$$\bar{P}(X,Y)Z+\bar{P}(Y,Z)X+\bar{P}(Z,X)Y=P(X,Y)Z+P(Y,Z)X+P(Z,X)Y \quad \ldots(43)$$

If

$$P(X, Y)Z + P(Y, Z)X + P(Z, X)Y = 0 \qquad \ldots(44)$$

Then

$$\bar{P}(X, Y)Z + \bar{P}(Y, Z)X + \bar{P}(Z, X)Y = 0 \qquad \ldots(45)$$

Therefore, we can state that:

Theorem 2. Let M^n be an n-dimensional SP-Sasakian manifold with the quarter-symmetric metric connection $\bar{\nabla}$. Then the projective curvature tensor of M^n with respect to quarter-symmetric metric connection $\bar{\nabla}$ is cyclic.

If we put S = 0 in (41), we get

$$\bar{P}(X, Y)Z = \bar{R}(X, Y)Z \qquad \ldots(46)$$

Therefore, we can state that:

Theorem 3. In an SP-Sasakian manifold with respect to a quarter-symmetric metric connection $\bar{\nabla}$, if the Ricci tensor vanishes, then the Riemannian curvature tensor is equal to the projective curvature tensor with respect to quarter-symmetric metric connection.

5. QUASI-CONFORMAL CURVATURE TENSOR

Let M^n be an n-dimensional SP-Sasakian manifold. The quasi-conformal curvature tensor of M^n with respect to quarter-symmetric metric connection ∇ is defined by

$$\bar{C}(X, Y, Z, U) = a\bar{R}(X, Y, Z, U) + b[\bar{S}(Y, Z)g(X, U) - \bar{S}(X, Z)g(Y.U)]$$

$$+ b[g(Y, Z)\bar{S}(X, U) - g(X, Z)\bar{S}(Y, U)] \frac{\bar{r}}{(2n+1)}\left[\frac{a}{2n} + b\right]\begin{bmatrix} g(Y, Z)g(X, U) \\ - g(X, Z)g(Y, U) \end{bmatrix} \ldots(47)$$

Using (27), (28) and (29) in (47), we obtain

$$\bar{C}(X, Y, Z, U) = -\bar{C}(Y, X, Z, U) \qquad \ldots(48)$$

Therefore, we can state that;

Theorem 4. In an SP-Sasakian manifold with respect to quarter-symmetric metric connection, the quasi-conformal curvature tensor is skew symmetric in X and Y.

If $\bar{S} = 0$ and $\bar{r} = 0$, (47) gives

$$\bar{C}(X, Y, Z, U) = a\bar{R}(X, Y, Z, U) \qquad \ldots(49)$$

Therefore, we can state that;

Theorem 5. In an SP-Sasakian manifold with quarter-symmetric metric connection $\overline{\nabla}$, if Ricci tensor and scalar curvature both vanishes, then quasi conformal curvature tensor is equal to the constant multiple of the Riemannian curvature tensor with respect to quarter-symmetric metric connection.

6. M-PROJECTIVE CURVATURE TENSOR

Let M^n be an n-dimensional SP-Sasakian manifold then the M-projective curvature tensor (w*) of M^n with respect to quarter-symmetric metric connection ∇ is defined by

$$w^*(X,Y,Z,U) = \overline{R}(X,Y,Z,U) - \frac{1}{2(n-1)}[\overline{S}(Y,Z)g(X,U) - \overline{S}(X,Z)g(Y,U)]$$

$$-\frac{1}{2(n-1)}[g(Y,Z)\overline{S}(X,U) - g(X,Z)\overline{S}(Y,U)] \qquad \ldots(50)$$

Using (27), (28) and (29)in (50), we obtain

$$\overline{w}^*(X,Y,Z,U) = -\overline{w}^*(Y,X,Z,U)$$

Therefore, we can state that;

Theorem 6. The M-projective curvature tensor is skew symmetric in two slots in an SP-Sasakian manifold with respect to quarter-symmetric metric connection.

And

If S = 0, (50) gives

$$w^*(X,Y,Z,U) = R(X,Y,Z,U) \qquad \ldots(51)$$

Therefore, we can state that;

Theorem 7. In an SP-Sasakian manifold with quarter-symmetric metric connection $\overline{\nabla}$, if Ricci tensor vanishes, then the Riemannian curvature tensor is equal to the M-projective curvature tensor o with respect to quarter symmetric metric connection.

7. CONHORMONIC CURVATURE TENSOR

Let M^n be n-dimensional SP-Sasakian manifold. The conhormonic curvature tensor (C*) of M^n with respect to quarter-symmetric metric connection $\overline{\nabla}$ is defined by

$$C^*(X,Y,Z,U) = \overline{R}(X,Y,Z,U) - \frac{1}{(n-2)}[\overline{S}(Y,Z)g(X,U) - \overline{S}(X,Z)g(Y,U)]$$

$$-\frac{1}{(n-2)}[g(Y,Z)\overline{S}(X,U)-g(X,Z)\overline{S}(Y,U)] \quad \ldots(52)$$

Using (27), (28) and (29), in (52), we obtain

$$\overline{C}*(X,Y,Z,U)=-\overline{C}*(Y,X,Z,U) \quad \ldots(53)$$

Therefore, we can state that;

Theorem 8. In an SP-Sasakian manifold with respect to quarter-symmetric metric connection, the conhormonic curvature tensor is skew-symmetric in X and Y

And

if S = 0 (53) gives

$$\overline{C}*(X,Y,Z,U)=\overline{R}(X,Y,Z,U) \quad \ldots(54)$$

Therefore, we can state that;

Theorem 9. In SP-Sasakian manifold with quarter-symmetric metric connection ∇, if Ricci tensor vanishes, then Riemannian curvature tensor is equal to the conhormonic curvature tensor with respect to quarter-symmetric metric connection.

From (51) and (54), we obtain

$$\overline{C}*(X,Y,Z,U)=\overline{w}*(X,Y,Z,U) \quad \ldots(55)$$

Hence, we state the theorem;

Theorem 10. In an SP-Sasakian manifold with quarter-symmetric metric connection, if Ricci tensor vanishes, then the conhormonic curvature tensor is equal to the M-projective curvature tensor with respect to quarter-symmetric metric connection.

8. PROJECTIVE RICCI CURVATURE TENSOR

Let M^n be n-dimensional SP-Sasakian manifold. The projective Ricci curvature tensor of M^n with respect to quarter-symmetric metric connection $\overline{\nabla}$ is defined by

$$\overline{P}*(X,Y)Z=\frac{(2n+1)}{2n}[\overline{S}(X,Z)Y]-\frac{\bar{r}}{2n}g(X,Y)Z \quad \ldots(56)$$

From (28) and (29), we obtain

$$\overline{P}*(X,Y)Z=P*(X,Y)Z+\left[\frac{-3n^2+9}{(2n)}\right]g(X,Y)Z+\left[\frac{(2n+1)(n-2)}{n}\right]\eta(X)\eta(Y)Z \quad \ldots(57)$$

From (57), we obtain

$$\overline{P}*(X,Y)Z = \overline{P}*(Y,X)Z \qquad \ldots(58)$$

If $\overline{S}$ = 0, then (57) gives

$$\overline{P}*(X,Y)Z = 0 \qquad \ldots(59)$$

Therefore, we can state that;

Theroem 11. In an SP-Sasakian manifold with quarter-symmetric metric connection, if Ricci tensor vanishes, then the projective Ricci curvature tensor becomes flat.

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