

ON CERTAIN SUBCLASSES OF FUNCTIONS WITH RESPECT TO $(2j, k)$ - SYMMETRIC CONJUGATE POINTS

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Abstract : In the present paper using the concept of $(2j, k)$ -symmetric conjugate points, the classes $\mathcal{S}_{sc}^{(2j,k)}(\lambda, \phi)$, $\mathcal{K}_{sc}^{(2j,k)}(\lambda, \phi)$ and $\mathcal{H}_{sc}^{(2j,k)}(\alpha, \phi)$ are introduced. The integral representations for functions belonging to these classes are provided and some properties for $\mathcal{H}_{sc}^{(2j,k)}(\alpha, \phi)$ are discussed.

Key words : Analytic functions, univalent functions, $(2j, k)$ -symmetric

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1. Introduction

Let A denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \dots(1)$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

and S denote the subclass of A consisting of all function which are univalent in U . The class P designates the class of function of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad (z \in \mathcal{U}),$$

with the condition $\Re\{p(z)\} > 0$. For f and g be analytic in u , we say that the function f is subordinate to g in \mathcal{U} , if there exists an analytic function ω in \mathcal{U} such that $|\omega(z)| < z$ and $f(z) = g(\omega(z))$, we denote by $f \prec g$. If g is univalent in \mathcal{U} , then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

We now introduce the concept of (j, k) -symmetrical functions which generalizes the concept of even, odd and k -symmetric functions. Consider,

$$e^{-i\alpha} f(e^{i\alpha} z) = z + \sum_{n=2}^{\infty} a_n e^{i(n-1)\alpha} z^n, \quad \alpha \in \mathbb{R}. \quad \dots(2)$$

$$\text{and } [f(z^k)]^{\frac{1}{k}} = z + \frac{a_2}{k} z^{k+1} + \frac{1}{2k^2} [2ka_3 - (k-1)a_2^2] z^{2k-1} + \dots \quad \dots(3)$$

where k is a positive integer. The transformation in (1.2) is a rotation of f because it rotates the unit disc in the z -plane through an angle α and rotates the image domain in the

w -plane in the reverse direction through an angle of the same magnitude. In (3) the transformation $u = z^k$ maps u onto k copies of u and $f(z)$ carries this surface onto k copies of $f(U)$ joined by a suitable branch point at $w = 0$. It is intuitively clear that the k^{th} root merely unwinds the symmetry. Precisely, we have

Definition 1.1. Let k be a positive integer. A domain D is said to be k -fold symmetric if a rotation of D about the origin through an angle $2\pi/k$ carries D onto itself. A function f is said to be k -fold symmetric in u if for every z in u

$$f(e^{2\pi i/k} z) = e^{2\pi i/k} f(z).$$

The family of all k -fold symmetric function is denoted by \mathcal{S}^k and for $k = 2$ we get the odd univalent function.

The notion of (j, k) -symmetrical functions ($k = 2, 3, \dots ; j = 0, 1, 2, \dots, k-1$) is a generalization of the notion of even, odd, k -symmetrical functions and generalize the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function.

The theory of (j, k) symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan uniqueness theorem for holomorphic mappings [6].

Definition 1.2. Let $\varepsilon = (e^{2\pi i/k} z)$ and $j = 0, 1, 2, \dots, k-1$ where $k \geq 2$ is a number. A function $f : \mathcal{U} \mapsto \mathbb{C}$ is called (j, k) -symmetrical if $f(\varepsilon z) = \varepsilon^j f(z)$, $z \in \mathcal{U}$.

The family of all (j, k) -symmetrical functions is denoted be $\mathcal{S}^{(j,k)}$. $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are respectively the classes of even, odd and k -symmetric functions. We have the following decomposition theorem.

Theorem 1.3. [6] For every mapping $f : \mathcal{U} \mapsto \mathbb{C}$, there exists exactly the sequence of (j, k) -symmetrical functions $f_{j,k}$,

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z)$$

where $f_{j,k}(z) = \frac{1}{k} \sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^v z)$,(4)

$$(f \in \mathcal{A}; k = 1, 2, \dots; j = 0, 1, 2, \dots, k-1).$$

By the uniqueness of above decomposition, the mappings $f_{j,k}$ are called (j, k) -symmetrical parts of the mapping f . For $k = 2$ the above decomposition is the

well-known partition of a function f onto the sum of its even part $f_{(0,2)}$ and its odd part $f_{(1,2)}$. Making use of the principle of subordination and the concept of (j, k) -symmetric points we now introduce the following classes of analytic functions and derive some interesting results.

We denote by $\mathcal{S}^*, \mathcal{K}, \mathcal{C}, \mathcal{C}^*$ the familiar subclasses consisting of functions which, respectively, starlike, convex, close-to-convex and quasi-convex in \mathcal{U} .

Al-Amiri et al. in [1] introduced and investigated a class of functions starlike with respect to $2k$ -symmetric conjugate points, which satisfy the following inequality

$$\operatorname{Re} \left(\frac{zf'(z)}{f_{2k}(z)} \right) > 0,$$

where f_{2k} defined by

$$f_{2k}(z) = \frac{1}{2k} \sum_{v=0}^{k-1} [\varepsilon^{-v} f(\varepsilon^v z) + \varepsilon^v \overline{f(\varepsilon^v \bar{z})}], \quad (f \in \mathcal{A}).$$

K. R. Karthikeyan in [5] once introduced and investigated a class of function starlike with respect to $(2j, k)$ -symmetric conjugate points, and $\mathcal{S}_{sc}^{(2j,k)}(\phi)$ denote the class of function $f \in \mathcal{A}$ and satisfy the subordination condition,

$$\frac{zf'(z)}{f_{2j,k}(z)} \prec \phi(z); \quad (z \in \mathcal{U}),$$

where $f_{2j,k}(z)$ is defined by (5). A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_{sc}^{(2j,k)}(\phi)$ if it satisfies the subordination condition

$$\frac{(zf'(z))'}{f'_{2j,k}(z)} \prec \phi(z); \quad (z \in \mathcal{U}),$$

where $k = 1, 2, 3, \dots$; $j = 0, 1, 2, \dots, (k-1)$; $\phi \in \mathcal{P}$, $f_{2j,k}$ is defined by

$$(f_{2j,k}(z) = \frac{1}{2k} \sum_{v=0}^{k-1} [\varepsilon^{-vj} f(\varepsilon^v z) + \varepsilon^{vj} \overline{f(\varepsilon^v \bar{z})}], \quad (f \in \mathcal{A}).) \quad \dots(5)$$

If v in an integer, then the following identities follow directly from (5)

$$f'_{2j,k}(z) = \frac{1}{2k} \sum_{v=0}^{k-1} [\varepsilon^{-vj+v} f'(\varepsilon^v z) + \varepsilon^{vj-v} \overline{f'(\varepsilon^v \bar{z})}], \quad \dots(6)$$

$$f''_{2j,k}(z) = \frac{1}{2k} \sum_{v=0}^{k-1} [\varepsilon^{-vj+2v} f''(\varepsilon^v z) + \varepsilon^{vj-2v} \overline{f''(\varepsilon^v \bar{z})}], \quad \dots(7)$$

and $f_{2j,k}(\varepsilon^v z) = \varepsilon^{vj} f_{2j,k}(z), \quad f_{2j,k}(z) = \overline{f_{2j,k}(\bar{z})} \quad \dots(8)$

$$f'_{2j,k}(\varepsilon^v z) = \varepsilon^{vj-v} f'_{2j,k}(z), \quad f'_{2j,k}(\bar{z}) = \overline{f'_{2j,k}(z)}. \quad \dots(9)$$

The classes $\mathcal{S}_{sc}^{(2j,k)}(\phi)$ of functions starlike with respect to $(2j, k)$ -symmetric conjugate points and $\mathcal{K}_{sc}^{(2j,k)}(\phi)$ of functions convex with respect to $(2j, k)$ -symmetric conjugate points were studied recently in [5].

Motivated by above-mentioned function classes we now introduce the following analytic functions with respect to $(2j, k)$ -symmetric conjugate points.

Definition 1.4. Let $\mathcal{S}_{sc}^{(2j,k)}(\lambda, \phi)$ denote the class of functions $f \in \mathcal{A}$ satisfies

$$\frac{f(z)f'(z)}{z} \neq 0, \quad (z \in \mathcal{U}) \text{ and the subordination condition}$$

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_{2j,k}(z) + \lambda z f'_{2j,k}(z)} \prec \phi(z), \quad (z \in \mathcal{U}), \quad \dots(10)$$

where $\phi \in \mathcal{P}$, $0 \leq \lambda \leq 1$, $k = 1, 2, 3, \dots$ is a fixed positive integer and $f_{2j,k}(z)$ is defined by (5). Moreover, a function $f \in \mathcal{A}$ is in the class $\mathcal{K}_{sc}^{(2j,k)}(\lambda, \phi)$ if and only if $zf' \in \mathcal{S}_{sc}^{(2j,k)}(\lambda, \phi)$.

We also define the following class.

Definition 1.5. Let $\mathcal{H}_{sc}^{(2j,k)}(\alpha, \phi)$ denote the class of functions f in \mathcal{A} satisfies

$\frac{f(z)f'(z)}{z} \neq 0$, ($z \in \mathcal{U}$) and the subordination condition

$$(1-\alpha) \frac{zf'(z)}{f_{2j,k}(z)} + \alpha \frac{(zf'(z))'}{f'_{2j,k}(z)} \prec \phi(z), \quad (z \in \mathcal{U}), \quad \dots(11)$$

where $\phi \in \mathcal{P}$, $\alpha \geq 0$, $k = 1, 2, 3, \dots$ is a fixed positive integer and $f_{2j,k}(z)$ is defined by (5).

In our paper we need the following lemmas.

Lemma 1.6. [3,7]. Let β, γ in \mathbb{C} . Suppose that ϕ is convex and univalent in \mathcal{U} with $\phi(0) = 1$ and $\Re\{\beta\phi(z) + \gamma\} > 0$, ($z \in \mathcal{U}$).

If $m(z)$ is analytic in \mathcal{U} with $m(0) = 1$, then the subordination

$$m(z) + \frac{zm'(z)}{\beta m(z) + \gamma} \prec \phi(z),$$

implies that $m(z) \prec \phi(z)$.

Lemma 1.7. [12]. Let β, γ in \mathbb{C} . Suppose that ϕ is convex and univalent in \mathcal{U} with $\phi(0) = 1$ and $\Re\{\beta\phi(z) + \gamma\} > 0$, ($z \in \mathcal{U}$).

Also, let $q(z) \prec \phi(z)$.

If $u \in \mathcal{P}$ and satisfies the subordination

$$u(z) + \frac{zu'(z)}{\beta q(z) + \gamma} \prec \phi(z),$$

then $u(z) \prec \phi(z)$.

Lemma 1.8. [5]. Let ϕ in \mathcal{P} . Then $\mathcal{S}_{sc}^{(2j,k)}(\phi) \subset \mathcal{C} \subset \mathcal{S}$.

Lemma 1.9. [5]. Let ϕ in \mathcal{P} . Then, $\mathcal{K}_{sc}^{(2,j,k)}(\phi) \subset \mathcal{C}^{\square} \subset \mathcal{C}$.

2. Main Results

Theorem 2.1 . Let $\phi \in \mathcal{P}$ and $0 \leq \lambda \leq 1$, then,

$$\mathcal{S}_{sc}^{(2,j,k)}(\lambda, \phi) \subset \mathcal{S}_{sc}^{(2,j,k)}(\phi) \subset \mathcal{C} \subset \mathcal{S}. \quad \dots(12)$$

Proof. Let $f \in \mathcal{S}_{sc}^{(2,j,k)}(\lambda, \phi)$. Then

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_{2j,k}(z) + \lambda z f'_{2j,k}(z)} \prec \phi(z), \quad (z \in \mathcal{U}). \quad \dots(13)$$

Let $z = z\varepsilon^v$ also holds true that is,

$$\frac{\varepsilon^v z f'(\varepsilon^v z) + \lambda (\varepsilon^v z)^2 f''(\varepsilon^v z)}{(1-\lambda)f_{2j,k}(\varepsilon^v z) + \lambda \varepsilon^v z f'_{2j,k}(\varepsilon^v z)} \prec \phi(z), \quad (z \in \mathcal{U}), \quad \dots(14)$$

and
$$\frac{\overline{\varepsilon^v z f'(\varepsilon^v \bar{z})} + \overline{\lambda (\varepsilon^v \bar{z})^2 f''(\varepsilon^v \bar{z})}}{(1-\lambda)\overline{f_{2j,k}(\varepsilon^v \bar{z})} + \overline{\lambda \varepsilon^v \bar{z} f'_{2j,k}(\varepsilon^v \bar{z})}} \prec \phi(z), \quad \dots(15)$$

By (14), (15), we have

$$\frac{\varepsilon^{v-vj} z f'(\varepsilon^v z) + \lambda \varepsilon^{2v-vj} z^2 f''(\varepsilon^v z)}{(1-\lambda)f_{2j,k}(z) + \lambda z f'_{2j,k}(z)} \prec \phi(z), \quad \dots(16)$$

$$\frac{\varepsilon^{vj-v} \overline{z f'(\varepsilon^v \bar{z})} + \lambda \varepsilon^{vj-2v} \overline{z^2 f''(\varepsilon^v \bar{z})}}{(1-\lambda)\overline{f_{2j,k}(z)} + \overline{\lambda z f'_{2j,k}(z)}} \prec \phi(z). \quad \dots(17)$$

Summing (16) and (17) we get.

$$\frac{\frac{z}{2} \left(\varepsilon^{v-vj} f'(\varepsilon^v z) + \varepsilon^{vj-v} \overline{f'(\varepsilon^v \bar{z})} \right) + \frac{\lambda z^2}{2} \left(\varepsilon^{2v-vj} f''(\varepsilon^v z) + \varepsilon^{vj-2v} \overline{f''(\varepsilon^v \bar{z})} \right)}{(1-\lambda)f_{2j,k}(z) + \lambda z f'_{2j,k}(z)} \prec \phi(z), \quad \dots(18)$$

Letting $v=0,1,2,\dots,k-1$ in (18), repeatedly, and summing the resulting equations, we easily get:

$$\frac{\frac{z}{2k} \sum_{v=0}^{k-1} \left(\varepsilon^{v-vj} f'(\varepsilon^v z) + \varepsilon^{vj-v} \overline{f'(\varepsilon^v \bar{z})} \right) + \frac{\lambda z^2}{2k} \sum_{v=0}^{k-1} \left(\varepsilon^{2v-vj} f''(\varepsilon^v z) + \varepsilon^{vj-2v} \overline{f''(\varepsilon^v \bar{z})} \right)}{(1-\lambda)f_{2j,k}(z) + \lambda z f'_{2j,k}(z)} \prec \phi(z), \quad \dots(19)$$

or equivalently
$$\frac{z f'_{2j,k}(z) + \lambda z^2 f''_{2j,k}(z)}{(1-\lambda)f_{2j,k}(z) + \lambda z f'_{2j,k}(z)} \prec \phi(z), \quad \dots(20)$$

If we set
$$q(z) = \frac{z f'_{2j,k}(z)}{f_{2j,k}(z)}, \quad (z \in \mathcal{U}),$$

then (20) can be written as follows:

$$\frac{z f'_{2j,k}(z) + \lambda z^2 f''_{2j,k}(z)}{(1-\lambda)f_{2j,k}(z) + \lambda z f'_{2j,k}(z)} = q(z) + \frac{\lambda z q'(z)}{(1-\lambda) + \lambda q(z)} \prec \phi(z). \quad \dots(21)$$

We now prove $q(z) \prec \phi(z)$. \dots(22)

Considering the following two cases.

(i) If $\lambda = 0$, from (20), we know that (22) obviously holds true.

(ii) If $0 < \lambda \leq 1$, by noting that

$$\Re \left\{ \phi(z) + \frac{1}{\lambda} - 1 \right\} > 0.$$

By Lemma 1.6, we get (2.11).

By setting
$$p(z) = \frac{z f'(z)}{f_{2j,k}(z)}, \quad (z \in \mathcal{U}),$$
 we get

$$\frac{z f'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_{2j,k}(z) + \lambda z f'_{2j,k}(z)} = \frac{(1-\lambda)z f'(z) + \lambda z (z f'(z))'}{(1-\lambda)f_{2j,k}(z) + \lambda z f'_{2j,k}(z)}$$

$$\begin{aligned}
 &= \frac{(1-\lambda)p(z)f_{2j,k}(z) + \lambda z[p'(z)f_{2j,k}(z) + p(z)f'_{2j,k}(z)]}{(1-\lambda)f_{2j,k}(z) + \lambda zf'_{2j,k}(z)} \\
 &= \frac{(1-\lambda)p(z) + \lambda zp'(z) + p(z)\frac{\lambda zf'_{2j,k}(z)}{f_{2j,k}(z)}}{(1-\lambda) + \frac{\lambda zf'_{2j,k}(z)}{f_{2j,k}(z)}} \quad \dots(23)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\lambda zp'(z) + p(z) \left[(1-\lambda) + \frac{\lambda zf'_{2j,k}(z)}{f_{2j,k}(z)} \right]}{(1-\lambda) + \frac{\lambda zf'_{2j,k}(z)}{f_{2j,k}(z)}} \\
 &= p(z) + \frac{\lambda zp'(z)}{(1-\lambda) + \lambda q(z)}. \quad \dots(24)
 \end{aligned}$$

Now, by similarly applying the above proof for $q(z) \prec \phi(z)$, and using Lemma 1.7 in (24), we have

$$p(z) = \frac{zf(z)}{f_{2j,k}(z)} \prec \phi(z),$$

which implies that $\mathcal{S}_{sc}^{(2j,k)}(\lambda, \phi) \subset \mathcal{S}_{sc}^{(2j,k)}(\phi)$.

Furthermore, by Lemma 1.8, we find that

$$\mathcal{S}_{sc}^{(2j,k)}(\lambda, \phi) \subset \mathcal{S}_{sc}^{(2j,k)}(\phi) \subset \mathcal{C} \subset \mathcal{S}.$$

Corollary 2.2. *Let $\phi \in \mathcal{P}$ and $0 \leq \lambda \leq 1$, then,*

$$\mathcal{K}_{sc}^{(2j,k)}(\lambda, \phi) \subset \mathcal{K}_{sc}^{(2j,k)}(\phi) \subset \mathcal{C}^{\square} \subset \mathcal{C}. \quad \dots(25)$$

We now give some integral representations for the function classes $\mathcal{S}_{sc}^{(2j,k)}(\lambda, \phi)$ and $\mathcal{K}_{sc}^{(2j,k)}(\lambda, \phi)$.

Theorem 2.3. *Let $f \in \mathcal{S}_{sc}^{(2j,k)}(\lambda, \phi)$ with $0 < \lambda \leq 1$.*

$$\text{Then, } f_{2,j,k}(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \exp \left(\frac{1}{2k} \sum_{v=0}^{k-1} \int_0^u \frac{\phi(w_j(\varepsilon^v \zeta)) + \overline{\phi(w_j(\varepsilon^v \bar{\zeta}))} - 2}{\zeta} d\zeta \right) u^{\frac{1}{\lambda}-1} du, \dots (26)$$

where, $f_{2,j,k}(z)$ is defined by (5), w_j is analytic in \mathcal{U} with

$$w_j(0) = 0 \text{ and } |w_j(z)| < 1, \quad j = 0, 1, 2, \dots, k-1, \quad (z \in \mathcal{U}) \quad \dots (27)$$

Proof. Suppose that $f \in \mathcal{S}_{sc}^{(2j,k)}(\lambda, \phi)$. Then

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_{2,j,k}(z) + \lambda z f'_{2,j,k}(z)} = \phi(w_j(z)), \quad (z \in \mathcal{U}), \quad \dots (28)$$

where, w_j is analytic in \mathcal{U} and satisfies $\frac{f(z)f'(z)}{z} \neq 0$.

By similarly applying the arguments given in the Theorem 2.1 to (28), we obtain

$$\frac{(1-\lambda)zf'_{2,j,k}(z) + \lambda z(zf'_{2,j,k}(z))'}{(1-\lambda)f_{2,j,k}(z) + \lambda z f'_{2,j,k}(z)} = \frac{1}{2k} \sum_{v=0}^{k-1} \phi(w_j(\varepsilon^v z)) + \overline{\phi(w_j(\varepsilon^v \bar{z}))}. \dots (29)$$

It now follows from

$$\frac{(1-\lambda)f'_{2,j,k}(z) + \lambda(zf'_{2,j,k}(z))'}{(1-\lambda)f_{2,j,k}(z) + \lambda z f'_{2,j,k}(z)} - \frac{1}{z} = \frac{1}{2k} \sum_{v=0}^{k-1} \frac{\phi(w_j(\varepsilon^v z)) + \overline{\phi(w_j(\varepsilon^v \bar{z}))} - 2}{z}. \quad \dots (30)$$

By integral, we have

$$\log \left(\frac{(1-\lambda)f_{2,j,k}(z) + \lambda z f'_{2,j,k}(z)}{z} \right) = \frac{1}{2k} \sum_{v=0}^{k-1} \int_0^z \frac{\left(\phi(w_j(\varepsilon^v \zeta)) + \overline{\phi(w_j(\varepsilon^v \bar{\zeta}))} \right) - 2}{\zeta} d\zeta, \dots (31)$$

that is,

$$(1-\lambda)f_{2,j,k}(z) + \lambda z f'_{2,j,k}(z) = z \cdot \exp \left(\frac{1}{2k} \sum_{v=0}^{k-1} \int_0^z \frac{\left(\phi(w_j(\varepsilon^v \zeta)) + \overline{\phi(w_j(\varepsilon^v \bar{\zeta}))} \right) - 2}{\zeta} d\zeta \right). \quad \dots (32)$$

From equality (32), we can get equality (26) easily. Hence the proof is complete.

Corollary 2.4. Let $f \in \mathcal{S}_{sc}^{(2,j,k)}(\lambda, \phi)$ and $F_{2j,k}(z) = (1-\lambda)f_{2j,k}(z) + \lambda z f'_{2j,k}(z)$ then it follows by (20) that

$$\frac{z F'_{2j,k}(z)}{F_{2j,k}(z)} \prec \phi(z). \text{ Then } F_{2j,k}(z) \in \mathcal{S}^*(\phi).$$

Theorem 2.5. Let $f \in \mathcal{S}_{sc}^{(2,j,k)}(\lambda, \phi)$ with $0 < \lambda \leq 1$. Then

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \int_0^u \exp \left(\frac{1}{2k} \sum_{v=0}^{k-1} \int_0^\xi \frac{\phi(w_j(\varepsilon^v \zeta)) + \overline{\phi(w_j(\varepsilon^v \bar{\zeta}))} - 2}{\zeta} d\zeta \right) \phi(w_j(\xi)) \alpha u^{(1-2\lambda)/\lambda} du \dots(33)$$

where, w_j is analytic in \mathcal{U} with

$$w_j(0) = 0 \text{ and } |w_j(z)| < 1, \quad j = 0, 1, 2, \dots, k-1, \quad (z \in \mathcal{U}) \dots(34)$$

Proof. Suppose that $f \in \mathcal{S}_{sc}^{(2,j,k)}(\lambda, \phi)$. Then in light of (2.17) and (28), we get

$$\begin{aligned} (1-\lambda)f'(z) + \lambda(zf'(z))' &= \frac{(1-\lambda)f_{2j,k}(z) + \lambda z f'_{2j,k}(z)}{z} \cdot \phi(w_j(z)). \\ &= \exp \left(\frac{1}{2k} \sum_{v=0}^{k-1} \int_0^z \frac{\phi(w_j(\varepsilon^v \zeta)) + \overline{\phi(w_j(\varepsilon^v \bar{\zeta}))} - 2}{\zeta} d\zeta \right) \cdot \phi(w_j(z)). \dots(35) \end{aligned}$$

Integrating equality (35), two times, we easily get (33). Hence the proof is complete.

Corollary 2.6. Let $f \in \mathcal{K}_{sc}^{(2,j,k)}(\lambda, \phi)$ with $0 < \lambda \leq 1$. Then,

$$f_{2j,k}(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \int_0^u \exp \left(\frac{1}{2k} \sum_{v=0}^{k-1} \int_0^\xi \frac{\phi(w_j(\varepsilon^v \zeta)) + \overline{\phi(w_j(\varepsilon^v \bar{\zeta}))} - 2}{\zeta} d\zeta \right) d\xi u^{\frac{1}{\lambda}-2} du, \dots(36)$$

where, $f_{2j,k}(z)$ is defined by (5), w_j is analytic in \mathcal{U} with

$$w_j(0) = 0 \text{ and } |w_j(z)| < 1, \quad j = 0, 1, 2, \dots, k-1, \quad (z \in \mathcal{U}) \dots(37)$$

Corollary 2.7. Let $f \in \mathcal{K}_{sc}^{(2,j,k)}(\lambda, \phi)$ with $0 < \lambda \leq 1$.

$$\text{Then } f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \int_0^u \frac{1}{r} \int_0^r \exp \left(\frac{1}{2k} \sum_{v=0}^{k-1} \int_0^\xi \frac{\phi(w_j(\varepsilon^v \zeta)) + \overline{\phi(w_j(\varepsilon^v \zeta))} - 2}{\zeta} d\zeta \right) \phi(w_j(\xi)) d\xi dr u^{\frac{1}{\lambda}-2} du$$

where, w_j is analytic in \mathcal{U} with

$$w_j(0) = 0 \text{ and } |w_j(z)| < 1, \quad j = 0, 1, 2, \dots, k-1, \quad (z \in \mathcal{U}). \quad \dots(38)$$

3. Properties of the class $\mathcal{H}_{sc}^{(2j,k)}(\alpha, \phi)$

Theorem 3.1. Let $\phi \in \mathcal{P}$ and $\alpha \geq 0$. Then,

$$\mathcal{H}_{sc}^{(2j,k)}(\alpha, \phi) \subset \mathcal{S}_{sc}^{(2j,k)}(\phi) \subset \mathcal{C} \subset \mathcal{S}. \quad \dots(39)$$

Proof. Suppose that $f \in \mathcal{K}_{sc}^{(2j,k)}(\alpha, \phi)$.

$$\text{Then } (1-\alpha) \frac{zf'(z)}{f_{2j,k}(z)} + \alpha \frac{f'(z) + zf''(z)}{f'_{2j,k}(z)} \prec \phi(z), \quad \dots(40)$$

let $z = \varepsilon^v z$ also holds true that is,

$$(1-\alpha) \frac{\varepsilon^v zf'(\varepsilon^v z)}{f_{2j,k}(\varepsilon^v z)} + \alpha \frac{f'(\varepsilon^v z) + \varepsilon^v zf''(\varepsilon^v z)}{f'_{2j,k}(\varepsilon^v z)} \prec \phi(z), \quad \dots(41)$$

$$\text{and } (1-\alpha) \frac{\overline{\varepsilon^v z f'(\varepsilon^v \bar{z})}}{f_{2j,k}(\varepsilon^v \bar{z})} + \alpha \frac{\overline{f'(\varepsilon^v \bar{z}) + \varepsilon^v \bar{z} f''(\varepsilon^v \bar{z})}}{f'_{2j,k}(\varepsilon^v \bar{z})} \prec \phi(z), \quad \dots(42)$$

Upon summing (41) and (42) and applying the similar arguments given with the proof for Theorem 2.1, we have

$$(1-\alpha) \frac{zf'_{2j,k}(z)}{f_{2j,k}(z)} + \alpha \frac{(zf'_{2j,k}(z))'}{f'_{2j,k}(z)} \prec \phi(z), \quad \dots(43)$$

By setting $q(z) = \frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}$. Then, (43) can be written as follows

$$(1-\alpha)\frac{zf'_{2j,k}(z)}{f_{2j,k}(z)} + \alpha\frac{(zf'_{2j,k}(z))'}{f'_{2j,k}(z)} = q(z) + \alpha\frac{zq'(z)}{q(z)} \prec \phi(z).$$

Thus by Lemma 1.6, we have

$$q(z) = \frac{zf'_{2j,k}(z)}{f_{2j,k}(z)} \prec \phi(z).$$

At the same time, if we let

$$p(z) = \frac{zf'(z)}{f_{2j,k}(z)}, \text{ We have}$$

$$(1-\alpha)\frac{zf'(z)}{f_{2j,k}(z)} + \alpha\frac{(zf'(z))'}{f'_{2j,k}(z)} = p(z) + \alpha\frac{zp'(z)}{q(z)} \prec \phi(z).$$

Since $q(z) \prec \phi(z)$, by Lemma 1.7, we know that

$$p(z) = \frac{zf'(z)}{f_{2j,k}(z)} \prec \phi(z),$$

which implies that $\mathcal{H}_{sc}^{(2,j,k)}(\alpha, \phi) \subset \mathcal{S}_{sc}^{(2,j,k)}(\phi)$.

Furthermore, by Lemma 1.8 we have $\mathcal{H}_{sc}^{(j2,k)}(\alpha, \phi) \subset \mathcal{S}_{sc}^{(2,j,k)}(\phi) \subset \mathcal{C} \subset \mathcal{S}$.

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