Journal of Rajasthan Academy of Physical Sciences
ISSN : 0972-6306; URL : http://raops.org.in
Vol. 13, No. 1, March. 2014, pp 51-66

ON CERTAIN TOPOLOGICAL STRUCTURES OF BANACH SPACE VALUED PARANORMED SEQUENCE $\ell_\infty((S, ||.||), \Phi, \bar{u})$ DEFINED BY ORLICZ FUNCTION

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Received : January 21, 2014

Abstract : The aim of this paper is to introduce and study a new class $\ell_\infty((S, ||.||), \Phi, \bar{u})$ of Banach space valued sequences using Orlicz function as a generalization of basic space of bounded complex sequences ℓ_∞. We investigate the condition of linearity of the class $\ell_\infty((S, ||.||), \Phi, \bar{u})$ and then explore its topological structures when topologized it with suitable natural paranorm.

Keywords and Phrases : Paranormed space, sequence space, solid space, GK-, GC-,space, Orlicz function, Orlicz space.

2010 Mathematics Subject Classification : Primary 46A45,Secondary-46B15

1. Preliminaries

So far, a bulk number of research works have been done on various types of paranormed spaces. The notion of paranormed space is closely related to linear metric space , see Wilansky [25]. The studies of paranorm on sequence spaces were initiated by Maddox [15] and many others. Bhardwaj and Bala [3], Parashar and Choudhary [16], Pahari [17,18], Srivastava et al [22,23, 24] and many others further studied various types of paranormed spaces of sequences and functions.

Before proceeding with the main results, we recall some of the basic notations and definitions that are used in this paper.

Definition 1: *A paranormed space* (S, ξ) *is a linear space* S *with zero element* θ *together with a function* $\xi : S \to R_+$ *(called a paranorm on S) which satisfies the following axioms:*

PN_1: $\xi(\theta) = 0$; PN_2: $\xi(s) = \xi(-s)$, for all $s \in S$;

PN_3: $\xi(s + t) \leq \xi(s) + \xi(t)$, for all $s, t \in S$; and

PN_4: *Scalar multiplication is continuous*

i.e., *if* $< \gamma_n >$ *is a sequence of scalars with* $\gamma_n \to \gamma$ *as* $n \to \infty$ *and* $< s_n >$ *a sequence of vectors with* $\xi(s_n - s) \to 0$ *as* $n \to \infty$ *then* $\xi(\gamma_n s_n - \gamma s) \to 0$ *as* $n \to \infty$.

Note that the continuity of scalar multiplication is equivalent to

(i) if $\xi(s_n) \to 0$ and $\gamma_n \to \gamma$ as $n \to \infty$, then $\xi(\gamma_n s_n) \to 0$ as $n \to \infty$; and

(ii) if $\gamma_n \to 0$ as $n \to \infty$ and s be any element in S, then $\xi(\gamma_n s) \to 0$, (see Wilansky [25]).

A paranorm is called total if $\xi(s) = 0$ implies $s = \theta$, (see Wilansky [25]).

Definition 2: *Let S be a normed space over* **C**, *the field of complex numbers. Let* $\omega(S)$ *denotes the linear space of all sequences* $\bar{s} = < s_k >$ *with* $s_k \in S$, $k \geq 1$ *with usual coordinate wise operations*

i.e., $\bar{s} + \bar{t} = < s_k + t_k >$ and $\gamma \bar{s} = < \gamma s_k >$,for all $\bar{s}, \bar{t} \in \omega(S)$ and $\gamma \in \mathbf{C}$.

We shall denote $\omega(C)$ by ω . Any linear subspace of ω is called a *sequence space*.

Further if $\bar{\gamma} = < \gamma_k > \in \omega$ and $\bar{s} \in \omega(S)$, we shall write $\bar{\gamma}\bar{s} = < \gamma_k s_k >$.

Definition 3: *A sequence space S is said to be solid if* $\bar{s} = < s_k > \in S$ *and* $\bar{\gamma} = < \gamma_k >$ *a sequence of scalars with* $|\gamma_k| \leq 1$, *for all* $k \geq 1$, *then* $\bar{\gamma}\bar{s} = < \gamma_k s_k > \in S$.

In studying various properties of a vector valued sequence space (see, [6]) , we have the following definitions:

Definition 4: *A normed space S - valued topological sequence space V(S) equipped with the linear topology* $\Im$ *is said to be a GK-space if the map*

$$P_k : V(S) \to S,\ P_k(\bar{s}) = s_k,$$

is continuous for each k. A GK-space is called a GFK - space if it is complete linear metric space and a GC – space if $R_k : S \to V(S),\ R_k(s) = \delta_k(s),\ s \in S$ *is continuous for each* $k \geq 1$*, where* $\delta_k(s) = (\theta, \theta, \theta, \ldots, \theta, s, \theta, \ldots)$*, s is at* k^{th} *place.*

Subsequently, various types of sequence spaces in normed space were introduced and studied in different directions ,(for instances, see [2], [5], [6], [9] , [17] , [20], [22], [23] and many others).

Definition 5*:* *By an Orlicz function we mean a continuous, non decreasing and convex function* Φ*:* $[0,\infty) \to [0,\infty)$ *satisfying* $\Phi(0) = 0$, $\Phi(t) > 0$ *for* $t > 0$ *and* $\Phi(t) \to \infty$ *as* $t \to \infty$.

Note that an Orlicz function is always unbounded. An Orlicz function satisfies the inequality $\Phi(\gamma t) \leq \gamma\, \Phi(t)$ for all γ satisfying $0 < \gamma < 1$.

An Orlicz function Φ is said to satisfy Δ_2-condition for all values of $t \geq 0$, if there exists a constant $Q > 0$ such that $\Phi(2t) \leq Q\, \Phi(t)$. The Δ_2-condition is equivalent to the satisfaction of inequality

$\Phi(lt) \leq Q\, l\, \Phi(t)$ for all values of t and $l > 1$,(see [10]).

Lindenstrauss and Tzafriri (see, [11], [12] [13], [14]) used the idea of Orlicz function to construct the sequence space ℓ_Φ of scalars $< s_k >$ such that $\sum_{k=1}^{\infty} \Phi\left(\frac{\| s_k \|^{s}_{k}}{r}\right) < \infty$ for some $r > 0$. They proved that the space ℓ_Φ equipped with the norm defined by

$$\|\bar{s}\|_\Phi = \inf\left\{ r > 0 : \sum_{k=1}^{\infty} \Phi\left(\frac{|s_k|}{r}\right) \leq 1 \right\}$$

becomes a Banach space. Clearly the space ℓ_Φ is closely related to the sequence space ℓ_p which is an Orlicz sequence space with $\Phi(t) = t^p$, $1 \leq p < \infty$. Subsequently various types of topological structures in sequence spaces using Orlicz function have been introduced and studied, for instances we refer a few ([1], [2], [3], [4], [7], [8], [16], [17], [19], [21], [22], [23], [24]).

2. The Class $\ell_\infty((S, \|\cdot\|), \Phi, \bar{u})$ of Normed Space Valued Vector Sequences

Let $\bar{u} = \langle u_k \rangle$ and $\bar{v} = \langle v_k \rangle$ be any sequences of strictly positive real numbers and assume that

$$0 < l \leq \inf_k u_k \leq \sup_k u_k = L < \infty.$$

We now introduce the following class of normed space S- valued sequences

$$\ell_\infty((S, \|\cdot\|), \Phi, \bar{u}) = \left\{ \bar{s} = \langle s_k \rangle : s_k \in S \text{ and } \sup_k \Phi\left(\frac{\|s_k\|}{r}\right)^{u_k} < \infty, \text{for some } r > 0 \right\}. \quad \text{...(1)}$$

Further when $u_k = 1$ for all k, then $\ell_\infty((S, \|\cdot\|), \Phi, \bar{u})$ will be denoted by $\ell_\infty((S, \|\cdot\|), \Phi)$.

Besides studying the class (1), we now introduce and study a new subclass $\bar{\ell}_\infty((S, \|\cdot\|), \Phi, \bar{u})$ of $\ell_\infty((S, \|\cdot\|), \Phi, \bar{u})$ as follows:

$$\bar{\ell}_\infty((S, \|\cdot\|), \Phi, \bar{u}) = \left\{ \bar{s} = \langle s_k \rangle : s_k \in S \text{ and } \sup_k \Phi\left(\frac{\|s_k\|}{r}\right)^{u_k} < \infty, \text{for every } r > 0 \right\}. \quad \text{...(2)}$$

3. Linear Topological Structure of $\ell_\infty((S, ||.||), \Phi, \bar{u})$

In this section, we shall investigate some results that characterize the linear topological structures of the class $\ell_\infty((S, ||.||), \Phi, \bar{u})$ by endowing it with suitable natural paranorm. As far as the linear space structure of the class $\ell_\infty((S, ||.||), \Phi, \bar{u})$ over the field C of complex numbers is concerned, we throughout take coordinate wise operations i.e., for sequences $\bar{s} = < s_k >$ and $\bar{t} = < t_k >$ and scalar α,

$$\bar{s} + \bar{t} = < s_k + t_k > \text{ and } \alpha\,\bar{s} = < \alpha\, s_k >.$$

The zero element of $\ell_\infty((S, ||.||), \Phi, \bar{u})$ is denoted by $\bar{\theta}$. Throughout the work, we shall use

$$|s + t|^{u_k} \le D\,\{|s|^{u_k} + |t|^{u_k}\}, \text{ where } s, t \in \mathbf{C}, 0 < u_k \le \sup_k u_k = L,$$

$$A\,[\alpha] = \max\,\{1, |\alpha|\} \text{ for scalar } \alpha, D = A[2^{L-1}].$$

Theorem 3.1 *The class* $\ell_\infty((S, ||.||), \Phi, \bar{u})$ *forms a solid.*

Proof : Let $\bar{s} = < s_k > \in \ell_\infty((S, ||.||), \Phi, \bar{u})$. So that

$$\sup_k \Phi\left(\frac{||s_k||^{u_k}}{r}\right) < \infty \text{ for some } r > 0.$$

Let $< \alpha_k >$ be a sequence of scalars such that $|\alpha_k| \le 1$ for all $k \ge 1$. Since Φ is non-decreasing, we have

$$\Phi\left(\frac{||\alpha_k s_k||^{u_k}}{r}\right) = \Phi\left(\frac{|\alpha_k|^{u_k}\,||s_k||^{u_k}}{r}\right) \le \Phi\left(\frac{||s_k||^{u_k}}{r}\right), \text{ and hence}$$

$$\sup_k \Phi\left(\frac{||\alpha_k s_k||^{u_k}}{r}\right) \le \sup_k \Phi\left(\frac{||s_k||^{u_k}}{r}\right) < \infty.$$

This shows that $< \alpha_k s_k > \in \ell_\infty((S, ||.||), \Phi, \bar{u})$. So $\ell_\infty((S, ||.||), \Phi, \bar{u})$ forms a solid.

Theorem 3.2 : $\ell_\infty((S, ||\,.\,||), \Phi, \bar{u})$ *forms a linear space over the complex numbers* C *if* $\bar{u} = <u_k>$ *is bounded above.*

Proof: Suppose that $\sup_k u_k < \infty$. Let $\bar{s} = <s_k>$, $\bar{t} = <t_k> \in \ell_\infty((S, ||\,.\,||), \Phi, \bar{u})$ and $\alpha, \beta \in C$.

Then there exist $r_1 > 0$ and $r_2 > 0$ such that

$$\sup_k \Phi\left(\frac{||s_k||}{r_1}^{u_k}\right) < \infty \quad \text{and} \quad \sup_k \Phi\left(\frac{||t_k||}{r_2}^{u_k}\right) < \infty.$$

Let us choose $r_3 > 0$ such that $2\,D\,r_1 A[\,|\alpha|^M\,] \le r_3$ and $2\,D\,r_2 A[\,|\beta|^M\,] \le r_3$.

For such r_3, using non decreasing and convex properties of Φ, we have

$$\Phi\left(\frac{||\alpha\, s_k + \beta t_k||^{u_k}}{r_3}\right) \le \Phi\left(\frac{D|\alpha|^{u_k}\,||s_k||^{u_k} + D|\beta|^{u_k}\,||t_k||^{u_k}}{r_3}\right)$$

$$= \Phi\left(\frac{D\,A[\,|\alpha|^L\,]\,||\,s_k||^{u_k}}{r_3} + \frac{D\,A[\,|\beta|^L\,]\,||\,t_k||^{u_k}}{r_3}\right)$$

$$\le \Phi\left(\frac{1}{2\,r_1}\,||\,s_k||^{u_k} + \frac{1}{2\,r_2}\,||\,t_k||^{u_k}\right)$$

$$\le \frac{1}{2}\,\Phi\left(\frac{||s_k||^{u_k}}{r_1}\right) + \frac{1}{2}\,\Phi\left(\frac{||t_k||^{u_k}}{r_2}\right)$$

and therefore,

$$\sup_k \Phi\left(\frac{||\alpha\, s_k + \beta t_k||^{u_k}}{r_3}\right) \le \frac{1}{2}\sup_k \Phi\left(\frac{||s_k||^{u_k}}{r_1}\right) + \frac{1}{2}\sup_k \Phi\left(\frac{||t_k||^{u_k}}{r_2}\right) < \infty.$$

This implies that $\ell_\infty((S, ||\,.\,||), \Phi, \bar{u})$ forms a linear space over C.

Theorem 3.3. *If* $\ell_\infty((S, ||\,.\,||), \Phi, \bar{u})$ *forms a linear space over* C *, then* $\bar{u} = <u_k>$ *is bounded above.*

Proof : Suppose that $\ell_\infty((S, \|.\|), \Phi, \bar{u})$ is a linear space over C but $\sup_k u_k = \infty$. Then there exists a sequence $< k(n) >$ of positive integers satisfying

$k(n+1) > k(n) \geq 1, n \geq 1$ for which

$u_{k(n)} > n$, for each $n \geq 1$. .... (3)

Now, corresponding to $s \in S$ with $\| s \| = 1$, we define a sequence $\bar{s} = < s_k >$ by

$$s_k = \begin{cases} s \text{ , for } k = k(n), n \geq 1, \text{and} \\ \theta, \text{ otherwise.} \end{cases} \qquad ....(4)$$

Let $r > 0$. Then for $k = k(n)$, $n \geq 1$,we have

$$\sup_k \Phi\left(\frac{\|s_k\|^{u_k}}{r}\right) = \sup_n \Phi\left(\frac{\| s \|^{u_{k(n)}}}{r}\right) = \Phi\left(\frac{1}{r}\right) < \infty,$$

and $\sup_k \Phi\left(\frac{\|s_k\|^{u_k}}{r}\right) = 0$,otherwise.

This shows that $\bar{s} \in \ell_\infty((S, \|.\|), \Phi, \bar{u})$. But on the other hand in view of (3) and (4) for any $r > 0$ and scalar $\beta = 4$ using non decreasing property of Φ ,we have

$$\Phi\left(\frac{\|\beta\, s_k\|^{u_k}}{r}\right) = \Phi\left(\frac{\| 4\, u \|^{u_{k(n)}}}{r}\right) \geq \Phi\left(\frac{4^n}{r}\right)$$

$$\geq \Phi\left(\frac{4}{r}\right), \text{ for each } k \geq 1$$

and therefore, $\sup_k \left(\frac{\|\beta\, s_k\|^{u_k}}{r}\right) \geq \Phi\left(\frac{4}{r}\right)$, for each $k \geq 1$.

This shows that $\beta \bar{s} \notin \ell_\infty((S, \|.\|), \Phi, \bar{u})$,a contradiction. This completes the proof.

After combining the Theorems 3.2 and 3.3, we get:

Theorem 3.4. $\ell_\infty((S, \|.\|), \Phi, \bar{u})$ forms a linear space over C if and only if $\sup_k u_k < \infty$.

Theorem 3.5. *If* Φ *satisfies* Δ_2*- condition then* $\ell_\infty((S, \|.\|), \Phi, \bar{u}) = \bar{\ell}_\infty((S, \|.\|), \Phi, \bar{u})$.

Proof: To prove the theorem, it suffices to show that

$\ell_\infty((S, \|\cdot\|), \Phi, \bar{u}) \subseteq \bar{\ell}_\infty((S, \|\cdot\|), \Phi, \bar{u})$,

since the reverse inclusion is always true.

Let $\bar{s} \in \ell_\infty((S, \|\cdot\|), \Phi, \bar{u})$. Then for some $r > 0$,

$$\sup_k \Phi\left(\frac{\|s_k\|^{u_k}}{r}\right) < \infty. \qquad \text{.... (5)}$$

Let us consider an arbitrary $r_1 > 0$.If $r \leq r_1$, then obviously by the non decreasing property of Φ, we have

$$\sup_k \Phi\left(\frac{\|s_k\|^{u_k}}{r_1}\right) \leq \sup_k \Phi\left(\frac{\|s_k\|^{u_k}}{r}\right) < \infty,$$

and hence we get $\bar{s} \in \bar{\ell}_\infty((S, \|\cdot\|), \Phi, \bar{u})$.

But on the other hand, if $r > r_1$,so that $\frac{r}{r_1} > 1$ then by using Δ_2 condition of Φ and in view of (5) ,we get

$$\Phi\left(\frac{\|s_k\|^{u_k}}{r_1}\right) = \Phi\left(\frac{\frac{r}{r_1}\|s_k\|^{u_k}}{r}\right) \leq K.\frac{r}{r_1}\ \Phi\left(\frac{\|s_k\|^{u_k}}{r}\right).$$ Hence

$$\sup_k \Phi\left(\frac{\|s_k\|^{u_k}}{r_1}\right) \leq K.\frac{r}{r_1}\sup_k \Phi\left(\frac{\|s_k\|^{u_k}}{r}\right) < \infty,$$

where K is the number involved in Δ_2- condition . Hence $\bar{s} \in \bar{\ell}_\infty((S, \|\cdot\|), \Phi, \bar{u})$.

Corollary 3.6. *If Φ satisfies the Δ_2 - condition, then $\bar{\ell}_\infty((S, \|\cdot\|), \Phi, \bar{u})$ is a linear space over **C** if and only if $\sup_k u_k < \infty$.*

Proof : Proof easily follows from the Theorems 3.4,3.5.

In what follows we shall take $< u_k >$ as bounded , $\sup_k u_k = L < \infty$ and $\inf_k u_k = l > 0$.

Denote $w_k = u_k / L$, and consider a set

$$\psi(\bar{s}) = \{ r > 0 : \sup_k \Phi\left(\frac{||s_k||^{w_k}}{r}\right) \le 1\}, \text{for } \bar{s} = <s_k> \in \ell_\infty((S, ||.||), \Phi, \bar{u}). \qquad .... (6)$$

Consider a real valued function ξ on $\ell_\infty((S, ||.||), \Phi, \bar{u})$ defined by

$$\xi(\bar{s}) = \inf \{ r > 0 : \sup_k \Phi\left(\frac{||s_k||^{w_k}}{r}\right) \le 1 \}, \bar{s} = \in \ell_\infty((S, ||.||), \Phi, \bar{u}). \qquad .... (7)$$

We prove below that $\ell((S, ||.||), \Phi, \bar{u})$ with respect to ξ forms a paranormed space.

Theorem 3.7. *$(\ell((S, ||.||), \Phi, \bar{u}), \xi)$ forms a paranormed space.*

Proof : Obviously for $\bar{s} = <s_k>$, $\bar{t} = <t_k> \in \ell_\infty((S, ||.||), \Phi, \bar{u})$, $\xi(\bar{\theta}) = 0$ and $\xi(-\bar{s}) = \xi(\bar{s})$.

Now in view of (6) consider $r_1 \in \psi(\bar{s})$ and $r_2 \in \psi(\bar{t})$ and $r_3 = r_1 + r_2$.

Then clearly by the convexity of Φ we have

$$\Phi\left(\frac{||s_k + t_k||^{w_k}}{r_3}\right) \le \Phi\left[\frac{||s_k||^{w_k}}{r_1} \times \frac{r_1}{r_3} + \frac{||t_k||^{w_k}}{r_2} \times \frac{r_2}{r_3}\right]$$

$$\le \frac{r_1}{r_3} . \sup_k \Phi\left(\frac{||s_k||^{w_k}}{r_1}\right) + \frac{r_2}{r_3} . \sup_k \Phi\left(\frac{||t_k||^{w_k}}{r_2}\right)$$

$$\le \frac{r_1}{r_3} + \frac{r_2}{r_3} = 1,$$

and consequently $\sup_k \Phi\left(\frac{||s_k + t_k||^{w_k}}{r_3}\right) \le 1$.

This shows that $r_3 = r_1 + r_2 \in \psi(\bar{s} + \bar{t})$.

Thus, $\xi(\bar{s} + \bar{t}) \le r_1 + r_2$ for each $r_1 \in \psi(\bar{s})$ and $r_2 \in \psi(\bar{t})$ implies that $\xi(\bar{s} + \bar{t}) \le \xi(\bar{s}) + \xi(\bar{t})$.

Finally we show the continuity of scalar multiplication. Let $\bar{s}^{(n)} = < s_k^{(n)} >$ be a sequence in $\ell_\infty((S, ||.||), \Phi, \bar{u})$ such that $\xi(\bar{s}^{(n)}) \to 0$ as $n \to \infty$ and $< \alpha_n >$ a sequence of scalars such that $\alpha_n \to \alpha$. We prove that $\xi(\alpha_n \bar{s}^{(n)}) \to 0$.

Now, $$\xi(\alpha_n \bar{s}^{(n)}) = \inf\left\{ r : \sup_k \Phi\left(\frac{\|\alpha_n\, s_k^{(n)}\|^{w_k}}{r}\right) \le 1 \right\}$$

$$= \inf\left\{ r : \sup_k \Phi\left(\frac{|\alpha_n|^{w_k}\, \| s_k^{(n)}\|^{w_k}}{r}\right) \le 1 \right\}$$

$$\le \inf\left\{ r : \sup_k \Phi\left(\frac{H^{w_k}\, \| s_k^{(n)}\|^{w_k}}{r}\right) \le 1 \right\}$$

where $H = \sup_n |\alpha_n|$ .Thus for $s = \max(1, H)$,then we get

$$\xi(\alpha_n \bar{s}^{(n)}) \le \inf\left\{ r : \sup_k \Phi\left(\frac{s\, \| s_k^{(n)}\|^{w_k}}{r}\right) \le 1 \right\}.$$

Let $r = ts$,so that

$$\xi(\alpha_n \bar{s}^{(n)}) \le \inf\left\{ ts : \sup_k \Phi\left(\frac{\|s_k^{(n)}\|^{w_k}}{t}\right) \le 1 \right\}$$

$$= s \times \xi(\bar{s}^{(n)})$$

implies that $\xi(\alpha_n \bar{s}^{(n)}) \to 0$, as $\xi(\bar{s}^{(n)}) \to 0$ as $n \to \infty$.

Let $\alpha_n \to 0$ as $n \to \infty$ and $\bar{s}$ be any element in $\ell_\infty((S, \|.\|), \Phi, \bar{u})$. We show that $\xi(\alpha_n \bar{s}) \to 0$.

Now for $0 < \varepsilon < 1$,we can find a positive integer N such that $|\alpha_n| \le \varepsilon$ for all $n \ge N$.

Since $\inf_k u_k = l > 0$, therefore

$$|\alpha_n|^{w_k} \le |\alpha_n|^{1/L} \le \varepsilon^{1/L} \text{ for all } n \ge N ..$$

So that $$\Phi\left(\frac{\|\alpha_n s_k\|^{w_k}}{r}\right) \le \Phi\left(\frac{|\alpha_n|^{w_k}\, \|s_k\|^{w_k}}{r}\right) \le \Phi\left(\frac{\varepsilon^{1/L}\, \|s_k\|^{w_k}}{r}\right).$$

For $\bar{s} \in \ell_\infty((S, \|.\|), \Phi, \bar{u})$, $\psi(\bar{s}) = \{ r > 0 : \sup_k \Phi\left(\frac{\|s_k\|^{w_k}}{r}\right) \le 1 \}$.

So that $\psi(\varepsilon^{1/L}\bar{s}) = \{ r > 0 : \sup_k \Phi\left(\frac{\varepsilon^{1/L}\,||s_k||}{r}^{w_k}\right) \leq 1\}$

and if $\sup_k \Phi\left(\frac{\varepsilon^{1/L}\,||s_k||}{r}^{w_k}\right) \leq 1$, then $\sup_k \Phi\left(\frac{||\alpha_n s_k||}{r}^{w_k}\right) \leq 1$.

So, if $r \in \psi(\varepsilon^{1/L}\bar{s})$,then $r \in \psi(\alpha_n \bar{s})$ i.e., $\psi(\varepsilon^{1/L}\bar{s}) \subseteq \psi(\alpha_n \bar{s})$.

Taking infimum over such r's, we get

$$\text{Inf}\{ r : r \in \psi(\alpha_n \bar{s})\} \leq \text{Inf}\{ r : r \in \psi(\varepsilon^{1/L}\bar{s})\}$$

$$= \varepsilon^{1/L}\,\text{Inf}\{ r : r \in \psi(\bar{s})\}$$

which shows that $\xi(\alpha_n \bar{s}) \leq \varepsilon^{1/L}\xi(\bar{s})$ for all $n \geq N$,

i.e., $\xi(\alpha_n \bar{s}) \to 0$ as $n \to \infty$.

Hence $\ell_\infty((S, ||.||), \Phi, \bar{u})$ forms a paranormed space. This completes the poof.

Corollary 3.8. *If* $\inf_k u_k = l > 0$ *, then* $(\ell_\infty((S, ||.||), \Phi, \bar{u}), \xi)$ *forms a total paranormed space.*

Proof : Suppose that $\xi(\bar{s}) = 0$. Then by definition of paranorm (7) , for every $\varepsilon > 0$ there exists some r_ε $(0 < r_\varepsilon < \varepsilon)$ such that

$$\sup_k \Phi\left(\frac{||s_k||}{r_\varepsilon}^{w_k}\right) \leq 1 .$$

This shows that $\frac{\in}{r} > o$ $\Phi\left(\frac{||s_k||}{\varepsilon}^{w_k}\right) \leq 1$,

for every $\varepsilon > 0$. This is possible only when $||s_k||^{w_k} = 0$ for each $k \geq 1$. Hence $\bar{s} = \theta$.

Hence in view of Theorem 3.7, $(\ell_\infty((S, ||.||), \Phi, \bar{u}), \xi)$ forms a total paranormed space.

Theorem 3.9. *Paranormed space* $(\ell_\infty((S, ||.||), \Phi, \bar{u}), \xi)$ *is complete.*

Proof: Let $< \bar{s}^{(i)} >$ be a Cauchy sequence in $\ell_\infty((S, ||.||), \Phi, \bar{u})$. Let r be a fixed positive real number such that $\Phi(r) \geq 1$. Then for each $\frac{\in}{r} > o$, there exists an integer $N \geq 1$ such that

$\xi\,(\bar{s}^{(i)} - \bar{s}^{(j)}) < \frac{\varepsilon}{r}$ for all $i, j \geq N$. .... (8)

Using definition of paranorm ξ, we see that

$$\sup_k \Phi\left(\frac{\|s_k^{(i)} - s_k^{(j)}\|^{w_k}}{\xi\,(\bar{s}^{(i)} - \bar{s}^{(j)})}\right) \leq 1 \text{ for all } i, j \geq N.$$

Thus, $\Phi\left(\frac{\|s_k^{(i)} - s_k^{(j)}\|^{w_k}}{\xi\,(\bar{s}^{(i)} - \bar{s}^{(j)})}\right) \leq 1 \leq \Phi\,(r)$, for all $i, j \geq N$ and $k \geq 1$ ....(9)

But Φ is non decreasing, therefore

$$\frac{\|s_k^{(i)} - s_k^{(j)}\|^{w_k}}{\xi\,(\bar{s}^{(i)} - \bar{s}^{(j)})} < r$$

Hence in view of (8), we have

$\| s_k^{(i)} - s_k^{(j)} \|^{w_k} < \varepsilon$. .... (10)

This shows that $(s_k^{(i)})$ is a Cauchy sequence in S for all $k \geq 1$. But S is complete, therefore there exists s_k (say)in S for each $k \geq 1$ such that $s_k^{(i)} \to s_k$ as $i \to \infty$. We show that

$$\bar{s} = < s_k > \in \ell_\infty\,((S, \|\,.\,\|)\,, \Phi, \bar{u}\,).$$

Let us choose $r > 0$ such that

$\xi\,(\bar{s}^{(i)} - \bar{s}^{(j)}) < r < \varepsilon$ for all $i, j \geq N$. .... (11)

Since Φ is non decreasing, therefore by (9) and (11), we have

$$\sup_k \Phi\left(\frac{\|s_k^{(i)} - s_k^{(j)}\|^{w_k}}{r}\right) \leq \sup_k \Phi\left(\frac{\|s_k^{(i)} - s_k^{(j)}\|^{w_k}}{\xi\,(\bar{s}^{(i)} - \bar{s}^{(j)})}\right)$$

$$\leq 1 \text{ for all } i, j \geq N.$$

Since Φ is continuous, taking limit as $j \to \infty$, we see that

$$\sup_k \Phi\left(\frac{||s_k^{(i)} - s_k||^{w_k}}{r}\right) \le 1 \text{ for all } i \ge N.$$

Taking infimum of such r's, we get

$$\xi(\bar{s}^{(i)} - \bar{s}) = \text{inf. } \left\{ r : \sup_k \Phi\left(\frac{||s_k^{(i)} - s_k||^{w_k}}{r}\right) \le 1 \text{ for all } i \ge N \right\}$$

$$\le r < \varepsilon.$$

$\Rightarrow \quad \xi(\bar{s}^{(i)} - \bar{s}) < \varepsilon$, for all $i \ge N$.

This shows that $\bar{s}^{(i)} \to \bar{s}$ as $i \to \infty$ and clearly $\bar{s}^{(i)} - \bar{s} \in \ell_\infty((S, ||.||), \Phi, \bar{u})$, for all $i \ge N$.

Also, $\bar{s}^{(N)}$ and $\bar{s}^{(N)} - \bar{s} \in \ell_\infty((S, ||.||), \Phi, \bar{u})$, therefore it follows that

$$\bar{s} = \bar{s}^{(N)} - (\bar{s}^{(N)} - \bar{s}) \in \ell_\infty((S, ||.||), \Phi, \bar{u}).$$

This completes the proof.

Theorem 3.10: *Let* $\bar{u} = < u_k >$ *such that* $\sup_k u_k < \infty$ *and* S *be a normed space. Then* $(\ell_\infty((S, ||.||), \Phi, \bar{u}), \xi)$ *forms a* GK*–space.*

Proof : Let $\bar{s} = < s_k > \in \ell_\infty((S, ||.||), \Phi, \bar{u})$. Then by definition of paranorm ξ in (7), we see that

$$\sup_k \Phi\left(\frac{||s_k||^{w_n}}{\xi(\bar{s})}\right) \le 1 \quad \text{and hence } \Phi\left(\frac{||s_k||^{w_k}}{\xi(\bar{s})}\right) \le 1.$$

Let k_0 be a fixed positive real number such that $\Phi(k_0) \ge 1$, then

$$\Phi\left(\frac{||s_k||^{w_k}}{\xi(\bar{s})}\right) \le \Phi(k_0).$$

Since Φ is non-decreasing therefore

$$||s_k||^{w_k} < k_0 \, \xi(\bar{s})$$

or $\quad ||s_k|| < [k_0 \, \xi(\bar{s})]^{1/w_k}$

and so $\; ||P_k(\bar{s})|| = ||s_k|| < [k_0 \, \xi(\bar{s})]^{1/w_k}$

shows that $P_k : \ell_\infty((S, ||.||), \Phi, \bar{u}) \to S$, where $P_k(\bar{s}) = s_k$ for each $s_k \in S$, $k \geq 1$,is continuous and hence $(\ell_\infty((S, ||.||), \Phi, \bar{u}), \xi)$ forms a GK- space.

In view of Theorem 3.9 and 3.10, we have

Corollary 3.11. $(\ell_\infty((S, ||.||), \Phi, \bar{u}), \xi)$ *forms a GFK–space.*

Theorem 3.12. *Let* $\bar{u} = < u_k >$ *such that* $\sup_k u_k < \infty$ *and S be a normed space. Then* $(\ell_\infty((S, ||.||), \Phi, \bar{u}), \xi)$ *forms a GC–space.*

Proof: Let $R_n : S \to \ell_\infty((S, ||.||), \Phi, \bar{u})$, defined by $R_n(s) = \delta_n(s)$, $n \geq 1$, $s \in S$, where $\delta_n(s) = (\theta, \theta, \theta, \ldots, \theta, s, \theta, \theta, \ldots)$, s at n^{th}place. We prove the continuity of R_n at origin.

Clearly $\delta_n(s) \in \ell_\infty((S, ||.||), \Phi, \bar{u})$. Now for given $\varepsilon > 0$,we choose $s \in S$ such that

$$\Phi\left(\frac{||s||^{w_n}}{\varepsilon}\right) < 1. \qquad \ldots.(12)$$

Then in view of (12),we have

$$\xi(R_n(s)) = \xi(\delta_n(s)) = \inf\left\{ r > 0 : \sup_n \Phi\left(\frac{||s||^{w_n}}{r}\right) \leq 1 \right\} \leq \varepsilon,$$

and therefore R_n is continuous for each $n \geq 1$.

Hence $(\ell_\infty((S, ||.||), \Phi, \bar{u}), \xi)$ is a GC space.

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