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BAYESIAN ANALYSIS OF A PROBABILITY MODEL FOR FIRST CONCEPTION

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Abstract : Probability models are widely used in different disciplinary fields. The present paper is an attempt to develop probability model of waiting time of first conception and also analyse under the Bayesian environment of waiting time of first conception under precautionary loss function.

Keywords: Probability model, waiting time, conception, precautionary loss function.

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1. Introduction

A Bayesian approach might be useful in addressing the issues related with the demographic events of society. By design, Bayesian methods natively consider the uncertainty associated with the parameters of a probability model (even if those uncertain parameters are believed to be fixed numbers). Bayesian methods are often recommended as the proper way to make formal use of subjective information such as expert opinion and personal judgments or beliefs of an analyst. An important advantage of Bayesian methods, unlike frequentist methods with which they are often contrasted, is that they can always yield a precise answer, even when no data at all are available. Finally, recent Bayesian literature has focused on the potential significance of model uncertainty and how it can be incorporated into quantitative analysis.

At the time of marriage a woman is susceptible to conception and the time elapsed before a conception is a random variable determined by fecundability, which is defined as the monthly chance of a conception. It is important here to note the time of first conception after the marriage because the analysis of waiting time of first conception signifies couple's fertility at early stages of married life. This variable is widely used to study fertility characteristic of a woman, since it is independent of effect of amenorrhea period and generally, woman does not like to use contraceptives to postpone first birth. There is little chance of recall lapse in reporting the time of first birth from the date of first marriage of first birth. Treating the first conception as the random phenomenon the probabilistic models can be developed and this variable can be treated as discrete as well as continuous variable depending upon the situation and assumptions made for the study.

For the first time this variable was considered as a discrete variable and Gini [6] derived the geometric distribution for the waiting time of first conception. He defined the term 'Fecund ability' and the monthly chance of conception for women living in the married, fecund and exposed state. Later the same variable was considered as a continuous variable and hence treating the time elapsed from the marriage or from the beginning of the reproductive process to first conception as continuous makes mathematical treatments more convenient and easy. Singh [10], Henry [7] and Vincent [12] developed some models treating the waiting time of first conception as continuous. The negative exponential distribution plays the role of geometric distribution for studying The waiting time of conceive after marriage. Thus if X denotes the time of first conception then the density function , say , F(x) is given by

$$F(x) = \delta e^{-\delta x}; \qquad (x > 0, \delta > 0)$$

where ` δ is instantaneous fecundability

A number of authors made modifications on the above simple distribution to study realistic situation. In the present model of waiting time of first conception the time elapsed is defined over the range $(0, \infty)$. But in practical problems the upper limit may be considered as finite, as women can conceive up to an age limit. So, there is a need of introducing a new continuous model with finite range. Keeping this in view, an attempt has been made to characterize an existing model derive by Mukeherjee-Islam [8], defined over a finite range for the purpose of life testing analysis but it suits in realistic or real life situations.

2. A FINITE RANGE CONTINUOUS MODEL

A new probability distribution has been considered in the section as a continuous model, introduced by Mukherjee-Islam [8] for the purpose of studying waiting time

$$f(x,\theta,p) = (p/\theta^{p})x^{p-1} \qquad (p,\theta > 0, x \geq 0)$$

The above model is monotonic decreasing and highly skewed to the right. The graph is J-shaped thereby the unimodel nature.

Let us consider the reparametrized finite range distribution whose pdf is given by

$$f(x,\sigma,\theta) = \frac{1}{\theta x}\left(\frac{x}{\sigma}\right)^{1/\theta} \ ; (\theta > 0, \sigma > 0, \text{ and } 0 < x \leq \sigma) \qquad \text{.... (1)}$$

where 'θ' is instantaneous fecundibility and 6 is considered as age limit beyond which a married woman cannot conceive.

$$F(x) = P(x \leq x) = \int_0^x f(x,\sigma,\theta)dx = \left(\frac{x}{\sigma}\right)^{1/\theta} \qquad \text{....(2)}$$

The survival function at time x, say S(x) = P (X>x) is given by

$$S(x) = P(X > x) = 1\text{-}P(X \leq x) = 1 - \left(\frac{x}{\sigma}\right)^{1/\theta} \qquad \text{..... (3)}$$

Also, the conception rate function, say, W(x) at time x is then given by

$$W(x)=\frac{P(X=x)}{P(X\geq x)}$$

$$=\frac{(x/\sigma)^{1/\theta}}{\theta_x\left\{1-(x/\sigma)^{1/\theta}\right\}}$$

$$=\frac{x^{(1/\theta)-1}}{\theta(\sigma^{1/\theta}-x^{1/\theta})} \qquad \text{.... (4)}$$

$$\mu_2^{i}=\frac{\sigma^n}{(1+n\theta)}$$

$$\mu_1^{i}=E(x)=\frac{\sigma^n}{1+\theta} \qquad \text{.... (5)}$$

and $$\mu_2^{i}=\frac{\sigma^2}{1+2\theta}$$

However $$\mu_2^{i}=V(x)=\frac{(\sigma\theta)^z}{(1+2\theta)(1+\theta)^z} \qquad \text{..... (6)}$$

3. Maximum Likelihood Estimator

The joint probability density function is given by

$$f(\underline{x}\,|\,\theta)=\theta^{-n}\left(\prod_{i=1}^{n}x_i\right)^{-1}e^{-z/\theta} \qquad \text{..... (7)}$$

where $$z=\left[\sum_{i=1}^{n}\log\left(\frac{\sigma}{x_i}\right)\right]$$

Thus the maximum likelihood estimater (MLE) $\hat{\theta}$ of θ is given by

$$\hat{\theta}=z/n \qquad \text{..... (8)}$$

4. Bayesian Analysis of The Model

The fundamental problems in Bayesian analysis are that of the choice of prior distribution of g (θ) and a loss function L (...). Let us consider three prior distribution of θ to obtain the Bayes estimators which are as follows:

(a) Quasi-prior

For the situation where experimenter has no prior information about the parameter θ, one may use the quasi density as given by

$$g_1(\theta) = 1/\theta^d; \quad (\theta > 0, d > 0) \qquad \text{... (9)}$$

Here d= 0 leads to diffuse prior and d=1, a non informative prior.

(b) Natural conjugate prior of θ

The most widely used prior distribution of θ is the inverted gamma distribution with parameters α and β (>0) with p.d.f. given by

$$g_2(\theta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta} & ;\theta > 0\,(\alpha,\beta) > 0 \\ 0 & ;otherwise \end{cases} \qquad \text{...(10)}$$

The main reason for general acceptability is the mathematical tractability resulting from the fact that inverted gamma distribution is conjugate for prior for θ.

(c) Uniform prior

It frequently happens that the life tester knows in advance that the probable values of θ lies over a finite range [α, β] but he does not have any strong opinion about any subset of values over this range.in such a case uniform distribution over [α, β] may be a good approximation.

$$g_3(\theta) = \begin{cases} \frac{1}{\beta-\alpha}; & 0<\alpha<\theta\leq\beta \\ 0 & ;\text{otherwise} \end{cases} \qquad \text{.... (11)}$$

(d) Loss function

The Bayes estimator $\hat{\theta}$ of θ is of course, optimal relative to the loss function chosen. A commonly used loss function is the squared error loss function (SELF)

$$L(\hat{\theta},\theta)=(\hat{\theta}-\theta)^2, \qquad \text{.... (12)}$$

which is a symmetrical loss function and assigns equal losses to over estimation and under estimation. Canfield [3] points out that the use of symmetric loss function may be inappropriate in the estimation of reliability function. Over estimation of reliability function or average lifetime is usually much more serious than under estimation of reliability function or main failure time. Also, an under estimation of the failure rate results in more serious consequence than an overestimation of the failure rate. This leads to statistician to think about asymmetrical loss function which has been proposed in statistical literature. It is well known that the Bayes estimator under the above loss function, say $\hat{\theta}$ s, is the posterior mean. The squared error loss function (SELF) is often use also because it does not lead to extensive numerical computation but several authors (Ferguson [5], Varian [11], Berger [12], Zellner [13] and Basu and Ebrahimi [14] have recognized the inappropriateness of using symmetric loss function is several estimation problems.

(e) Precautionary loss function

Norstrom [9] introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss function with quadratic loss function as a special case. These loss function approach infinitely near the origin to prevent underestimation and thus giving a conservative estimators, especially when low failure rates are being estimated. These estimators are very useful when under estimation may lead to serious consequences. A very useful and simple asymmetric precautionary loss function is

$$L(\overset{\wedge}{\theta},\theta)=\frac{(\overset{\wedge}{\theta}-\theta)^2}{\overset{\wedge}{\theta}} \qquad \text{.... (13)}$$

The posterior expectation of loss function in (13) is

$$E_\pi\left[L(\overset{\wedge}{\theta},\theta)\right]=E_\pi\left(\frac{\theta^2}{\overset{\wedge}{\theta}}\right)+E_\pi(\overset{\wedge}{\theta})-2E_\pi(\theta) \qquad \text{... (14)}$$

The value of $\overset{\wedge}{\theta}$ that minimises (14), is denoted by $\hat{\theta}_P$, Bayes estimater of θ under precautionary loss function is obtained by solving equation

$$\frac{d}{d\theta}E_\pi[L(\overset{\wedge}{\theta},\theta)=0 \Rightarrow \left[E_\pi\left\{\theta^2\left(-\frac{1}{\overset{\wedge}{\theta}{}_P^2}\right)\right\}+E_\pi(1)\right]=0$$

$$\Rightarrow\left(-\frac{1}{\overset{\wedge}{\theta}{}_P^2}\right)E_\pi(\theta^2)=-1$$

$$\Rightarrow \overset{\wedge}{\theta}_P=[E_\pi(\theta^2)]^{1/2} \qquad \text{.... (15)}$$

5. Bayes Estimator Under g_1 (θ)

Under g_1 (θ), the posterior distribution is defined by

$$f(\theta\,|\,\underline{x})=\frac{f((\underline{x}\,|\,\theta))g_1(\theta)}{\int_0^\infty f((\underline{x}\,|\,\theta))g_1(\theta)d\theta} \qquad \text{.... (16)}$$

Substituting the values of $g_1(\theta)$ and $f(\underline{x}\,|\,\theta)$ from (9) and (7) in (16) we get, after simplification, as

$$f(\theta\,|\,\underline{x})\frac{\theta^{-(n+d)}e^{-z/\theta}}{\int_0^\infty \theta^{-(n+d)}e^{-z/\theta}d\theta} \qquad \text{... (17)}$$

$$= \frac{z^{n+d-1}}{\Gamma(n+d-1)} \theta^{-(n+d)} e^{-z/\theta} \quad (\theta > 0, n+d > 1)$$

The Bayes estimator under squared error loss function is the posterior mean given by

$$\hat{\theta}_s = \int_0^\infty \theta f(\theta) \mid \underline{x}) d\theta. \qquad .... (18)$$

Substituting the values of $f(\theta \mid \underline{x})$ from (17) in (18) and on solving we get

$$\hat{\theta}_s = \frac{z}{n+d-2} \ ; \ (n+d > 2). \qquad .... (19)$$

The Bayes estimator under precautionary loss function, say $\hat{\theta}_p$ using the value of $f(\theta|\underline{x})$ from equation (17) is the solution of equation (15) given by

$$\hat{\theta}_p = [E-(\theta^z)]^{1/2} \quad = \left[\int_0^\infty \theta^2 f(\theta \mid \underline{x}) d\theta \right]^{1/2}$$

$$= \left[\frac{z^{n+d-1}}{\Gamma(n+d-1)} \int_0^\infty \theta^{-(n+d-2)} e^{-z/\theta} d\theta \right]^{1/2}$$

On simplification leads to

$$\hat{\theta}_p = \frac{z}{[n+d-2)(n+d-3)]^{1/2}} \qquad .... (20)$$

6. Bayes Estimator Under $g_2(\theta)$

Under $g_2(\theta)$, the posterior distribution is defined by

$$f(\theta \mid \underline{x}) = \frac{f(\underline{x} \mid \theta) g_2(\theta)}{f_o^\infty f(\underline{x} \mid \theta) g_2(\theta) d\theta} \qquad .... (21)$$

Substituting the values of $g_2(\theta)$ and $f(\theta \mid x)$ from equations (10) and (7) in (21) we get, after simplication,

$$f(\theta \mid \underline{x}) = \frac{\theta^{-n=\alpha-1} e^{-(\beta+z)/\theta}}{\int_0^\infty \theta^{-n-\alpha-1} e^{-(\beta+z)|\theta} d\theta} \qquad \text{..... (22)}$$

which on simplification yields

$$f(\theta \mid \underline{x}) = \frac{(\beta+z)^{n+\alpha}}{\Gamma(n+\alpha)} \theta^{-(n+\alpha+1) e^{-(\beta+z)/\theta}} \qquad \text{.... (23)}$$

The Bayesian estimator under squared error loss function is the posterior mean given by

$$\hat{\theta}_S = \int_0^\infty \theta f(\theta \mid \underline{x}) d\theta \qquad \text{.... (24)}$$

Substituting the values of $f(\theta \mid \underline{x})$ from equation (23) in equation (24) and on solving we get

$$\hat{\theta}_S = \frac{\beta + z}{(n + \alpha - 1)} \qquad \text{..... (25)}$$

the Bayes estimator under precautionary loss function using the value of $f(\theta \mid \underline{x})$ from equation (23) is given by

$$\hat{\theta}_P = \frac{\beta + z}{[n + a - 1)(n + a - 2)]^{1/2}} \qquad \text{..... (26)}$$

7. Bayes Estimator Under $g_3(\theta)$

We have $f(\theta \mid \underline{x}) = \frac{\theta^{-n} e^{-z/\theta}}{\int_\alpha^\beta \theta^{-n} e^{-z/\theta} d\theta} = \frac{z^{n-1} \theta^{-n} e^{-z/\theta}}{I_g(z/\alpha, n-1) - I_g(z/\beta, n-1)}$ .... (27)

where $I_g(x,n)=\int_0^x e^{-x}x^{n-1}\,dt$ is the incomplete gama function.

The Bayes estimator under squared error loss function is the posterior mean given by

$$\hat{\theta}_P=\int_\alpha^\beta \theta f(\theta\,|\,\underline{x})d\theta \qquad \text{.... (28)}$$

Substituting the values of $f(\theta\,|\,\underline{x})$ from (26) in (28), we get

$$\hat{\theta}_S=\left(\frac{I_g\left(\frac{z}{\alpha},n-2\right)-I_g\left(\frac{z}{\beta},n-2\right)}{I_g\left(\frac{z}{\alpha},n-1\right)-I_g\left(\frac{z}{\beta},n-1\right)}\right)z \qquad \text{.... (29)}$$

The Bayes estimator under precautionary loss function

$$\hat{\theta}_P=\left[\frac{I_g\left(\frac{z}{\alpha},n-3\right)-I_g\left(\frac{z}{\beta},n-3\right)}{I_g\left(\frac{z}{\alpha},n-1\right)-I_g\left(\frac{z}{\beta},n-1\right)}\right]^{1/2} z \qquad \text{.... (30)}$$

The equation (29) and (30) can be solved numerically.

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