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E–BAYESIAN AND HIERARCHICAL BAYESIAN ESTIMATION OF POWER HAZARD DISTRIBUTION

Sanjay Goyal[1*] and **Anil Kumar Bhardwaj**[2]
[1]Research Scholar, Department of Statistics, University of Rajasthan, Jaipur
[2]Professor, Department of Statistics, University of Rajasthan, Jaipur
Email: sanjaygoyal4phd@gmail.com[1*], dranilkumarbhardwaj@gmail.com

Abstract: In this paper E-Bayesian and Hierarchical Bayesian estimation methods are used to estimate the scale parameter of Power Hazard Distribution under 4 different loss functions - Squared Error Loss Function (SELF), Entropy Loss Function, Weighted Balance Loss function (WBLF), Minimum Expected Loss Function (MELF). The definition and properties of E-Bayesian and Hierarchical Bayesian estimation are provided. Relations among E-Bayesian estimators along with relations between E-Bayesian and Hierarchical Bayesian estimators are derived. A simulation study is carried out to compare these estimators using Mean Square Error (MSE) with the help of tool of R language on simulated data. To demonstrate the applicability of derived estimators, a real dataset is analyzed.

Keywords: E-Bayesian estimation, Hierarchical Bayesian estimation, Power Hazard distribution, Mean Square Error, Loss function.

1. Introduction

The probability density function of Power Hazard Distribution is given by Khan and Mustafa [9].

$$f(x) = \alpha x^{\theta} e^{-\frac{\alpha}{\theta+1}x^{\theta+1}}$$

$x \geq 0$

The cumulative density function (c.d.f.) of exponentiated odd lomax exponential distribution is given by Ansari and Nofal [1], Ijaz and Asim [6].

$$F(x) = 1 - e^{-\frac{\alpha}{\theta+1}x^{\theta+1}}$$

$x \geq 0, \alpha > 0, \theta > -1$

Where α and θ are respectively scale and shape parameters.

The likelihood function is given such as –

$$L = \alpha^n \prod \mathrm{x_i}^{\theta} \, e^{-\frac{\alpha}{\theta+1}\sum \mathrm{x_i}^{\theta+1}}$$

OR

$$L = \alpha^n e^{\theta \sum \log \mathrm{x_i}} e^{-\alpha T}$$

Where T = $\frac{\sum \mathrm{x_i}^{\theta+1}}{\theta+1}$

Survival function -

$$S(x) = 1 - F(x)$$

$$S(x) = e^{-\frac{\alpha}{\theta+1}x^{\theta+1}}$$

Power Hazard distribution is introduced by Mugdadi [11], Khan and Mustafa [8] which is extremely helpful for modeling the lifetime dataset for survival analysis. This distribution can further reduced to rayleigh, Weibull and exponential distribution.

Hierarchical Bayesian estimation process requires prior distribution for at least two stages which makes Hierarchical Bayesian estimation more efficient than Bayesian Estimation. The initial idea of Hierarchical Bayesian prior is given by Lindley and Smith in 1972. This method contains complicated numerical integrations which makes this problematic and time consuming.

E-Bayesian Estimation process is introduced by Han [5]. E-Bayesian Estimation overcome this drawback. Estimating parameters using E-Bayesian estimation process is comparatively simple than Hierarchical Bayesian Estimation. Han also developed the method of Hierarchical Bayesian and E-Bayesian estimation for failure rate of exponential distribution.

E-Bayesian and Hierarchical estimators for various distributions and models were derived by several researchers.

Han [3, 4] developed E-Bayesian and Hierarchical estimators for parameter of Pareto distribution. E-Bayesian estimator of Burr Type – XII model is generated by Jaheen and Okasha [7], Kumar and Sharma [10] explored the E-Bayesian and Hierarchical Bayesian estimator for hazard rate of Kumaraswamy distribution. Pathak et al. [12] conducted a study to obtain E-Bayesian estimator for Poisson Inverse Exponential distribution. El-Sagheer [2] proposed E-Bayesian estiamator under progressively type II censoring for Rayleigh Model. Reyad and Ahmed [13] explored E-Bayesian estimator for Kumaraswamy distribution based on Type-II censoring scheme.

Bayes Estimators of parameter α of power hazard distribution for gamma prior are given as –

For Gamma Prior

$$P(\alpha) = \frac{b^a e^{-\alpha b} \alpha^{a-1}}{\lceil a}$$

Under Square Error Loss Function

$$\hat{\alpha}_S = \frac{\boldsymbol{a+n}}{\boldsymbol{b+T}}$$

Under Entropy Loss function

$$\hat{\alpha}_E = \frac{\boldsymbol{a+n-1}}{\boldsymbol{b+T}}$$

Under Weighted Balance Loss function

$$\hat{\alpha}_W = \frac{\boldsymbol{a+n+1}}{\boldsymbol{b+T}}$$

Under Minimum Expected Loss function

$$\hat{\alpha}_M = \frac{\boldsymbol{a+n-2}}{\boldsymbol{b+T}}$$

Where $T = \frac{\sum \mathrm{x_i}^{\theta+1}}{\theta+1}$

2. E-Bayesian Estimation

In this section, E-bayesian estimators for the scale parameter of distribution are derived for different loss function - Squared Error Loss Function (SELF), Entropy Loss Function, Weighted Balance Loss function (WBLF), Minimum Expected Loss Function(MELF).

According to han prior parameters for the distribution a and b should be selected in such a manner that prior distribution be a decreasing function of parameter α.

$$\frac{dg(\alpha|a,b)}{d\alpha} = \frac{d}{d\alpha}\left(\frac{b^a e^{-\alpha \mathrm{b}} \alpha^{a-1}}{\lceil a}\right)$$

$$\frac{dg(\alpha|a,b)}{d\alpha} = \frac{b^a}{\lceil a}\left[(a-1)e^{-\alpha \mathrm{b}}\alpha^{a-2} - be^{-\alpha \mathrm{b}}\alpha^{a-1}\right]$$

$$\frac{dg(\alpha|a,b)}{d\alpha} = \frac{b^a e^{-\alpha \mathrm{b}} \alpha^{a-2}}{\lceil a}\left[(a-1) - b\alpha\right]$$

$$\hat{\alpha}_{EB} = \int_0^1 \int_0^p \hat{\alpha}\, \pi(\alpha, \mathrm{a}, \mathrm{b})\, \mathrm{da}\ \ \mathrm{db}$$

Here

$$\pi_1(\alpha, \mathrm{a}, \mathrm{b}) = \frac{2(\mathrm{p}-\mathrm{b})}{\mathrm{p}^2}$$

$$\pi_2(\alpha, \mathrm{a}, \mathrm{b}) = \frac{1}{\mathrm{p}}$$

$$\pi_3(\alpha, \mathrm{a}, \mathrm{b}) = \frac{2\mathrm{b}}{\mathrm{p}^2}$$

0 < a < 1 , 0 < b < p

1.1 Under SELF –

1.1.1 For π_1

$$\hat{\alpha}_{EBS1} = \int_0^1 \int_0^p \hat{\alpha}_s \, \pi_1(\alpha, \mathrm{a}, \mathrm{b}) \, \mathrm{da} \; \mathrm{db}$$

$$\hat{\alpha}_{EBS1} = \frac{2}{p^2} \int_0^1 \int_0^p \frac{(a+n)(p-b)}{(b+T)} \mathrm{da} \; \mathrm{db}$$

$$\hat{\alpha}_{EBS1} = \frac{(2n+1)}{p^2} \left[(p+T) log\left(\frac{p+T}{T}\right) - p \right]$$

1.1.2 For π_2

$$\hat{\alpha}_{EBS2} = \int_0^1 \int_0^p \hat{\alpha}_s \, \pi_2(\alpha, \mathrm{a}, \mathrm{b}) \, \mathrm{da} \; \mathrm{db}$$

$$\hat{\alpha}_{EBS2} = \frac{1}{p} \int_0^1 \int_0^p \frac{(a+n)}{(b+T)} \mathrm{da} \; \mathrm{db}$$

$$\hat{\alpha}_{EBS2} = \frac{(2n+1)}{2p} log\left(\frac{p+T}{T}\right)$$

1.1.3 For π_3

$$\hat{\alpha}_{EBS3} = \int_0^1 \int_0^p \hat{\alpha}_s \, \pi_3(\alpha, \mathrm{a}, \mathrm{b}) \, \mathrm{da} \; \mathrm{db}$$

$$\hat{\alpha}_{EBS3} = \frac{2}{p^2} \int_0^1 \int_0^p \frac{(a+n)\, b}{(b+T)} \mathrm{da} \; \mathrm{db}$$

$$\hat{\alpha}_{EBS3} = \frac{(2n+1)}{p^2} [p - T \log\left(\frac{p+T}{T}\right)]$$

1.2 Under Entropy Loss Function –

1.2.1 For π_1

$$\hat{\alpha}_{EBE1} = \int_0^1 \int_0^p \hat{\alpha}_E\, \pi_1(\alpha, \mathrm{a}, \mathrm{b})\, \mathrm{da}\ \mathrm{db}$$

$$\hat{\alpha}_{EBE1} = \frac{2}{p^2} \int_0^1 \int_0^p \frac{(a+n-1)(p-b)}{(b+T)} \mathrm{da}\ \mathrm{db}$$

$$\hat{\alpha}_{EBE1} = \frac{(2n-1)}{p^2}\left[(p+T)log\left(\frac{p+T}{T}\right) - p\right]$$

1.2.2 For π_2

$$\hat{\alpha}_{EBE2} = \int_0^1 \int_0^p \hat{\alpha}_E\, \pi_2(\alpha, \mathrm{a}, \mathrm{b})\, \mathrm{da}\ \mathrm{db}$$

$$\hat{\alpha}_{EBE2} = \frac{1}{p} \int_0^1 \int_0^p \frac{(a+n-1)}{(b+T)} \mathrm{da}\ \mathrm{db}$$

$$\hat{\alpha}_{EBE2} = \frac{(2n-1)}{2p} log\left(\frac{p+T}{T}\right)$$

1.2.3 For π_3

$$\hat{\alpha}_{EBE3} = \int_0^1 \int_0^p \hat{\alpha}_E\, \pi_3(\alpha, \mathrm{a}, \mathrm{b})\, \mathrm{da}\ \mathrm{db}$$

$$\hat{\alpha}_{EBE3} = \frac{2}{p^2} \int_0^1 \int_0^p \frac{(a+n-1)\, b}{(b+T)} \mathrm{da}\ \mathrm{db}$$

$$\hat{\alpha}_{EBE3} = \frac{(2n-1)}{p^2}\left[p - T \log\left(\frac{p+T}{T}\right)\right]$$

1.3 Under WBLF –

1.3.1 For π_1

$$\hat{\alpha}_{EBW1} = \int_0^1 \int_0^p \hat{\alpha}_W\, \pi_1(\alpha, \mathrm{a}, \mathrm{b})\, \mathrm{da}\ \mathrm{db}$$

$$\hat{\alpha}_{EBW1} = \frac{2}{p^2} \int_0^1 \int_0^p \frac{(a+n+1)(p-b)}{(b+T)} \mathrm{da}\ \mathrm{db}$$

$$\hat{\alpha}_{EBW1} = \frac{(2n+3)}{p^2}\left[(p+T)log\left(\frac{p+T}{T}\right) - p\right]$$

1.3.2 For π_2

$$\hat{\alpha}_{EBW2} = \int_0^1 \int_0^p \hat{\alpha}_W \, \pi_2(\alpha, \mathrm{a}, \mathrm{b}) \, \mathrm{da} \;\; \mathrm{db}$$

$$\hat{\alpha}_{EBW2} = \frac{1}{p}\int_0^1 \int_0^p \frac{(a+n+1)}{(b+T)} \mathrm{da} \;\; \mathrm{db}$$

$$\hat{\alpha}_{EBW2} = \frac{(2n+3)}{2p} log\left(\frac{p+T}{T}\right)$$

1.3.3 For π_3

$$\hat{\alpha}_{EBW3} = \int_0^1 \int_0^p \hat{\alpha}_W \, \pi_3(\alpha, \mathrm{a}, \mathrm{b}) \, \mathrm{da} \;\; \mathrm{db}$$

$$\hat{\alpha}_{EBW3} = \frac{2}{p^2}\int_0^1 \int_0^p \frac{(a+n+1)\, b}{(b+T)} \mathrm{da} \;\; \mathrm{db}$$

$$\hat{\alpha}_{EBW3} = \frac{(2n+3)}{p^2}\left[p - T \log\left(\frac{p+T}{T}\right)\right]$$

1.4 Under MELF –

1.4.1 For π_1

$$\hat{\alpha}_{EBM1} = \int_0^1 \int_0^p \hat{\alpha}_M \, \pi_1(\alpha, \mathrm{a}, \mathrm{b}) \, \mathrm{da} \;\; \mathrm{db}$$

$$\hat{\alpha}_{EBM1} = \frac{2}{p^2}\int_0^1 \int_0^p \frac{(a+n-2)(p-b)}{(b+T)} \mathrm{da} \;\; \mathrm{db}$$

$$\hat{\alpha}_{EBM1} = \frac{(2n-3)}{p^2}\left[(p+T)log\left(\frac{p+T}{T}\right) - p\right]$$

1.4.2 For π_2

$$\hat{\alpha}_{EBM2} = \int_0^1 \int_0^p \hat{\alpha}_M \, \pi_2(\alpha, \mathrm{a}, \mathrm{b}) \, \mathrm{da} \;\; \mathrm{db}$$

$$\hat{\alpha}_{EBM2} = \frac{1}{p}\int_0^1\int_0^p \frac{(a+n-2)}{(b+T)}\,\mathrm{da\ db}$$

$$\hat{\alpha}_{EBM2} = \frac{(2n-3)}{2p}\,log\left(\frac{p+T}{T}\right)$$

1.4.3 For π_3

$$\hat{\alpha}_{EBM3} = \int_0^1\int_0^p \hat{\alpha}_M\,\pi_3(\alpha,\mathrm{a},\mathrm{b})\,\mathrm{da\ db}$$

$$\hat{\alpha}_{EBM3} = \frac{2}{p^2}\int_0^1\int_0^p \frac{(a+n-2)\,b}{(b+T)}\,\mathrm{da\ db}$$

$$\hat{\alpha}_{EBM3} = \frac{(2n-3)}{p^2}\left[p - T\log\left(\frac{p+T}{T}\right)\right]$$

3. Hierarchical Bayesian Estimation

Hierarchical bayesian estimators for the distribution are derived using different loss functions - Squared Error Loss Function (SELF), Entropy Loss Function, Weighted Balance Loss function (WBLF), Minimum Expected Loss Function (MELF) in this section.

Hierarchical Prior distribution can be generated such as -

$$\pi_4 = \int_0^1\int_0^p g(\alpha|a,b)\,\pi_1(a,b)\,\mathrm{da\ db}$$

$$\pi_4 = \int_0^1\int_0^p \frac{b^a e^{-\alpha \mathrm{b}}\alpha^{a-1}}{\lceil a}\,\frac{2(\mathrm{p}-\mathrm{b})}{\mathrm{p}^2}\,\mathrm{da\ db}$$

$$\pi_4 = \frac{2}{\mathrm{p}^2}\int_0^1\int_0^p \frac{b^a e^{-\alpha \mathrm{b}}\alpha^{a-1}(\mathrm{p}-\mathrm{b})}{\lceil a}\,\mathrm{da\ db}$$

$$\pi_5 = \int_0^1\int_0^p g(\alpha|a,b)\,\pi_2(a,b)\,\mathrm{da\ db}$$

$$\pi_5 = \int_0^1\int_0^p \frac{b^a e^{-\alpha \mathrm{b}}\alpha^{a-1}}{\lceil a}\,\frac{1}{\mathrm{p}}\,\mathrm{da\ db}$$

$$\pi_5 = \frac{1}{\mathrm{p}}\int_0^1\int_0^p \frac{b^a e^{-\alpha \mathrm{b}}\alpha^{a-1}}{\lceil a}\,\mathrm{da\ db}$$

$$\pi_6 = \int_0^1\int_0^p g(\alpha|a,b)\,\pi_3(a,b)\,\mathrm{da\ db}$$

$$\pi_6 = \int_0^1\int_0^p \frac{b^a e^{-\alpha \mathrm{b}}\alpha^{a-1}}{\lceil a}\frac{2\mathrm{b}}{\mathrm{p}^2}\,\mathrm{da\ db}$$

$$\pi_6 = \frac{2}{\mathrm{p}^2}\int_0^1\int_0^p \frac{b^{a+1} e^{-\alpha \mathrm{b}}\alpha^{a-1}}{\lceil a}\,\mathrm{da\ db}$$

Now Hierarchical Posterior distribution can be derived with the help of Bayes Theorem-

$$\pi_1(\alpha|\mathrm{a,b}) = \frac{L(\alpha|\mathrm{x})\pi_4(\alpha)}{\int_0^\infty L(\alpha|\mathrm{x})\pi_4(\alpha)\mathrm{d}\alpha}$$

$$L(\alpha|\mathrm{x})\pi_4(\alpha) = \frac{2}{\mathrm{p}^2}\int_0^1\int_0^p \frac{b^a e^{-\alpha \mathrm{b}}\alpha^{a-1}(\mathrm{p}-\mathrm{b})}{\lceil a}\alpha^n e^{\theta\sum\log \mathrm{x_i}}e^{-\alpha T}\,\mathrm{da\ db}$$

$$L(\alpha|\mathrm{x})\pi_4(\alpha) = \frac{2}{\mathrm{p}^2}e^{\theta\sum\log \mathrm{x_i}}e^{-\alpha T}\int_0^1 \frac{\alpha^{n+a-1}}{\lceil a}\mathrm{da}\int_0^p b^a e^{-\alpha \mathrm{b}}(\mathrm{p}-\mathrm{b})\,\mathrm{db}$$

$$\int_0^\infty L(\alpha|\mathrm{x})\pi_4(\alpha)\,\mathrm{d}\alpha = \frac{2}{\mathrm{p}^2}e^{\theta\sum\log \mathrm{x_i}}\int_0^1\int_0^p \frac{b^a(\mathrm{p}-\mathrm{b})}{\lceil a}\mathrm{dadb}\int_0^\infty \alpha^{n+a-1}\,e^{-\alpha(b+\mathrm{T})}\mathrm{d}\alpha$$

$$\int_0^\infty L(\alpha|\mathrm{x})\pi_4(\alpha)\,\mathrm{d}\alpha = \frac{2}{\mathrm{p}^2}e^{\theta\sum\log \mathrm{x_i}}\int_0^1\int_0^p \frac{b^a(\mathrm{p}-\mathrm{b})\lceil(a+n)}{\lceil a(T+b)^{a+n}}\mathrm{dadb}$$

$$\pi_1(\alpha|\mathrm{a,b}) = \frac{\int_0^1\int_0^p \frac{b^a(\mathrm{p}-\mathrm{b})\alpha^{n+a-1}e^{-\alpha(b+\mathrm{T})}}{\lceil a}\mathrm{dadb}}{\int_0^1\int_0^p \frac{b^a(\mathrm{p}-\mathrm{b})\lceil(a+n)}{\lceil a(T+b)^{a+n}}\mathrm{dadb}}$$

$$\pi_2(\alpha|\mathrm{a,b}) = \frac{L(\alpha|\mathrm{x})\pi_5(\alpha)}{\int_0^\infty L(\alpha|\mathrm{x})\pi_5(\alpha)\,\mathrm{d}\alpha}$$

$$\pi_2(\alpha|\mathrm{a,b}) = \frac{\int_0^1 \int_0^p \frac{b^a \alpha^{n+a-1} e^{-\alpha(b+\mathrm{T})}}{\Gamma a}\,\mathrm{dadb}}{\int_0^1 \int_0^p \frac{b^a \Gamma(a+n)}{\Gamma a (T+b)^{a+n}}\,\mathrm{dadb}}$$

$$\pi_3(\alpha|\mathrm{a,b}) = \frac{L(\alpha|\mathrm{x})\pi_6(\alpha)}{\int_0^\infty L(\alpha|\mathrm{x})\pi_6(\alpha)\,\mathrm{d}\alpha}$$

$$\pi_3(\alpha|\mathrm{a,b}) = \frac{\int_0^1 \int_0^p \frac{b^{a+1} \alpha^{n+a-1} e^{-\alpha(b+\mathrm{T})}}{\Gamma a}\,\mathrm{dadb}}{\int_0^1 \int_0^p \frac{b^{a+1} \Gamma(a+n)}{\Gamma a (T+b)^{a+n}}\,\mathrm{dadb}}$$

1.5 Now Under SELF-

1.5.1 For π_1

$$\hat{\alpha}_{HS1} = E[\pi_1(\alpha|\mathrm{a,b})]$$

$$\hat{\alpha}_{HS1} = \int_0^\infty \alpha\pi_1(\alpha|\mathrm{a,b})\mathrm{d}\alpha$$

$$\hat{\alpha}_{HS1} = \frac{\int_0^1 \int_0^p \frac{b^a (\mathrm{p-b}) \Gamma n+a+1}{\Gamma a (T+b)^{a+n+1}}\,\mathrm{dadb}}{\int_0^1 \int_0^p \frac{b^a (\mathrm{p-b}) \Gamma(a+n)}{\Gamma a (T+b)^{a+n}}\,\mathrm{dadb}}$$

1.5.2 For π_2

$$\hat{\alpha}_{HS2} = E[\pi_2(\alpha|\mathrm{a,b})]$$

$$\hat{\alpha}_{HS2} = \int_0^\infty \alpha\pi_2(\alpha|\mathrm{a,b})\,\mathrm{d}\alpha$$

$$\hat{\alpha}_{HS2} = \frac{\int_0^1 \int_0^p \frac{b^a \Gamma n+a+1}{\Gamma a (T+b)^{a+n+1}}\,\mathrm{dadb}}{\int_0^1 \int_0^p \frac{b^a \Gamma(a+n)}{\Gamma a (T+b)^{a+n}}\,\mathrm{dadb}}$$

1.5.3 For π_3

$$\hat{\alpha}_{HS3} = E[\pi_3(\alpha|\mathrm{a,b})]$$

$$\hat{\alpha}_{HS3} = \int_0^\infty \alpha\pi_3(\alpha|\mathrm{a,b})\,\mathrm{d}\alpha$$

$$\hat{\alpha}_{HS3} = \frac{\int_0^1 \int_0^p \frac{b^{a+1} \Gamma n+a+1}{\Gamma a (T+b)^{a+n+1}}\,\mathrm{dadb}}{\int_0^1 \int_0^p \frac{b^{a+1} \Gamma(a+n)}{\Gamma a (T+b)^{a+n}}\,\mathrm{dadb}}$$

1.6 Under Entropy Loss Function –

1.6.1 For π_1

$$\hat{\alpha}_{HE1} = E[\pi_1(\alpha^{-1}|a,b)]^{-1}$$

$$E[\pi_1(\alpha^{-1}|a,b)] = \int_0^\infty \frac{1}{\alpha}\pi_1(\alpha|\mathrm{a,b})\,\mathrm{d}\alpha$$

$$\hat{\alpha}_{HE1} = \frac{\int_0^1\int_0^p \frac{b^a(\mathrm{p-b})\Gamma(a+n)}{\Gamma a(T+b)^{a+n}}\,\mathrm{dadb}}{\int_0^1\int_0^p \frac{b^a(\mathrm{p-b})\Gamma n+a-1}{\Gamma a(T+b)^{a+n-1}}\,\mathrm{dadb}}$$

1.6.2 For π_2

$$\hat{\alpha}_{HE2} = E[\pi_2(\alpha^{-1}|a,b)]^{-1}$$

$$E[\pi_2(\alpha^{-1}|a,b)] = \int_0^\infty \frac{1}{\alpha}\pi_2(\alpha|\mathrm{a,b})\,\mathrm{d}\alpha$$

$$\hat{\alpha}_{HE2} = \frac{\int_0^1\int_0^p \frac{b^a\Gamma(a+n)}{\Gamma a(T+b)^{a+n}}\,\mathrm{dadb}}{\int_0^1\int_0^p \frac{b^a\Gamma n+a-1}{\Gamma a(T+b)^{a+n-1}}\,\mathrm{dadb}}$$

1.6.3 For π_3

$$\hat{\alpha}_{HE3} = E[\pi_3(\alpha^{-1}|a,b)]^{-1}$$

$$E[\pi_3(\alpha^{-1}|a,b)] = \int_0^\infty \frac{1}{\alpha}\pi_3(\alpha|\mathrm{a,b})\,\mathrm{d}\alpha$$

$$\hat{\alpha}_{HE3} = \frac{\int_0^1\int_0^p \frac{b^{a+1}\Gamma(a+n)}{\Gamma a(T+b)^{a+n}}\,\mathrm{dadb}}{\int_0^1\int_0^p \frac{b^{a+1}\Gamma n+a-1}{\Gamma a(T+b)^{a+n-1}}\,\mathrm{dadb}}$$

1.7 Under WBLF –

1.7.1 For π_1

$$\hat{\alpha}_{HW1} = \frac{E[\pi_1(\alpha^2|a,b)]}{E[\pi_1(\alpha|a,b)]}$$

$$E[\pi_1(\alpha^2|a,b)] = \int_0^\infty \alpha^2\pi_1(\alpha|\mathrm{a,b})\,\mathrm{d}\alpha$$

$$\hat{\alpha}_{HW1} = \frac{\int_0^1 \int_0^p \frac{b^a(\text{p}-\text{b})\lceil(a+n+2)}{\lceil a(T+b)^{a+n+2}} \text{dadb}}{\int_0^1 \int_0^p \frac{b^a(\text{p}-\text{b})\lceil n+a+1}{\lceil a(T+b)^{a+n+1}} \text{dadb}}$$

1.7.2 For π_2

$$\hat{\alpha}_{HW2} = \frac{E[\pi_2(\alpha^2|a,b)]}{E[\pi_2(\alpha|a,b)]}$$

$$E[\pi_2(\alpha^2|a,b)] = \int_0^\infty \alpha^2 \pi_2(\alpha|\text{a},\text{b})\, \text{d}\alpha$$

$$\hat{\alpha}_{HW2} = \frac{\int_0^1 \int_0^p \frac{b^a\lceil(a+n+2)}{\lceil a(T+b)^{a+n+2}} \text{dadb}}{\int_0^1 \int_0^p \frac{b^a\lceil n+a+1}{\lceil a(T+b)^{a+n+1}} \text{dadb}}$$

1.7.3 For π_3

$$\hat{\alpha}_{HW3} = \frac{E[\pi_3(\alpha^2|a,b)]}{E[\pi_3(\alpha|a,b)]}$$

$$E[\pi_3(\alpha^2|a,b)] = \int_0^\infty \alpha^2 \pi_3(\alpha|\text{a},\text{b})\, \text{d}\alpha$$

$$\hat{\alpha}_{HW3} = \frac{\int_0^1 \int_0^p \frac{b^{a+1}\lceil(a+n+2)}{\lceil a(T+b)^{a+n+2}} \text{dadb}}{\int_0^1 \int_0^p \frac{b^{a+1}\lceil n+a+1}{\lceil a(T+b)^{a+n+1}} \text{dadb}}$$

1.8 Under MELF –

1.8.1 For π_1

$$\hat{\alpha}_{HM1} = \frac{E[\pi_1(\alpha^{-1}|a,b)]}{E[\pi_1 1(\alpha^{-2}|a,b)]}$$

$$E[\pi_1(\alpha^{-2}|a,b)] = \int_0^\infty \alpha^{-2} \pi_1(\alpha|\text{a},\text{b})\, \text{d}\alpha$$

$$\hat{\alpha}_{HM1} = \frac{\int_0^1 \int_0^p \frac{b^a(\text{p}-\text{b})\lceil(a+n-1)}{\lceil a(T+b)^{a+n-1}} \text{dadb}}{\int_0^1 \int_0^p \frac{b^a(\text{p}-\text{b})\lceil n+a-2}{\lceil a(T+b)^{a+n-2}} \text{dadb}}$$

1.8.2 For π_2

$$\hat{\alpha}_{HM2} = \frac{E[\pi_2(\alpha^{-1}|a,b)]}{E[\pi_2(\alpha^{-2}|a,b)]}$$

$$E[\pi_2(\alpha^{-2}|a,b)] = \int_0^\infty \alpha^{-2}\pi_2(\alpha|\mathrm{a},\mathrm{b})\,\mathrm{d}\alpha$$

$$\hat{\alpha}_{HM2} = \frac{\int_0^1\int_0^p \frac{b^a\lceil(a+n-1)}{\lceil a(T+b)^{a+n-1}}\mathrm{dadb}}{\int_0^1\int_0^p \frac{b^a\lceil n+a-2}{\lceil a(T+b)^{a+n-2}}\mathrm{dadb}}$$

1.8.3 For π_3

$$\hat{\alpha}_{HM3} = \frac{E[\pi_3(\alpha^{-1}|a,b)]}{E[\pi_3(\alpha^{-2}|a,b)]}$$

$$E[\pi_3(\alpha^{-2}|a,b)] = \int_0^\infty \alpha^{-2}\pi_3(\alpha|\mathrm{a},\mathrm{b})\,\mathrm{d}\alpha$$

$$\hat{\alpha}_{HM3} = \frac{\int_0^1\int_0^p \frac{b^{a+1}\lceil(a+n-1)}{\lceil a(T+b)^{a+n-1}}\mathrm{dadb}}{\int_0^1\int_0^p \frac{b^{a+1}\lceil n+a-2}{\lceil a(T+b)^{a+n-2}}\mathrm{dadb}}$$

4. Properties of the E-Bayesian and Hierarchical Bayesian Estimation

a) Relation between E Bayesian Estimators -

1.9 E-Bayes for SELF -

$$\hat{\alpha}_{EBS1} < \hat{\alpha}_{EBS2} < \hat{\alpha}_{EBS3}$$

$$\hat{\alpha}_{EBS1} = \frac{(2n+1)}{p^2}\left[(p+T)log\left(\frac{p+T}{T}\right) - p\right]$$

$$\hat{\alpha}_{EBS2} = \frac{(2n+1)}{2p}log\left(\frac{p+T}{T}\right)$$

$$\hat{\alpha}_{EBS3} = \frac{(2n+1)}{p^2}[p - T\log\left(\frac{p+T}{T}\right)]$$

$$\hat{\alpha}_{EBS1} - \hat{\alpha}_{EBS2} = \frac{(2n+1)}{p^2}\left[(p+T)log\left(\frac{p+T}{T}\right) - p\right] - \frac{(2n+1)}{2p}log\left(\frac{p+T}{T}\right)$$

$$\hat{\alpha}_{EBS1} - \hat{\alpha}_{EBS2} = \frac{(2n+1)}{p}\left[log\left(\frac{p+T}{T}\right)\left(\frac{T}{p}+\frac{1}{2}\right) - 1\right]$$

Similarly

$$\hat{\alpha}_{EBS2} - \hat{\alpha}_{EBS3} = \frac{(2n+1)}{p}\left[log\left(\frac{p+T}{T}\right)\left(\frac{T}{p}+\frac{1}{2}\right) - 1\right]$$

$$\hat{\alpha}_{EBS1} - \hat{\alpha}_{EBS2} = \hat{\alpha}_{EBS2} - \hat{\alpha}_{EBS3} = \frac{(2n+1)}{p}\left[log\left(1+\frac{p}{T}\right)\left(\frac{T}{p}+\frac{1}{2}\right) - 1\right]$$

For -1<x<1

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots = \sum_{i=1}^{\infty}(-1)^i\frac{x^i}{i}$$

$$log\left(1+\frac{p}{T}\right)\left(\frac{T}{p}+\frac{1}{2}\right) - 1 = \left(\frac{T}{p}+\frac{1}{2}\right)\left[\frac{p}{T}-\frac{p^2}{2T^2}+\frac{p^3}{3T^3}-\frac{p^4}{4T^4}+\cdots\right]-1$$

$$log\left(1+\frac{p}{T}\right)\left(\frac{T}{p}+\frac{1}{2}\right) - 1 = \frac{1}{12}\left(\frac{p}{T}\right)^2\left[1-\frac{p}{T}\right]+\frac{3}{40}\left(\frac{p}{T}\right)^4\left[1-\frac{8p}{9T}\right]+\cdots > 0$$

Since $0 < \frac{p}{T} < 1$

$$\hat{\alpha}_{EBS1} - \hat{\alpha}_{EBS2} = \hat{\alpha}_{EBS2} - \hat{\alpha}_{EBS3} = \frac{(2n+1)}{p}\left[log\left(1+\frac{p}{T}\right)\left(\frac{T}{p}+\frac{1}{2}\right) - 1\right] > 0$$

Hence

$$\hat{\alpha}_{EBS1} < \hat{\alpha}_{EBS2} < \hat{\alpha}_{EBS3}$$

$$\lim_{T\to\infty}\hat{\alpha}_{EBS1} - \hat{\alpha}_{EBS2} = \lim_{T\to\infty}\hat{\alpha}_{EBS2} - \hat{\alpha}_{EBS3}$$

$$= \lim_{T\to\infty}\frac{(2n+1)}{p}\left[\frac{1}{12}\left(\frac{p}{T}\right)^2\left[1-\frac{p}{T}\right]+\frac{3}{40}\left(\frac{p}{T}\right)^4\left[1-\frac{8p}{9T}\right]+\cdots\right]$$

$$\lim_{T\to\infty}\hat{\alpha}_{EBS1} < \lim_{T\to\infty}\hat{\alpha}_{EBS2} < \lim_{T\to\infty}\hat{\alpha}_{EBS3}$$

$$\lim_{T\to\infty}\hat{\alpha}_{EBS1} = \lim_{T\to\infty}\frac{(2n+1)}{p}\left[\frac{p}{2T}-\frac{p^2}{6T^2}+\frac{p^3}{12T^3}-\frac{p^4}{20T^4}+\cdots\right] = 0$$

$$\lim_{T\to\infty}\hat{\alpha}_{EBS2} = \lim_{T\to\infty}\frac{(2n+1)}{2p}\left[\frac{p}{T}-\frac{p^2}{2T^2}+\frac{p^3}{3T^3}-\frac{p^4}{4T^4}+\cdots\right] = 0$$

$$\lim_{T\to\infty}\hat{\alpha}_{EBS3} = \lim_{T\to\infty}\frac{(2n+1)}{p}\left[\frac{p}{2T}-\frac{p^2}{3T^2}+\frac{p^3}{4T^3}-\frac{p^4}{5T^4}+\cdots\right] = 0$$

$$\lim_{T\to\infty}\hat{\alpha}_{EBSi} = 0, \qquad i = 1,2,3$$

1.10 E-Bayes For Entropy loss function

$$\hat{\alpha}_{EBE1} < \hat{\alpha}_{EBE2} < \hat{\alpha}_{EBE3}$$

$$\lim_{T\to\infty}\hat{\alpha}_{EBE1} < \lim_{T\to\infty}\hat{\alpha}_{EBE2} < \lim_{T\to\infty}\hat{\alpha}_{EBE3}$$

$$\lim_{T\to\infty} \hat{\alpha}_{EBEi} = 0\,, \qquad i = 1,2,3$$

1.11 E-Bayes for Weighted Balance loss function

$$\hat{\alpha}_{EBW1} < \hat{\alpha}_{EBW2} < \hat{\alpha}_{EBW3}$$

$$\lim_{T\to\infty} \hat{\alpha}_{EBW1} < \lim_{T\to\infty} \hat{\alpha}_{EBW2} < \lim_{T\to\infty} \hat{\alpha}_{EBW3}$$

$$\lim_{T\to\infty} \hat{\alpha}_{EBWi} = 0\,, \qquad i = 1,2,3$$

1.12 E-Bayes for Minimum Expected loss function

$$\hat{\alpha}_{EBM1} < \hat{\alpha}_{EBM2} < \hat{\alpha}_{EBM3}$$

$$\lim_{T\to\infty} \hat{\alpha}_{EBM1} < \lim_{T\to\infty} \hat{\alpha}_{EBM2} < \lim_{T\to\infty} \hat{\alpha}_{EBM3}$$

$$\lim_{T\to\infty} \hat{\alpha}_{EBMi} = 0\,, \qquad i = 1,2,3$$

b) Relation between Hierarchical estimators -

1.13 For SELF

$$\lim_{T\to\infty} \hat{\alpha}_{HSi} = 0\,, \qquad i = 1,2,3$$

$$\hat{\alpha}_{HS1} = \frac{\int_0^1 \int_0^p \frac{b^a (\mathrm{p-b}) \lceil a+n+1}{\lceil a (T+b)^{a+n+1}} \mathrm{dadb}}{\int_0^1 \int_0^p \frac{b^a (\mathrm{p-b}) \lceil (a+n)}{\lceil a (T+b)^{a+n}} \mathrm{dadb}}$$

Here $\lceil a + n + 1 = (a+n)\lceil a + n$

$$\int_0^1 \int_0^p \frac{b^a (\mathrm{p-b}) \lceil a+n+1}{\lceil a (T+b)^{a+n+1}} \mathrm{dadb} = \int_0^1 \int_0^p \frac{b^a (\mathrm{p-b}) (a+n) \lceil a+n}{\lceil a (T+b)^{a+n+1}} \mathrm{dadb}$$

For $a \in (0,1),\ b \in (0,p),\ \lceil (a+n)(T+b)^{-1}$ is continuous and $\frac{b^a \lceil a+n}{\lceil a (T+b)^{a+n}} > 0$

Hence by using generalised mean value theorem , we can find at least number $a_1 \in (0,1)$, $b_1 \in (0,p)$

$$\int_0^1 \int_0^p \frac{b^a (\mathrm{p-b}) \lceil a+n+1}{\lceil a (T+b)^{a+n+1}} \mathrm{dadb} = \frac{(n+a_1)}{(T+b_1)} \int_0^1 \int_0^p \frac{b^a (\mathrm{p-b}) \lceil a+n}{\lceil a (T+b)^{a+n}} \mathrm{dadb}$$

$$\hat{\alpha}_{HS1} = \frac{n + a_1}{T + b_1}$$

$$\lim_{T\to\infty} \hat{\alpha}_{HS1} = 0$$

Similarly

$$\hat{\alpha}_{HS2} = \frac{\int_0^1 \int_0^p \frac{b^a \lceil a+n+1}{\lceil a(T+b)^{a+n+1}} \text{dadb}}{\int_0^1 \int_0^p \frac{b^a \lceil (a+n)}{\lceil a(T+b)^{a+n}} \text{dadb}}$$

$for\ a_2 \in (0,1), \qquad b_2 \in (0,p)$

$$\hat{\alpha}_{HS2} = \frac{n+a_2}{T+b_2}$$

$$\lim_{T\to\infty} \hat{\alpha}_{HS2} = 0$$

Similarly

$$\hat{\alpha}_{HS2} = \frac{\int_0^1 \int_0^p \frac{b^{a+1} \lceil a+n+1}{\lceil a(T+b)^{a+n+1}} \text{dadb}}{\int_0^1 \int_0^p \frac{b^{a+1} \lceil (a+n)}{\lceil a(T+b)^{a+n}} \text{dadb}}$$

$for\ a_3 \in (0,1), \qquad b_3 \in (0,p)$

$$\hat{\alpha}_{HS3} = \frac{n+a_3}{T+b_3}$$

$$\lim_{T\to\infty} \hat{\alpha}_{HS3} = 0$$

1.14 For Entropy loss function

$$\lim_{T\to\infty} \hat{\alpha}_{HEi} = 0 \qquad i = 1,2,3$$

1.15 for Weighted Balance loss function

$$\lim_{T\to\infty} \hat{\alpha}_{HWi} = 0 \qquad i = 1,2,3$$

1.16 for Minimum Expected loss function

$$\lim_{T\to\infty} \hat{\alpha}_{HMi} = 0 \qquad i = 1,2,3$$

5. Simulation Study

In this section, R-code has been used to conduct a Simulation study to compare different E-Bayesian and Hierarchical Bayesian estimates for Gamma Prior under 4 loss functions for the parameter of distribution.

Steps to perform Simulation -

I. Initial value of $\alpha\ and \theta$ have been taken as –

$\alpha = 1 \ , \theta = 1$

II. Value of hyper parameters and p are also assumed as –

$p = 0.5 \ , a = 0.2 \ , b = 0.3$

III. We generate random sample of different sizes 5 , 10 , 20 , 40 , 50 , 100 , 200 , 400 , 500 , 1000 from quantile function of distribution

IV. Find the mean and MSE values for estimators

V. Repeat steps upto 10000 times.

N = 10000 and n = 5 , 10 , 20 , 40 , 50 , 100 , 200 , 400 , 500 , 1000

And Quantile function -

$$x = [\frac{(\theta + 1)\ln(\frac{1}{1-z})}{\alpha}]^{1/(\theta+1)}$$

where z ~ U(0 , 1)

Table 1. E-Bayes estimates of α and their MSE for Square Error Loss Function

n	SELF		
	$\hat{\alpha}_{EBS1}$	$\hat{\alpha}_{EBS2}$	$\hat{\alpha}_{EBS3}$
5	1.30093 (0.54686)	1.27033 (0.48003)	1.23974 (0.41830)
10	1.14138 (0.17622)	1.13006 (0.16621)	1.11875 (0.15663)
20	1.06842 (0.06447)	1.06358 (0.06270)	1.05875 (0.06099)
40	1.03273 (0.02768)	1.03049 (0.02730)	1.02825 (0.02694)
50	1.02797 (0.02247)	1.02619 (0.02222)	1.02442 (0.02198)
100	1.01441 (0.01069)	1.01355 (0.01063)	1.01269 (0.01057)
200	1.0076 (0.00517)	1.00717 (0.00515)	1.00675 (0.00514)
400	1.00394 (0.00253)	1.00373 (0.00253)	1.00352 (0.00252)
500	1.00313 (0.00203)	1.00296 (0.00203)	1.00280 (0.00203)
1000	1.00135 (0.00101)	1.00127 (0.00101)	1.00118 (0.00101)

Table 2. E-Bayes estimates of α and their MSE for Entropy Loss Function

n	ENTROPY		
	$\hat{\alpha}_{EBE1}$	$\hat{\alpha}_{EBE2}$	$\hat{\alpha}_{EBE3}$
5	1.06440 (0.30961)	1.03936 (0.27397)	1.01433 (0.24175)
10	1.03268 (0.12896)	1.02244 (0.12271)	1.01220 (0.11682)
20	1.01630 (0.05436)	1.01170 (0.05321)	1.00710 (0.05216)
40	1.00723 (0.02536)	1.00505 (0.02511)	1.00287 (0.02487)
50	1.00761 (0.02090)	1.00587 (0.02073)	1.00413 (0.02057)
100	1.00432 (0.01029)	1.00346 (0.01025)	1.00261 (0.01021)
200	1.00257 (0.00507)	1.00215 (0.00506)	1.00173 (0.00505)
400	1.00144 (0.00251)	1.00123 (0.00250)	1.00102 (0.00250)
500	1.00113 (0.00201)	1.00096 (0.00201)	1.00079 (0.00201)
1000	1.00035 (0.00101)	1.00027 (0.00101)	1.00018 (0.00101)

Table 3. E-Bayes estimates of α and their MSE for Weighted Balance loss function

n	WBLF		
	$\hat{\alpha}_{EBW1}$	$\hat{\alpha}_{EBW2}$	$\hat{\alpha}_{EBW3}$
5	1.53746 (0.92618)	1.50130 (0.81969)	1.46515 (0.72032)
10	1.25009 (0.22995)	1.23769 (0.23558)	1.22529 (0.22173)
20	1.12054 (0.08029)	1.11546 (0.07785)	1.11039 (0.07548)
40	1.05823 (0.03133)	1.05593 (0.03082)	1.05364 (0.03033)
50	1.04832 (0.02489)	1.04651 (0.02456)	1.04470 (0.02424)
100	1.02450 (0.01129)	1.02364 (0.01122)	1.02277 (0.01114)
200	1.01262 (0.00532)	1.01219 (0.00530)	1.01177 (0.00528)
400	1.00645 (0.00257)	1.00624 (0.00257)	1.00603 (0.00256)
500	1.00514 (0.00206)	1.00497 (0.00206)	1.00480 (0.00205)
1000	1.00235 (0.00101)	1.00227 (0.00101)	1.00218 (0.00101)

Table 4. E-Bayes estimates of α and their MSE for Minimum Expected loss function

n	MELF		
	$\hat{\alpha}_{EBM1}$	$\hat{\alpha}_{EBM2}$	$\hat{\alpha}_{EBM3}$
5	0.82786 (0.21442)	0.80839 (0.20151)	0.78893 (0.19067)
10	0.92398 (0.10816)	0.91482 (0.10509)	0.90565 (0.10230)
20	0.96418 (0.04997)	0.95982 (0.04939)	0.95545 (0.04885)
40	0.98173 (0.02438)	0.97960 (0.02425)	0.97747 (0.02413)
50	0.98725 (0.02017)	0.98555 (0.02008)	0.98384 (0.01999)
100	0.99422 (0.01010)	0.99338 (0.01008)	0.99254 (0.01006)
200	0.99754 (0.00502)	0.99712 (0.00501)	0.99670 (0.00500)
400	0.99893 (0.00249)	0.99872 (0.00249)	0.99851 (0.00249)
500	0.99912 (0.00200)	0.99896 (0.00200)	0.99879 (0.00201)
1000	0.99935 (0.00101)	0.99927 (0.00101)	0.99918 (0.00101)

Table 5. Hierarchical Bayes estimates of α and their MSE for Squared error loss function

n	SELF		
	$\hat{\alpha}_{HS1}$	$\hat{\alpha}_{HS2}$	$\hat{\alpha}_{HS3}$
5	1.30616 (0.56621)	1.28252 (0.51182)	1.26221 (0.46023)
10	1.14318 (0.17841)	1.13470 (0.17025)	1.12804 (0.16346)
20	1.06921 (0.06485)	1.06569 (0.06342)	1.06302 (0.06231)
40	1.03310 (0.02776)	1.03150 (0.02746)	1.03031 (0.02724)
50	1.02826 (0.02253)	1.02700 (0.02233)	1.02605 (0.02218)
100	1.01456 (0.01071)	1.01395 (0.01066)	1.01350 (0.01062)
200	1.00767 (0.00517)	1.00737 (0.00516)	1.00715 (0.00515)
400	1.00398 (0.00253)	1.00383 (0.00253)	1.00372 (0.00253)
500	1.00316 (0.00203)	1.00304 (0.00203)	1.00296 (0.00203)
1000	1.00138 (0.00101)	1.00130 (0.00101)	1.00122 (0.00101)

Table 6. Hierarchical Bayes estimates of α and their MSE for Entropy loss function

n	ENTROPY		
	$\hat{\alpha}_{HE1}$	$\hat{\alpha}_{HE2}$	$\hat{\alpha}_{HE3}$
5	1.06730 (0.31921)	1.04849 (0.28917)	1.03302 (0.26239)
10	1.03409 (0.13042)	1.02661 (0.12520)	1.02081 (0.12090)
20	1.01701 (0.05464)	1.01371 (0.05368)	1.01123 (0.05293)
40	1.00758 (0.02542)	1.00604 (0.02522)	1.00489 (0.02506)
50	1.00789 (0.02094)	1.00666 (0.02080)	1.00575 (0.02069)
100	1.00447 (0.01031)	1.00387 (0.01027)	1.00342 (0.01024)
200	1.00264 (0.00507)	1.00235 (0.00506)	1.00213 (0.00506)
400	1.00147 (0.00251)	1.00132 (0.00250)	1.00122 (0.00250)
500	1.00116 (0.00201)	1.00104 (0.00201)	1.00095 (0.00201)
1000	1.00037 (0.00101)	1.00029 (0.00101)	1.00020 (0.00101)

Table 7. Hierarchical Bayes estimates of α and their MSE for Weighted balance loss function

n	WBLF		
	$\hat{\alpha}_{HW1}$	$\hat{\alpha}_{HW2}$	$\hat{\alpha}_{HW3}$
5	1.54498 (0.96079)	1.51678 (0.87638)	1.49143 (0.79128)
10	1.25223 (0.25308)	1.24279 (0.24149)	1.23523 (0.23168)
20	1.12140 (0.08079)	1.11767 (0.07886)	1.11481 (0.07734)
40	1.05862 (0.03144)	1.05697 (0.03104)	1.05573 (0.03074)
50	1.04863 (0.02496)	1.04733 (0.02471)	1.04636 (0.02451)
100	1.02465 (0.01131)	1.02404 (0.01125)	1.02358 (0.01121)
200	1.01269 (0.00533)	1.01239 (0.00531)	1.01217 (0.00530)
400	1.00649 (0.00257)	1.00634 (0.00257)	1.00623 (0.00256)
500	1.00516 (0.00206)	1.00505 (0.00206)	1.00496 (0.00205)
1000	1.00236 (0.00101)	1.00228 (0.00101)	1.00220 (0.00101)

Table 8. Hierarchical Bayes estimates of α and their MSE for Minimum Expected loss function

n	MELF		
	$\hat{\alpha}_{HM1}$	$\hat{\alpha}_{HM2}$	$\hat{\alpha}_{HM3}$
5	0.82805 (0.21914)	0.81433 (0.20697)	0.80352 (0.19677)
10	0.92495 (0.10911)	0.91849 (0.10631)	0.91355 (0.10401)
20	0.96480 (0.05016)	0.96173 (0.04962)	0.95943 (0.04919)
40	0.98206 (0.02442)	0.98057 (0.02430)	0.97946 (0.02421)
50	0.98752 (0.02020)	0.98633 (0.02011)	0.98544 (0.02005)
100	0.99436 (0.01011)	0.99377 (0.01009)	0.99333 (0.01007)
200	0.99761 (0.00502)	0.99732 (0.00501)	0.99710 (0.00501)
400	0.99896 (0.00249)	0.99882 (0.00249)	0.99871 (0.00249)
500	0.99915 (0.00201)	0.99903 (0.00201)	0.99895 (0.00201)
1000	0.99936 (0.00101)	0.99928 (0.00101)	0.99920 (0.00101)

6. Conclusion

In this section we conclude the findings of the study. It is shown in study that complexity of integration of Hierarchical Bayesian method is eliminated in E-Bayesian Method.

In table 1, 2, 3 and 4 E-Bayes estimates are given for different values of n. In table 5, 6, 7 and 8 Hierarchical Bayes estimates are given for different values of n.

For small samples MSE of H-Bayesian is greater than MSE of E-Bayesian. Hence E-Bayesian is preferable and efficient method to use. For large samples MSE of H-Bayesian is Equivalent to MSE of E-Bayesian but due to complexity of integration makes H-Bayesian method difficult.

Hence E-Bayesian is preferable method to use. Also, the MSE under MELF is minimum compared to SELF, Entropy and WBLF. Hence MELF is most suitable loss function than other three loss functions.

Conflict of Interest

The authors declare that they have no conflict of interest.

Statement and Declaration

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