

A MULTI-SERVER QUEUEING MODEL WITH FEEDBACK AND BALKING

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Abstract : We consider an M/M/C queueing system with feedback and balking. The customers are assumed to arrive in a Poisson fashion, balks with constant probability and their service times follow exponential distribution. The purpose of this research paper is to discuss the M/M/C queueing system with feedback and balking. The generating function of time dependent probabilities is obtained in terms of their Laplace transforms and steady state solution is also obtained. In order to match our results with earlier published work, some special cases are discussed.

Keywords: Feedback, balking, multi-server, Laplace transformation, generating function, exponential distribution.

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1. Introduction

In the analysis of queuing problems, it is assumed that customers arrive and stay in the queue until they receive service. But in real life, this does not always happen. The customer may become impatient, if they tend to join the queue only when a short waiting time is expected and remain in queue if the wait has been sufficiently small. On arrival a customer may immediately decide not to join the queue because of the longer queue length or of other information about the length of the service which is known as balking. Balking can be of two types that is forced balking and unforced balking. For example, when a customer enters the queuing

system, he observes that there is a long queue and it will take a lot of time for service. Therefore, he decides not to join the queue and this is called unforced balking. On the other hand, if a customer goes to a mechanic and his waiting space is full, then the customer has to leave the system. This is called forced balking.

If a customer is dissatisfied due to incomplete or inappropriate quality of service, then he can rejoin the queue to repeat the service, that is how feedback is defined in queuing theory. These types of queues are known as ‘queues with feedback’. The concept of feedback in queues was introduced by Finch [5]. Takacs [11] also studied queues with feedback. He considered single server queuing problem with feedback.

Haight [7] first presented an infinite M/M/1 queue with balking. The combined effects of balking and reneging in the M/M/1/N queues have been investigated by Ancker and Gafarian [3]. Al-Seed et al. [2] studied an M/M/C queue with balking and reneging and derived its transient solution by using the probability generating function technique and properties of Bessel functions. Choedhary and Medhi [4] studied balking & reneging a position dependent system and steady state probabilities are derived. Abou et al. [1] considered a multi-server finite capacity markovian queue with balking and reneging. Jain and Singh [8] also worked on reneging and gave the stationary solution of a multi-server queueing problem. Santhakumaran and Thangaraj [10] studied an M/M/1 feedback queueing problem with impatience for queue length and obtained results for stationary distribution.

In the present paper, we considered a multi-server feedback queueing problem with balking having infinite waiting capacity. We have obtained steady-state queue length probabilities for the problem. Transient-state queue length probabilities have also been obtained.

Practical situation which corresponds to the above problem can be that of a bank. An arriving customer on seeing a long queue at the service counter may leave the system. The customer after getting the first service, if found unsatisfactory (e.g.

Draft is issued in the wrong name) may rejoin the queue for service. The Bank administration can know the various probabilities for the number of customers to be served at any time. The result of the model may help the bank administration for better management.

The queueing system investigated in this paper is governed by the following assumptions:

- (i) Arrivals are Poisson with parameter λ and the service time distribution of every customer is exponential with parameter μ .
- (ii) The probability of rejoining the customer to the system is p and that of leaving the system is q for the customers getting first service, so that $p + q = 1$. However, the customer will have to leave the system after getting second service.
- (iii) After current departure, the next customer will depart the service channel for the first time with probability c_1 and for the second time with probability, c_2 so that $c_1 + c_2 = 1$.
- (iv) The arriving customer joins the queue definitely, if the number of customers in the system is less than 'k'. It may balk with probability $(1-\beta)$ or joins the queue with probability β if the number of customers in the system is greater than or equal to k .
- (v) The waiting space is infinite.
- (vi) The stochastic processes involved, viz.
 - (a) arrivals of units, (b) departures of units,are statistically independent.

2. Definitions

$P_n^{(o)}(t)$ = Probability that there are n customers in the system at time t and the next customer is to depart for the first time.

$P_n^{(1)}(t)$ = Probability that there are n customers in the system at time t and the next customer is to depart for the second time.

$P_n(t)$ = Probability that there are n customers in the system at time t .

we have
$$P_n(t) = P_n^{(o)}(t) + P_n^{(1)}(t), \quad n \geq 0 \quad \dots (1)$$

Initially, $P_0^{(o)}(0) = 1$ and $P_0^{(1)}(t) = 0, \quad t \geq 0$

The difference differential equations describing the system are

$$\frac{d}{dt} P_0^{(o)}(t) = -\lambda P_0^{(o)}(t) + \mu q P_1^{(o)}(t) + \mu P_1^{(1)}(t), \quad (n = 0) \quad \dots (2)$$

$$\begin{aligned} \frac{d}{dt} P_n^{(o)}(t) = & -(\lambda + n\mu)P_n^{(o)}(t) + \lambda P_{n-1}^{(o)}(t) + (n+1)\mu c_1 \{qP_{n+1}^{(2)}(t) + P_{n+1}^{(1)}(t)\} \\ & + n\mu c_1 p(1 - \delta_{n,1})P_n^{(o)}(t), \quad (0 < n < C) \end{aligned} \quad \dots (3)$$

$$\begin{aligned} \frac{d}{dt} P_n^{(1)}(t) = & -(\lambda + n\mu)P_n^{(1)}(t) + \lambda P_{n-1}^{(1)}(t) + (n+1)\mu c_2 \{qP_{n+1}^{(o)}(t) + P_{n+1}^{(1)}(t)\} \\ & + n\mu(c_1\delta_{n,1} + c_2)pP_n^{(o)}(t), \quad (0 < n < C) \end{aligned} \quad \dots(4)$$

$$\begin{aligned} \frac{d}{dt} P_n^{(o)}(t) = & -(\lambda + C\mu)P_n^{(o)}(t) + \lambda P_{n-1}^{(o)}(t) + C\mu c_1 \{qP_{n+1}^{(o)}(t) + P_{n+1}^{(1)}(t)\} + C\mu c_1 pP_n^{(o)}(t), \\ & (C \leq n < k) \end{aligned} \quad \dots(5)$$

$$\begin{aligned} \frac{d}{dt} P_n^{(1)}(t) = & -(\lambda + C\mu)P_n^{(1)}(t) + \lambda P_{n-1}^{(1)}(t) + C\mu c_2 \{qP_{n+1}^{(o)}(t) + P_{n+1}^{(1)}(t)\} + C\mu c_2 pP_n^{(o)}(t), \\ & (C \leq n < k) \end{aligned} \quad \dots (6)$$

$$\begin{aligned} \frac{d}{dt} P_n^{(o)}(t) = & -(\lambda\beta + C\mu)P_n^{(o)}(t) + [\lambda\beta + \lambda\delta_{n-k+1,1}(1-\beta)]P_{n-1}^{(o)}(t) + C\mu c_1 \{qP_{n+1}^{(o)}(t) + P_{n+1}^{(1)}(t)\} \\ & + C\mu c_1 pP_n^{(o)}(t), \quad (n \geq k) \end{aligned} \quad \dots (7)$$

$$\begin{aligned} \frac{d}{dt} P_n^{(1)}(t) = & -(\lambda\beta + C\mu)P_n^{(1)}(t) + [\lambda\beta + \lambda\delta_{n-k+1,1}(1-\beta)]P_{n-1}^{(1)}(t) + C\mu c_2 \{qP_{n+1}^{(o)}(t) + P_{n+1}^{(1)}(t)\} \\ & + C\mu c_2 pP_n^{(o)}(t), \quad (n \geq k) \end{aligned} \quad \dots (8)$$

where, $\delta_{n,1} = \begin{cases} 1, & \text{for } n=1 \\ 0, & \text{otherwise.} \end{cases}$

The steady-state difference equations describing the system are

$$\rho P_0^{(0)} = qP_1^{(0)} + P_1^{(1)} \quad \dots (9)$$

$$(\rho + n)P_n^{(0)} = \rho P_{n-1}^{(0)} + (n+1)c_1 \{qP_{n+1}^{(0)} + P_{n+1}^{(1)}\} + nc_1 p(1 - \delta_{n,1})P_n^{(0)}, \quad (0 < n < C) \quad \dots (10)$$

$$(\rho + n)P_n^{(1)} = \rho P_{n-1}^{(1)} + (n+1)c_2 \{qP_{n+1}^{(0)} + P_{n+1}^{(1)}\} + (c_1 \delta_{n,1} + c_2)npP_n^{(0)}, \quad (0 < n < C) \quad \dots (11)$$

$$(\rho + C)P_n^{(0)} = \rho P_{n-1}^{(0)} + Cc_1 \{qP_{n+1}^{(0)} + P_{n+1}^{(1)}\} + Cc_1 pP_n^{(0)}, \quad (C \leq n < k) \quad \dots (12)$$

$$(\rho + C)P_n^{(1)} = \rho P_{n-1}^{(1)} + Cc_2 \{qP_{n+1}^{(0)} + P_{n+1}^{(1)}\} + Cc_2 pP_n^{(0)}, \quad (C \leq n < k) \quad \dots (13)$$

$$(\rho\beta + C)P_n^{(0)} = [\rho\beta + \rho\delta_{n-k+1,1}(1-\beta)]P_{n-1}^{(0)} + Cc_1 \{qP_{n+1}^{(0)} + P_{n+1}^{(1)}\} + Cc_1 pP_n^{(0)}, \quad (n \geq k) \quad \dots (14)$$

$$(\rho\beta + C)P_n^{(1)} = [\rho\beta + \rho\delta_{n-k+1,1}(1-\beta)]P_{n-1}^{(1)} + Cc_2 \{qP_{n+1}^{(0)} + P_{n+1}^{(1)}\} + Cc_2 pP_n^{(0)}, \quad (n \geq k) \quad \dots (15)$$

Taking the Laplace transformation $\bar{P}_n(s) = \int_0^\infty e^{-st} P_n(t) dt$; ($\text{Re } s > 0$)

of (2) - (6) and dividing by μ , we get

$$(\rho + s/\mu)\bar{P}_0^{(0)}(s) = \frac{1}{\mu} + q\bar{P}_1^{(1)}(s) + \bar{P}_1^{(1)}(s) \quad \dots (16)$$

$$(\rho + s/\mu + n)\bar{P}_n^{(0)}(s) = \rho\bar{P}_{n-1}^{(0)}(s) + (n+1)c_1 \{q\bar{P}_{n+1}^{(0)}(s) + \bar{P}_{n+1}^{(1)}(s)\} + nc_1 p(1 - \delta_{n,1})p\bar{P}_n^{(0)}(s), \quad (0 < n < c) \quad \dots (17)$$

$$(\rho + s/\mu + n)\bar{P}_n^{(1)}(s) = \rho\bar{P}_{n-1}^{(1)}(s) + (n+1)c_2 \{q\bar{P}_{n+1}^{(0)}(s) + \bar{P}_{n+1}^{(1)}(s)\} + (c_1 \delta_{n,1} + c_2)np\bar{P}_n^{(0)}(s), \quad (0 < n < C) \quad \dots (18)$$

$$(\rho + s/\mu + C)\bar{P}_n^{(0)}(s) = \rho\bar{P}_{n-1}^{(0)}(s) + Cc_1 \{q\bar{P}_{n+1}^{(0)}(s) + \bar{P}_{n+1}^{(1)}(s)\} + Cc_1 p\bar{P}_n^{(0)}(s), \quad (C \leq n < k) \quad \dots (19)$$

$$(\rho + s/\mu + c)\bar{P}_n^{(1)}(s) = \rho\bar{P}_{n-1}^{(1)}(s) + Cc_2 \{q\bar{P}_{n+1}^{(0)}(s) + \bar{P}_{n+1}^{(1)}(s)\} + Cc_2 p\bar{P}_n^{(0)}(s), \quad (C \leq n < k) \quad \dots (20)$$

$$(\rho\beta + s/\mu + C)\bar{P}_n^{(0)}(s) = [\rho\beta + \rho\delta_{n-k+1,1}(1-\beta)]\bar{P}_{n-1}^{(0)}(s) + Cc_1\{q\bar{P}_{n+1}^{(0)}(s) + \bar{P}_{n+1}^{(1)}(s)\} + Cc_1p\bar{P}_n^{(0)}(s),$$

$$(n \geq k) \dots (21)$$

$$(\rho\beta + s/\mu + C)\bar{P}_n^{(1)}(s) = [\rho\beta + \rho\delta_{n-k+1,1}(1-\beta)]\bar{P}_{n-1}^{(1)}(s) + Cc_2\{q\bar{P}_{n+1}^{(0)}(s) + \bar{P}_{n+1}^{(1)}(s)\} + Cc_2p\bar{P}_n^{(0)}(s),$$

$$(n \geq k) \dots (22)$$

Throughout the paper we assume that

$$P^{(0)}(z, t) = \sum_{n=0}^{\infty} P_n^{(0)}(t)z^n, \quad \bar{P}^{(0)}(z, s) = \int_0^{\infty} e^{-st}P^{(0)}(z, t)dt$$

$$P^{(1)}(z, t) = \sum_{n=0}^{\infty} P_n^{(1)}(t)z^n, \quad \bar{P}^{(1)}(z, s) = \int_0^{\infty} e^{-st}P^{(1)}(z, t)dt$$

$$P(z, t) = P^{(0)}(z, t) + P^{(1)}(z, t), \text{ and } \bar{P}(z, s) = \int_0^{\infty} e^{-st}P(z, t)dt \quad \text{with } |z| \leq 1$$

3. Steady-State Solution of the Problem

Using $Ef(x) = f(x+1)$, equation (14) and (15) becomes,

$$[Cc_1qE^2 + \{Cc_1p - (\rho\beta + C)\}E + \rho\beta]P_n^{(0)} + Cc_1E^2P_n^{(1)} = 0, (n \geq k+1) \dots (23)$$

$$[Cc_2qE^2 + Cc_2pE]P_n^{(0)} + [Cc_2E^2 - (\rho\beta + C)E + \rho\beta]P_n^{(1)} = 0, (n \geq k+1) \dots (24)$$

To have solution of above system of equations, we must have:

$$(E-1)\{(\rho\beta + C)(Cc_1q + Cc_2)E^2 - \rho\beta(\rho\beta + 2C - Cc_1p)E + (\rho\beta)^2\} = 0, (n \geq k+1) \dots (25)$$

The values of $P_n^{(0)}$ and $P_n^{(1)}$ are given by

$$P_n^{(0)} = \sum_{i=0}^2 a_i z_1^i \quad \text{and} \quad P_n^{(1)} = \sum_{i=0}^2 b_i z_2^i, (n \geq k+1)$$

where z_0, z_1, z_2 are the roots of equation (25) with $z_0 = 1$ and $a_i, b_i; i = 0, 1, 2$ are arbitrary constants to be evaluated. The other two roots of equation (25) are

$$z_1 = \frac{\rho\beta}{\rho\beta + C} \text{ and } z_2 = \frac{\rho\beta}{C[c_2 + c_1q]}, \quad \left(\rho = \frac{\lambda}{\mu} < c \right) \quad \dots (26)$$

z_1 is always less than 1 but z_2 is less than 1 only when $\rho\beta < C(c_2 + c_1q)$. So we have two or one root < 1 only when $\rho\beta < C(c_2 + c_1q)$ or not. In case $z_2 \geq 1$ take $a_2 = b_2 = 0$. To have convergence of the solution, root ≥ 1 must be rejected. Thus rejecting z_0 , we have

$$P_n^{(0)} = \sum_{i=1}^2 a_i z_1^n \text{ and } P_n^{(1)} = \sum_{i=1}^2 b_i z_2^n, \quad (n \geq k + 1)$$

From equation (14) and (15) for $n = k + 1$, we can get probabilities $P_k^{(0)}$ and $P_k^{(1)}$. Subsequently putting the values in equations (14) and (15) for $n = k$ we can get $P_{k-1}^{(0)}$ and $P_{k-1}^{(1)}$. Further substituting these values in equations (12) and (13), we get the probabilities $P_{k-2}^{(0)}, P_{k-2}^{(1)}, P_{k-3}^{(0)}, P_{k-3}^{(1)}, \dots, P_C^{(0)}, P_C^{(1)}, P_{C-1}^{(0)}, P_{C-1}^{(1)}$ and using these in equations (10) and (11) we get, $P_{C-2}^{(0)}, P_{C-2}^{(1)}, \dots, P_1^{(0)}, P_1^{(1)}$ in terms of $P_0^{(0)}$. Four unknowns a_1, a_2, b_1 and b_2 (two unknown a_1 and b_1 in case of $z_2 \geq 1$ can be evaluated from equations (9) and (10) & (11) for $n=1$ and (14) for $n = k + 2$ in terms of $P_0^{(0)}$ and the value of $P_0^{(0)}$ can be found using the relation

$$P_0^{(0)} = 1 - \sum_{n=1}^{\infty} (P_n^{(0)} + P_0^{(1)})$$

Hence by using the value of a_1, a_2, b_1 and b_2 and $P_0^{(0)}$, the probabilities $P_n^{(0)}$ and $P_n^{(1)}$ are completely known for various values of n .

Special Cases

(i) When number of server is one, i.e. $M/M/1/\infty$ with feedback and balking :

Put $C=1$ in equation (25), we get

$$(E-1)\{(\rho\beta+1)(c_1q+c_2)E^2 - \rho\beta(\rho\beta+2-c_1p)E + (\rho\beta)^2\} = 0, (n \geq k+1) \dots (27)$$

this coincides with the equation (19) of [7].

(ii) When there is no feedback and balking :

Putting $q=1, p=0, c_1 = 1, c_2 = 0; P_n^{(1)} = 0, P_n^{(0)} = P_n, \beta = 1$ then equations (9)-(15) becomes

$$\rho P_0 = P_1 \dots (28)$$

$$(\rho + n)P_n = \rho P_{n-1} + (n+1)P_{n+1}, (0 < n < C) \dots (29)$$

$$(\rho + C)P_n = \rho P_{n-1} + CP_{n+1}, (n \geq C) \dots (30)$$

Solving equations (28)-(30) using recursive method, we get

$$P_n = \begin{cases} \frac{\rho^n}{n!} P_0, & \text{if } 0 < n < C \\ \frac{\rho^n}{C^{n-C} C!} P_0, & \text{if } n \geq C \end{cases}$$

And the value of P_0 can be found by using the relation $\sum_{n=0}^{\infty} P_n = 1$

$$\text{Therefore } P_0 = \left[\sum_{n=0}^{C-1} \frac{(C\rho)^n}{C!} - \frac{C^C}{C!} \left(\frac{\rho^C}{1-\rho} \right) \right]^{-1}$$

which is same as $M/M/C/\infty$ in classical Model.

4. Laplace Transformation of Probability Generating Function of Transient-State Queue Length Probabilities

We have

$$\begin{aligned} \bar{P}^{(0)}(z, s) &= \frac{1}{Q(z)} [(A(z) - C) \{z(c_2q - c_1pz)(A(z) - C) \bar{P}_1^{(0)}(s) \\ &+ c_2z \bar{P}_1^{(1)}(s)\} + (z/\mu)(A(z) - Cc_2) - \{\rho z(1-z)(A(z) - Cc_2)(1-\beta) \sum_{n=0}^{k-1} \bar{P}_n^{(0)}(s)z^n \end{aligned}$$

$$\begin{aligned}
 &+(c_1qA(z)(1-z) - zc_2(1-C)) \sum_{n=0}^{c-1} (C-n) \bar{P}_n^{(0)}(s)z^n - \{Cc_1\rho z(1-z)(1-\beta) \sum_{n=0}^{k-1} \bar{P}_n^{(1)}(s)z^n \\
 &+C(A(z) - c_1z) \sum_{n=0}^{C-1} (C-n) \bar{P}_n^{(1)}(s)z^n \}, \quad (\rho = \frac{\lambda}{\mu} < C, |z| \leq 1) \quad \dots (31)
 \end{aligned}$$

$$\begin{aligned}
 \bar{P}^{(1)}(z,s) = &\frac{1}{(Q(z))} [\{c_1pz^2((A(z) - Cc_2q) - Cpz) - C_2qz((A(z) - Cc_2pz) + Cq)\} \bar{P}_1^{(0)}(s) \\
 &-c_2(q+pz)C\rho z(1-z)(1-\beta) \sum_{n=0}^{k-1} \bar{P}_n^{(0)}(s)z^n -c_2z(A(z) - C(q+pz)) \bar{P}_1^{(1)}(s) + (Cc_2z/\mu)(q+pz) \\
 &-\rho z(1-z)(1-\beta)(A(z) - Cc_1(q+pz)) \sum_{n=0}^{k-1} \bar{P}_n^{(1)}(s)z^n - (A(z) - Cz) \sum_{n=0}^{C-1} (C-n) \bar{P}_n^{(0)}(s)z^n \\
 &-\{A(z)(z - c_2) - Cc_1(q+pz)\} \sum_{n=0}^{C-1} (C-n) \bar{P}_n^{(1)}(s)z^n], (\rho = \frac{\lambda}{\mu} < C, |z| \leq 1) \quad \dots (32)
 \end{aligned}$$

$$\begin{aligned}
 \bar{P}(z,s) = &\frac{1}{Q(z)} [(1-z)Cpz\{(c_2q - c_1pz) \bar{P}_1^{(0)}(s) + c_2 \bar{P}_1^{(1)}(s)\} + (z/\mu)\{A(z) - c_2p(1-z)\}] \\
 &-(1-z) \left\{ pz(A(z) - (1-z)Cc_2p)(1-\beta) \sum_{n=0}^{k-1} \bar{P}_n^{(0)}(s)z^n + (Cc_2pz + qA(z)) \sum_{n=0}^{C-1} (C-n) \bar{P}_n^{(0)}(s)z^n \right\} \\
 &-(1-z) \{pz(A(z) + (1-z)Cc_1p)(1-\beta) \sum_{n=0}^{k-1} \bar{P}_n^{(1)}(s)z^n + (A(z) - Cc_1pz) \\
 &\sum_{n=0}^{C-1} [(C-n) \bar{P}_n^{(1)}(s)z^n], (\rho = \lambda/\mu < C, |z| \leq 1) \quad \dots (33)
 \end{aligned}$$

where, $Q(z) = \{(A(z) - Cc_1(q+pz))(A(z) - Cc_2)\} - (q+pz)cC^2c_1c_2$

$A(z) = \{-\rho\beta z^2 + (s/\mu + \rho\beta + C)z\}$, $D(z) = K_1(z)K_2(z) - C^2c_1c_2(q+pz)$,

$K_1(z) = (-\rho\beta z^2 + (s/\mu + \rho\beta + C - Cc_1p)z - Cc_1q)$, and

$K_2(z) = (-\rho\beta z^2 + (s/\mu + \rho\beta + C)z - Cc_2)$

Obviously $K_1(z)$ and $K_2(z)$ have two zeroes inside the unit circle.

Let $f(z) = K_1(z)K_2(z)$ and $g(z) = (q + pz)C^2c_1c_2$

Then

$$|f(z)| = \left| -\rho\beta z^2 + (s/\mu + \rho\beta + C - Cc_1p)z - Cc_1q \right| \left| (-\rho\beta z^2 + (s/\mu + \rho\beta + C)z - Cc_2) \right| \\ = (\zeta + Cc_2)(\zeta + Cc_1), \text{ for } s/\mu = \zeta + i\eta, |z|=1$$

$$> C^2c_1c_2 \geq |g(z)|$$

Hence $|f(z)| > |g(z)|$ on $|z|=1$

Since all the conditions of Rouché's Theorem are satisfied, so D has two zeroes inside the unit circle. Let these zeroes be z_m ($m = 0, 1$). Numerator must vanish for these two zeroes since $\bar{P}(z, s)$ is an analytical function of z . For $k=3$ these two equations along with equation (16) and (17) & (18) for $n=1$ will determine the five unknowns $\bar{P}_0^{(0)}(s), \bar{P}_1^{(0)}(s), \bar{P}_1^{(1)}(s), \bar{P}_2^{(0)}(s)$ and $\bar{P}_2^{(1)}(s)$ (in case $C=1, C=2$ or $C=3$).

For $k=4$, along with these two equations and equation (16) and (17)&(18) for $n=1, 2$ will determine the seven unknowns $\bar{P}_0^{(0)}(s), \bar{P}_1^{(0)}(s), \bar{P}_1^{(1)}(s), \bar{P}_2^{(0)}(s), \bar{P}_2^{(1)}(s), \bar{P}_3^{(0)}(s)$ and

$\bar{P}_3^{(1)}(s)$ (in case $C=1, C=2, C=3$ or $C=4$). In general along with equation (16) and (17) and (18) for $n = 1, 2, 3, \dots, C-2$ will determine the $(2C-1)$ unknowns $\bar{P}_0^{(0)}(s), \bar{P}_1^{(0)}(s), \bar{P}_1^{(1)}(s), \dots, \bar{P}_{C-1}^{(0)}(s), \bar{P}_{C-1}^{(1)}(s)$, (when number of servers = C).

Following the above procedure, we find all the probabilities of the above generating function, whatever the value of k and C as according to the defined model. Hence the generating function $\bar{P}(z, s)$ is completely known.

$\bar{P}_n(s)$ can be obtained by using the following formula

$$\bar{P}_n(s) = \frac{1}{n!} \frac{d^{(n)}\bar{P}(z, s)}{dz^n} \quad \text{at } z=0$$

In either case $P_n(t)$ can be found by inverting the Laplace transform $\bar{P}_n(s)$.

Further $\bar{P}(1, s) = \frac{1}{s}$, as desired and $\bar{P}(0, s) = \lim_{z \rightarrow 0} \bar{P}(z, s) = \frac{\text{zero}}{\text{zero}}$

On using L'Hospital's rule, it can be shown that $\bar{P}(0, s) = \bar{P}_0^{(0)}(s)$.

SPECIAL CASES

(i) When there is no feedback i.e. M/M/C/∞ model with balking:

Putting $q = 1, p = 0; \bar{P}^{(0)}(z, s) = \bar{P}(z, s), \bar{P}^{(1)}(z, s) = 0$ and $\bar{P}_o^{(0)} = \bar{P}_o(s)$

in equation (33), we find that

$$\bar{P}(z, s) = \frac{1}{A(z) - C} \left[\frac{z}{\mu} - (1 - z) \left\{ \rho z (1 - \beta) \sum_{n=0}^{k-1} \bar{P}_n(s) z^n + \sum_{n=0}^{C-1} (C - n) \bar{P}_n(s) z^n \right\} \right] \dots (34)$$

where $A(z) = \{-\rho\beta z^2 + (s/\mu + \rho\beta + C)z\}$, and $\rho = \frac{\lambda}{\mu} > C; |z| \leq 1$

(ii) M/M/C/∞ model without feedback and balking :

Putting $q = 1, p = 0, \beta = 1, \bar{P}^{(0)}(z, s) = \bar{P}(z, s), \bar{P}^{(1)}(z, s) = 0$ and $\bar{P}_o^{(0)} = \bar{P}_o(s)$

in equation (33), we find that.

$$\bar{P}(z, s) = \frac{1}{A(z) - C} \left[\frac{z}{\mu} - (1 - z) \sum_{n=0}^{C-1} (C - n) \bar{P}_n(s) z^n \right], (\rho = \frac{\lambda}{\mu} < C; |z| \leq 1) \dots (35)$$

This coincides with equation (27) of [5].

(iii) When there is no feedback i.e. M/M/1/∞ model with balking:

Putting $C=1, q = 1, p = 0; \bar{P}^{(0)}(z, s) = \bar{P}(z, s), \bar{P}^{(1)}(z, s) = 0$ and $\bar{P}_o^{(0)} = \bar{P}_o(s)$

in equation (33), we get

$$\bar{P}(z, s) = \frac{(z/\mu) - (1 - z)\bar{P}_o(s) - (1 - z)(1 - \beta)\rho z \sum_{n=0}^{k-1} \bar{P}_n(s) z^n}{(A(z) - 1)}, (\rho = \frac{\lambda}{\mu} < 1; |z| \leq 1)$$

..... (36)

This coincides with equation (30) of [7].

(iv) **When there is no feedback and no balking with one server i.e. $M/M/1/\infty$ model without balking:**

Putting $\beta = 1$ in equation (36), we find that.

$$\bar{P}(z, s) = \frac{(z/\mu) - (1-z)\bar{P}_0(s)}{A(z) - 1} = \frac{(z/\mu) - (1-z)\bar{P}_0(s)}{[-\rho z^2 + \{\rho + (s/\mu)z + (z-1)\}]},$$

$$(\rho = (\lambda/\mu < 1; |z| \leq 1) \quad \dots (37)$$

This coincides with equation (28) of [5].

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