Journal of Rajasthan Academy of Physical Sciences
ISSN: 0972-6306; URL: http://raops.org.in
Vol.24, No.1&2, January-June, 2025, 140-153

P-K MULTI-INDEX MITTAG LEFFLER FUNCTION

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Abstract: In this paper, we introduce a new generalised $\boldsymbol{p-k}$ multi-index Mittag Leffler function and derive the integral representation of "$\boldsymbol{p-k}$ multi-index Mittag Leffler function". We also establish its relation with other functions, obtain Laplace, k-Laplace and Mellin transforms of the function and explain its basic properties.

Mathematics Subject Classification (2020) : 33E12.

Keywords: k-Integral Transform, $p-k$ Pochhammer and Gamma function.

1. Introduction

The main aim of this paper is to introduce $p-k$ multi-index Mittag Leffler function and find its integral representative, relation with other functions and integral transform of it.

Section 2 describes definitions of $p-k$ Pochhammer symbol, $p-k$ Gamma function, Mittag Leffler function, $p-k$ Mittag Leffler function, multi-index Mittag Leffler function.

Section 3 Introduces new $p-k$ multi-index Mittag Leffler function, its integral representation, various integral transform of the $p-k$ multi-index Mittag Leffler function and its some basic properties.

2. Preliminaries

2.1 Definition

Two parameter Pochhammer symbol introduced by Gehlot [3] in the year 2017.

$$ {}_p(x)_{n,k} = (\tfrac{px}{k})(\tfrac{px}{k}+p)(\tfrac{px}{k}+2p)\ldots(\tfrac{xp}{k}+(n-1)p), \tag{1} $$

where $x \in \mathbb{C}$; $k, p \in R^{+}-(0)$ and $Re(x) > 0$; $n \in \mathbb{N}$.

Two parameter Gamma function introduced by Gehlot [3] in the year 2017.

$$ {}_p\Gamma_k(x) = \frac{1}{k} lim_{\to\infty} \frac{n! p^{n+1}(np)^{\frac{x}{k}}}{{}_p(x)_{n+1,k}} \tag{2} $$

The **Mittag- Leffler function** $E_\alpha(z)$ introduced by Leffler [10] in 1903, is defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} \quad (3)$$

where $z \in \mathbb{C},\ \alpha \geq 0$

Wiman function introduced by Wiman [18] in the year 1905 is generalisation of $E_\alpha(z)$ and is defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}. \quad (4)$$

Here $z,\alpha,\beta \in \mathbb{C};\ Re(\alpha) > 0,\ Re(\beta) > 0.$

Prabhakar [12] introduced function $E_{\alpha,\beta}^{\gamma}(z)$ in 1971 in the form of

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(n\alpha+\beta)} \frac{z^n}{n!}, \quad (5)$$

where $z,\alpha,\beta,\gamma \in C;\ Re(\alpha) > 0,\ Re(\beta) > 0,\ Re(\gamma) > 0$ and $(\gamma)_n$ is the Pochhammer symbol.

In 2007, Shukla and Prajapati [15] introduced more **generalised Mittag Leffler function** $E_{\alpha,\beta}^{\gamma,q}(z)$ and defined as

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma(n\alpha+\beta)} \frac{z^n}{n!}, \quad (6)$$

where $\alpha,\beta,\gamma \in C;\ Re(\alpha) > 0,\ Re(\beta) > 0,\ Re(\gamma) > 0$ and $q \in (0,1) \cup N.$

Gehlot [4] introduced **k-Mittag Leffler function** in the year 2012. The k-Mittag Leffler function is expressed as

$$GE_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k} z^n}{\Gamma_k(n\alpha+\beta)(n!)}, \quad (7)$$

where $k \in \mathbb{R}\ \ z,\alpha,\beta \in \mathbb{C};\ Re(\alpha) > 0,\ Re(\beta) > 0$ and $q \in (0,1) \cup N$, $(\gamma)_{nq,k}$ is the k-Pochhammer symbol and $\Gamma_k(x)$ is the k-Gamma function given by [1].

Kiryakova [9] in the year 2010 introduced and studied a class of special function of Mittag Leffler type that are **multi-index** analogues of $E_{\alpha,\beta}$ by changing $\alpha = \frac{1}{\rho},\ \beta = \mu$ by two sets of indices $(\alpha = \frac{1}{\rho_1}, \frac{1}{\rho_2} \cdots \frac{1}{\rho_m}), (\beta = \mu_1, \mu_2, \cdots \mu_m)$ for integer $m > 1$ and $\rho_1, \rho_2, \cdots \rho_m > 0$ and $\mu_1, \mu_2, \cdots \mu_m$ are arbitrary real (complex) numbers.The multi-index Mittag Leffler functions are defined as

$$E_{(\frac{1}{\rho_i}),(\mu_i)}(z) = E_{(\frac{1}{\rho_i}),(\mu_i)}^{(m)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}). \quad (8)$$

The same functions have also been studied by Luchko [11], called by him Mittag Leffler function of vector index.

Saxena and Nishimotto [14] studied in the year 2010 the **generalised multi-index Mittag Leffler function** and defined as

$$E_{\gamma,k}[(\alpha_1,\beta_1)\cdots(\alpha_m,\beta_m)z] = \sum_{k=0}^{\infty} \frac{(\gamma)_{kn} z^k}{\prod_{j=1}^{m} \Gamma(n\alpha_j+\beta_j) n!}, \tag{9}$$

where $\alpha_j, \beta_j, \gamma, k \in \mathbb{C}$, $Re(\alpha_j) > 0$, $Re(\beta_j) > 0$, $(j = 1, \cdots, m)$, $Re(k) > 0$, $Re(\sum_{j=1}^{m} \alpha_j) > Max[0, k-1]$.

Wright generalised Hypergeometric function [10]

$${}_p\psi_q \left[\begin{matrix}(\alpha_1, A_1)\cdots(\alpha_p, A_p)\\ (\beta_1, B_1)\cdots(\beta_q, B_q)\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + A_i n) z^n}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n)(n!)}. \tag{10}$$

$${}_p\psi_q \left[\begin{matrix}(\alpha_1, A_1)\cdots(\alpha_p, A_p)\\ (\beta_1, B_1)\cdots(\beta_q, B_q)\end{matrix}; z\right] = H_{p,q+1}^{1,p}\left[-z\middle|\begin{matrix}(1-\alpha_1, A_1)\cdots(1-\alpha_p, A_p)\\ (0,1)(1-\beta_1, B_1)\cdots(1-\beta_q, B_q)\end{matrix}\right], \tag{11}$$

where $H_{p,q}^{m,n}[.]$ denotes the fox $H-$ function.

Euler Beta transform [17],

$$B[f(t){:}\, m, n] = \int_0^1 t^{m-1}(1-t)^{n-1} f(t)\, dt. \tag{12}$$

Laplace transform ([16],equation 3.1.1)

$$L[f(t){:}\, s] = \int_0^{\infty} e^{-st} f(t)\, dt. \tag{13}$$

Mellin Transform,([16], equation 4.1.1)

$$M[f(t){:}\, s] = \int_0^{\infty} t^{s-1} f(t)\, dt = f^*(s), \quad \mathrm{Re}(s) > 0. \tag{14}$$

$$f(t) = M^{-1}[f^*(s); t] = \frac{1}{2\pi i}\int f^*(s) t^{-s}\, ds. \tag{15}$$

3. $p-k$ multi-index Mittag Leffler function

3.1 Definition

$p-k$ multi-index Mittag Leffler function The new $p-k$ multi-index Mittag Leffler function is generalisation of multi-index Mittag Leffler function. It is defined by [5]

$${}_pE_k^{\gamma,q}[(\mu_1, \tfrac{1}{\rho_1})(\mu_2, \tfrac{1}{\rho_2})\cdots(\mu_m, \tfrac{1}{\rho_m}); z] = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(\mu_1 + \frac{n}{\rho_1})\cdots {}_p\Gamma_k(\mu_m + \frac{n}{\rho_m})} \frac{z^n}{n!}. \tag{16}$$

Where z is any complex variable, $m \geq 1$ and $m \in \mathbb{Z}$ and $\mu_1, \mu_2, \cdots \mu_m$ are arbitrary real (complex) numbers and $Re(\mu_i) > 0 \quad \forall i = 1,2\cdots m$; $q \in \mathbb{N}$; $k, p \in \mathbb{R}^+ - (0)$, $\gamma \in C/Z^-$, $\rho_1, \rho_2, \cdots \rho_m > 0$ and $\frac{1}{\rho_1}, \frac{1}{\rho_2}, \cdots \frac{1}{\rho_m} \in \mathbb{R}$

The convergence criteria of $p-k$ multi-index Mittag Leffler function is

(i) the function converges absolutely for all $z \in \mathbb{C}$ if $(\frac{1}{k\rho_1} + \frac{1}{k\rho_2} + \cdots + \frac{1}{k\rho_m} + 1) > q$.

(ii) if $(\frac{1}{k\rho_1} + \frac{1}{k\rho_2} + \cdots + \frac{1}{k\rho_m}) + 1 = q$ then the series is absolutely convergent for $|z| < \frac{\prod_{i=1}^{m} (\frac{1}{k\rho_i})^{\frac{1}{k\rho_i}}}{q^q p^{(q - \frac{1}{\rho_1 k} - \frac{1}{\rho_2 k} \cdots - \frac{1}{\rho_m k})}}$, then

(iii) if $(\frac{1}{k\rho_1} + \frac{1}{k\rho_2} + \cdots + \frac{1}{k\rho_m}) + 1 = q$ and $|z| = \frac{\prod_{i=1}^{m} (\frac{1}{k\rho_i})^{\frac{1}{k\rho_i}}}{q^q p^{(q - \frac{1}{\rho_1 k} - \frac{1}{\rho_2 k} \cdots - \frac{1}{\rho_m k})}}$, then the function is convergent when $\sum_{j=1}^{m} (\frac{\mu_j}{k}) + \frac{\gamma}{k} > \frac{m}{2}$

Particular cases: for some particular values of parameters of equation (16)

(a) For $m = 1,\ \mu_1 = \beta,\ \rho_1 = \frac{1}{\alpha}$ equation (16) is reduced to $p - k$ Mittag Leffler function defined by Gehlot [5]

$$ {}_pE_k^{\gamma,q}[(\beta,\alpha);(z)] = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^n}{n!} = {}_p\ E_{k,\alpha,\beta}^{\gamma,q}(z). \quad (17)$$

(b) For $p = k,\ m = 1,\ \mu_1 = \beta,\ \rho_1 = \frac{1}{\alpha}$ equation (16) is reduced to generalised k Mittag Leffler function defined by Gehlot [4]

$$ {}_kE_k^{\gamma,q}[(\beta,\alpha);(z)] = \sum_{n=0}^{\infty} \frac{{}_k(\gamma)_{nq,k}}{{}_k\Gamma_k(n\alpha+\beta)} \frac{z^n}{n!} = GE_{k,\alpha,\beta}^{\gamma,q}(z). \quad (18)$$

(c) For $p = k$,and $k = 1$ $m = 1$, $\mu_1 = \beta$, $\rho_1 = \frac{1}{\alpha}$ equation (16) is reduced to equation (6) defined by [15]

$$ {}_1E_1^{\gamma,q}[(\beta,\alpha);(z)] = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma(n\alpha+\beta)} \frac{z^n}{n!} = E_{\alpha,\beta}^{\gamma,q}(z). \quad (19)$$

Special case of the equation (19) when we substitute $p = k$, $k = 1$, $\gamma = \frac{\gamma}{\xi}$; $\beta = \frac{\mu_1}{\xi}, \frac{\mu_2}{\xi}, \cdots \frac{\mu_m}{\xi}$ and $\alpha = \frac{1}{\xi\rho_1}, \frac{1}{\xi\rho_2}, \cdots \frac{1}{\xi\rho_m}$, in the eqation (16) , then

$$ {}_1E_1^{(\frac{\gamma}{\xi}),q}[(\frac{\mu_1}{\xi}, \frac{1}{\xi\rho_1}) \cdots (\frac{\mu_m}{\xi}, \frac{1}{\xi\rho_m}); z] = \sum_{n=0}^{\infty} \frac{(\frac{\gamma}{\xi})_{nq}}{\Gamma(\frac{\mu_1}{\xi} + \frac{n}{\xi\rho_1})\Gamma(\frac{\mu_2}{\xi} + \frac{n}{\xi\rho_2}) \cdots \Gamma(\frac{\mu_m}{\xi} + \frac{n}{\xi\rho_m})} \frac{z^n}{n!}$$

$$ = E^{(\frac{\gamma}{\xi}),q}[(\frac{\mu_1}{\xi}, \frac{1}{\xi\rho_1}) \cdots (\frac{\mu_m}{\xi}, \frac{1}{\xi\rho_m}); z]. \quad (20)$$

(d) For $p = k$,and $k = 1, q = 1,\ \gamma = 1,\ m = 1,\ \mu_1 = 1,\ \rho_1 = \frac{1}{\alpha}$ equation (21) is reduced to Mittag Leffler function defined by [10]

$$ {}_1E_1^{1,1}[(1,\alpha);(z)] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)} = E_\alpha(z). \quad (21)$$

3.2 Integral Representation of p-k multi-index Mittag Leffler function

Theorem 3.1 Let $m \geq 1$ and $m \in \mathbb{Z}$ and $\mu_1, \mu_2, \cdots \mu_m$ are arbitrary real (complex) numbers and $Re(\mu_i) > 0 \quad \forall i = 1,2 \cdots m;\ q \in \mathbb{N};\ k, p \in \mathbb{R}^+ - (0),\ \gamma \in C/Z^-,\ \rho_1, \rho_2, \cdots \rho_m > 0$ and $\frac{1}{\rho_1}, \frac{1}{\rho_2}, \cdots \frac{1}{\rho_m} \in \mathbb{R}$ then the function ${}_pE_k^{\gamma,q}[(\mu_1, \frac{1}{\rho_1})(\mu_2, \frac{1}{\rho_2}) \cdots (\mu_m, \frac{1}{\rho_m}); z]$ is represented by the Mellin-Barnes integral [2] as

$${}_pE_k^{\gamma,q}\left[\left(\mu_1, \frac{1}{\rho_1}\right)\left(\mu_2, \frac{1}{\rho_2}\right) \cdots \left(\mu_m, \frac{1}{\rho_m}\right); z\right] =$$

$$\frac{(k^m)p^{-(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k})}}{2\pi i\ \Gamma\frac{\gamma}{k}} \int_L \frac{\Gamma s\ \Gamma\left(\frac{\gamma}{k}-qs\right)(-z)^{-s}\ p^{(q-\frac{1}{\rho_1 k}-\frac{1}{\rho_2 k}\cdots-\frac{1}{\rho_m k})^{-s}}}{\Gamma\left(\frac{\mu_1}{k}-\frac{s}{\rho_1 k}\right)\Gamma(\frac{\mu_2}{k}-\frac{s}{\rho_2 k})\cdots\Gamma(\frac{\mu_m}{k}-\frac{s}{\rho_m k})}\ ds. \tag{22}$$

Where $|arg \quad z| < \pi$ The contour integration is from $-i\infty$ to $+i\infty$. It separates the poles of integrand as $s = -n \quad \forall\, n \in N_0$ (to the left) from those $s = \frac{\frac{\gamma}{k}+n}{q} \quad \forall\, n \in N_0$ (to the right)

Proof: Taking R.H.S. of the equation (22)

$$= \frac{(k^m)p^{-(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k})}}{2\pi i\Gamma\frac{\gamma}{k}} \int_L \frac{\Gamma s\ \Gamma\left(\frac{\gamma}{k}-qs\right)\ (-z)^{-s}\ p^{(q-\frac{1}{\rho_1 k}-\frac{1}{\rho_2 k}\cdots-\frac{1}{\rho_m k})^{-s}}}{\Gamma(\frac{\mu_1}{k}-\frac{s}{\rho_1 k})\Gamma(\frac{\mu_2}{k}-\frac{s}{\rho_2 k})\cdots\Gamma(\frac{\mu_m}{k}-\frac{s}{\rho_m k})}\ ds$$

Using the theorem of calculus of residue

$= 2\pi i[$ sum of residues at the poles $s = 0, -1, -2 \cdots]$

$$= \frac{(k^m 2\pi i)p^{-(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k})}}{2\pi i\Gamma\frac{\gamma}{k}} \sum_{n=0}^{\infty} lim_{s\to -n} \frac{(s+n)\ \Gamma s\ \Gamma\left(\frac{\gamma}{k}-qs\right)\ (-z)^{-s}\ p^{(q-\frac{1}{\rho_1 k}-\frac{1}{\rho_2 k}\cdots-\frac{1}{\rho_m k})^{-s}}}{\Gamma\left(\frac{\mu_1}{k}-\frac{s}{\rho_1 k}\right)\Gamma(\frac{\mu_2}{k}-\frac{s}{\rho_2 k})\cdots\Gamma(\frac{\mu_m}{k}-\frac{s}{\rho_m k})}$$

$$= \frac{(k^m)p^{-(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k})}}{\Gamma\frac{\gamma}{k}} \sum_{n=0}^{\infty} lim_{s\to -n} \frac{\pi\ (s+n)\ \Gamma\left(\frac{\gamma}{k}-qs\right)\ (-z)^{-s}\ p^{(q-\frac{1}{\rho_1 k}-\frac{1}{\rho_2 k}\cdots-\frac{1}{\rho_m k})^{-s}}}{\sin\pi s\ \Gamma(1-s)\ \Gamma\left(\frac{\mu_1}{k}-\frac{s}{\rho_1 k}\right)\Gamma(\frac{\mu_2}{k}-\frac{s}{\rho_2 k})\cdots\Gamma(\frac{\mu_m}{k}-\frac{s}{\rho_m k})}$$

$$= \frac{k^m}{\Gamma\frac{\gamma}{k}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma\left(\frac{\gamma}{k}+qn\right)\ (-z)^n\ p^{qn}\ p^{-(\frac{\mu_1}{k}+\frac{n}{\rho_1 k})} p^{-(\frac{\mu_2}{k}+\frac{n}{\rho_2 k})} \ldots p^{-(\frac{\mu_m}{k}+\frac{n}{\rho_m k})}}{\Gamma(\frac{\mu_1}{k}+\frac{n}{\rho_1 k})\Gamma(\frac{\mu_2}{k}+\frac{n}{\rho_2 k})\cdots\Gamma(\frac{\mu_m}{k}+\frac{n}{\rho_m k})}$$

Using the equations (2.19) and (2.20) of [3] and the equation (16), we get

$$={}_p\ E_k^{\gamma,q}[(\mu_1, \frac{1}{\rho_1})(\mu_2, \frac{1}{\rho_2}) \cdots (\mu_m, \frac{1}{\rho_m}); z]$$

Hence the result.

3.3 Relation of p-k multi-index Mittag Leffler function with other functions

Relation of $p-k$ multi-index Mittag Leffler function with Fox H-function

Let $m \geq 1$ and $m \in \mathbb{Z}$ and $\mu_1, \mu_2, \cdots \mu_m$ are arbitrary real (complex) numbers and $Re(\mu_i) > 0 \quad \forall\, i = 1,2 \cdots m;\ q \in \mathbb{N};\ k, p \in \mathbb{R}^+ - (0),\ \gamma \in C/Z^-,\ \rho_1, \rho_2, \cdots \rho_m > 0$ and $\frac{1}{\rho_1}, \frac{1}{\rho_2}, \cdots \frac{1}{\rho_m} \in \mathbb{R}$

Using the equation (22)

$$ {}_pE_k^{\gamma,q}\left[\left(\mu_1, \frac{1}{\rho_1}\right)\left(\mu_2, \frac{1}{\rho_2}\right)\cdots\left(\mu_m, \frac{1}{\rho_m}\right); z\right] = $$

$$ \frac{(k^m)\, p^{-\left(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k}\right)}}{2\pi i \Gamma \frac{\gamma}{k}} \int_L \frac{\Gamma s\, \Gamma\left(\frac{\gamma}{k} - qs\right)\ (-z)^{-s} p^{\left(q - \frac{1}{\rho_1 k} - \frac{1}{\rho_2 k} \cdots - \frac{1}{\rho_m k}\right)^{-s}}}{\Gamma\left(\frac{\mu_1}{k} - \frac{s}{\rho_1 k}\right)\Gamma\left(\frac{\mu_2}{k} - \frac{s}{\rho_2 k}\right)\cdots\Gamma\left(\frac{\mu_m}{k} - \frac{s}{\rho_m k}\right)}\, ds $$

$$ = \frac{(k^m) p^{-\left(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k}\right)}}{\Gamma \frac{\gamma}{k}} H_{1,m+1}^{1,1}\left[-zp^{\left(q - \frac{1}{\rho_1 k} - \frac{1}{\rho_2 k} \cdots - \frac{1}{\rho_m k}\right)} \,\middle|\, \begin{matrix} (1 - \frac{\gamma}{k}, q) \\ (0,1)(1 - \frac{\mu_1}{k}, \frac{1}{\rho_1 k}) \cdots (1 - \frac{\mu_m}{k}, \frac{1}{\rho_m k}) \end{matrix}\right]. \tag{23} $$

Relation of $p-k$ multi-index Mittag Leffler function with Wright function

Let $m \geq 1$ and $m \in \mathbb{Z}$ and $\mu_1, \mu_2, \cdots \mu_m$ are arbitrary real (complex) numbers and $Re(\mu_i) > 0\ \forall\, i = 1,2 \cdots m; q \in \mathbb{N};\ k, p \in \mathbb{R}^+ - (0),\ \gamma \in C/Z^-,\ \rho_1, \rho_2, \cdots \rho_m > 0$ and $\frac{1}{\rho_1}, \frac{1}{\rho_2}, \cdots \frac{1}{\rho_m} \in \mathbb{R}$

Using the equation (22)

$$ {}_pE_k^{\gamma,q}[(\mu_1, \frac{1}{\rho_1})(\mu_2, \frac{1}{\rho_2}) \cdots (\mu_m, \frac{1}{\rho_m}); z] $$

$$ = \frac{(k^m) p^{-\left(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k}\right)}}{\Gamma \frac{\gamma}{k}} H_{1,m+1}^{1,1}\left[-zp^{\left(q - \frac{1}{\rho_1 k} - \frac{1}{\rho_2 k} \cdots - \frac{1}{\rho_m k}\right)} \,\middle|\, \begin{matrix} (1 - \frac{\gamma}{k}, q) \\ (0,1)(1 - \frac{\mu_1}{k}, \frac{1}{\rho_1 k} \cdots (1 - \frac{\mu_m}{k}, \frac{1}{\rho_m k}) \end{matrix}\right] $$

$$ = \frac{(k^m) p^{-\left(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k}\right)}}{\Gamma \frac{\gamma}{k}}\ {}_1\psi_m\left[\begin{matrix} (\frac{\gamma}{k}, q) \\ \left(\frac{\mu_1}{k}, \frac{1}{\rho_1 k}\right) \cdots \left(\frac{\mu_m}{k}, \frac{1}{\rho_m k}\right) \end{matrix} \,\middle|\, zp^{\left(q - \frac{1}{\rho_1 k} - \frac{1}{\rho_2 k} \cdots - \frac{1}{\rho_m k}\right)}\right]. \tag{24} $$

3.4 Integral transform of $p-k$ multi-index Mittag Leffler function

Theorem 3.2

Let $m \geq 1$ and $m \in \mathbb{Z}$ and $\mu_1, \mu_2, \cdots \mu_m$ are arbitrary real (complex) numbers and $Re(\mu_i) > 0 \quad \forall\, i = 1,2 \cdots m$; $q \in \mathbb{N}$; $k, p \in \mathbb{R}^+ - (0)$, $\rho_1, \rho_2, \cdots \rho_m > 0$ and $\frac{1}{\rho_1}, \frac{1}{\rho_2}, \cdots \frac{1}{\rho_m} \in \mathbb{R}$, $a, b, \gamma \in \mathbb{C}/\mathbb{Z}^-$; Re(a) > 0, Re(b) > 0, x,z $\in \mathbb{R}$ and $\sigma \in \mathbb{C}$, the **Euler Beta** transform of $p-k$ multi-index Mittag Leffler function is

$$\int_0^1 z^{a-1}(1-z)^{b-1}\,{}_pE_k^{\gamma,q}[(\mu_1,\frac{1}{\rho_1})(\mu_2,\frac{1}{\rho_2})\cdots(\mu_m,\frac{1}{\rho_m}); xz^\sigma]\,dz$$

$$= \frac{(k^m)\,\Gamma(b)\,p^{-(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k})}}{\Gamma\frac{\gamma}{k}}\,{}_2\psi_{m+1}\left[\begin{matrix}(\frac{\gamma}{k},q)(a,\sigma) \\ (\frac{\mu_1}{k},\frac{1}{\rho_1 k})\cdots(\frac{\mu_m}{k},\frac{1}{\rho_m k})(a+b,\sigma)\end{matrix}\,\Big|\, xp^{(q-\frac{1}{\rho_1 k}-\frac{1}{\rho_2 k}\cdots-\frac{1}{\rho_m k})}\right]. \tag{25}$$

Proof: Using LHS side of the equation (25)

$$= \int_0^1 z^{a-1}(1-z)^{b-1}\,{}_pE_k^{\gamma,q}[(\mu_1,\frac{1}{\rho_1})(\mu_2,\frac{1}{\rho_2})\cdots(\mu_m,\frac{1}{\rho_m}); xz^\sigma]\,dz$$

Using the equation (16), we have

$$= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(\mu_1+\frac{n}{\rho_1})\cdots{}_p\Gamma_k(\mu_m+\frac{n}{\rho_m})}\int_0^1 \frac{(xz^\sigma)^n}{n!} z^{a-1}(1-z)^{b-1}\,dz$$

Order of integration and summation can be interchanged under the conditions of uniform convergence of the series as discussed in the subsection (16).

$$\sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}(x)^n}{{}_p\Gamma_k(\mu_1+\frac{n}{\rho_1})\cdots{}_p\Gamma_k(\mu_m+\frac{n}{\rho_m})(n!)}\int_0^1 z^{(\sigma n+a)-1}(1-z)^{b-1}\,dz$$

$$= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}(x)^n}{{}_p\Gamma_k(\mu_1+\frac{n}{\rho_1})\cdots{}_p\Gamma_k(\mu_m+\frac{n}{\rho_m})(n!)} B(a+\sigma n, b)$$

Using equation (2.19) and (2.20) of [3], we get

$$= \frac{(k^m)\Gamma(b)p^{-(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k})}}{\Gamma\frac{\gamma}{k}}\,{}_2\psi_{m+1}\left[\begin{matrix}(\frac{\gamma}{k},q)(a,\sigma) \\ (\frac{\mu_1}{k},\frac{1}{\rho_1 k})\cdots(\frac{\mu_m}{k},\frac{1}{\rho_m k})(a+b,\sigma)\end{matrix}\,\Big|\, xp^{(q-\frac{1}{\rho_1 k}-\frac{1}{\rho_2 k}\cdots-\frac{1}{\rho_m k})}\right].$$

Hence the result.

Particular Cases

a For $m = 1,\ \mu_1 = \beta,\ \rho_1 = \frac{1}{\alpha}$ equation (25) is reduced to equation (30) defined by Gehlot[6]

$$\int_0^1 z^{a-1}(1-z)^{b-1}{}_pE_k^{\gamma,q}[(\beta,\alpha);xz^{\sigma}]\,dz = \frac{(k)\Gamma(b)p^{-\left(\frac{\beta}{k}\right)}}{\Gamma\frac{\gamma}{k}}\,{}_2\psi_2\left[\begin{matrix}\left(\frac{\gamma}{k},q\right)(a,\sigma)\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right)(a+b,\sigma)\end{matrix}\Big|xp^{\left(q-\frac{\alpha}{k}\right)}.\right] \quad (26)$$

b For $m = 1,\ \mu_1 = \beta,\ \rho_1 = \frac{1}{\alpha}$ and $p = k$ equation (25) is reduced to equation (31) defined by Gehlot [6]

$$\int_0^1 z^{a-1}(1-z)^{b-1}{}_kE_k^{\gamma,q}[(\beta,\alpha);xz^{\sigma}]\,dz = \frac{\Gamma(b)k^{1-\left(\frac{\beta}{k}\right)}}{\Gamma\frac{\gamma}{k}}\,{}_2\psi_2\left[\begin{matrix}\left(\frac{\gamma}{k},q\right)(a,\sigma)\\ \left(\frac{\beta}{k},\frac{\alpha}{k}\right)(a+b,\sigma)\end{matrix}\Big|xk^{\left(q-\frac{\alpha}{k}\right)}\right] \quad (27)$$

Theorem 3.3

Let $m \geq 1$ and $m \in \mathbb{Z}$ and $\mu_1, \mu_2, \cdots \mu_m$ are arbitrary real (complex) numbers and $Re(\mu_i) > 0\ \ \forall\, i = 1,2 \cdots m$; $q \in \mathbb{N}$; $k, p \in \mathbb{R}^+ - (0)$, $\rho_1, \rho_2, \cdots \rho_m > 0$ and $\frac{1}{\rho_1}, \frac{1}{\rho_2}, \cdots \frac{1}{\rho_m} \in \mathbb{R},\ a, \gamma \in \mathbb{C}/\mathbb{Z}^-$, Re(a) > 0; x,z $\in \mathbb{R}$ $\sigma \in \mathbb{C},\ \alpha > 0$ and $l \geq 1$ then **k-Laplace transform** [8] of $p - k$ multi-index Mittag Leffler function is (here k of k Laplace transform is replaced by l to avoid confusion with k of multi-index Mittag Leffler function) is

$$\int_0^{\infty} z^{a-1}e^{-s^{\frac{1}{\alpha}}z^l}{}_pE_k^{\gamma,q}[(\mu_1,\frac{1}{\rho_1})(\mu_2,\frac{1}{\rho_2})\cdots(\mu_m,\frac{1}{\rho_m});xz^{\sigma l}]\,dz$$

$$= \frac{(k^m)s^{-\left(\frac{a}{\alpha l}\right)}p^{-\left(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k}\right)}}{l\Gamma\frac{\gamma}{k}}\,{}_2\psi_m\left[\begin{matrix}\left(\frac{\gamma}{k},q\right)\left(\frac{a}{l},\sigma\right)\\ \left(\frac{\mu_1}{k},\frac{1}{\rho_1 k}\right)\cdots\left(\frac{\mu_m}{k},\frac{1}{\rho_m k}\right)\end{matrix}\Big|\frac{xp^{\left(q-\frac{1}{\rho_1 k}-\frac{1}{\rho_2 k}\cdots-\frac{1}{\rho_m k}\right)}}{s^{\frac{\sigma}{\alpha}}}\right]. \quad (28)$$

Where $\left|\frac{x}{s^{\frac{\sigma}{\alpha}}}\right| < 1$

Proof Using LHS side of the equation (28)

$$= \int_0^{\infty} z^{a-1}e^{-s^{\frac{1}{\alpha}}z^l}{}_pE_k^{\gamma,q}[(\mu_1,\frac{1}{\rho_1})(\mu_2,\frac{1}{\rho_2})\cdots(\mu_m,\frac{1}{\rho_m});xz^{\sigma l}]\,dz$$

Using the equation (16), we have

$$= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}(x)^n}{{}_p\Gamma_k(\mu_1 + \frac{n}{\rho_1}) \cdots {}_p\Gamma_k(\mu_m + \frac{n}{\rho_m})(n!)} \int_0^\infty z^{(l\sigma n+a)-1} e^{-z^l s^{\frac{1}{\alpha}}}\, dz$$

Order of integration and summation can be interchanged under the conditions of uniform convergence of the series as discussed in the subsection (16).

Let $z^l = u$ then $z = u^{1/l}$ and $dz = \frac{u^{(1/l)-1}}{l} du$

$$= \frac{1}{l} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}(x)^n}{{}_p\Gamma_k(\mu_1 + \frac{n}{\rho_1}) \cdots {}_p\Gamma_k(\mu_m + \frac{n}{\rho_m})(n!)} \int_0^\infty u^{\frac{(ln\sigma+a)}{l}-1} e^{-us^{\frac{1}{\alpha}}}\, du$$

Using definition of gamma function

$$= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}(x)^n}{{}_p\Gamma_k(\mu_1 + \frac{n}{\rho_1}) \cdots {}_p\Gamma_k(\mu_m + \frac{n}{\rho_m})(n!\, l)} \frac{\Gamma(\frac{a}{l} + n\sigma)}{(s^{\frac{1}{\alpha}})^{\frac{a}{l}+n\sigma}}$$

Using equation of (2.19) and (2.20) of [3] and using the equation (2.10), we get

$$= \frac{(k^m) s^{-(\frac{a}{\alpha l})} p^{-(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k})}}{l\Gamma\frac{\gamma}{k}} {}_2\psi_m \left[\begin{matrix} (\frac{\gamma}{k}, q)(\frac{a}{l}, \sigma) \\ (\frac{\mu_1}{k}, \frac{1}{\rho_1 k}) \cdots (\frac{\mu_m}{k}, \frac{1}{\rho_m k}) \end{matrix} \Big| \frac{xp^{(q - \frac{1}{\rho_1 k} - \frac{1}{\rho_2 k} \cdots - \frac{1}{\rho_m k})}}{s^{\frac{\sigma}{\alpha}}} \right]$$

Hence the result.

Particular Cases

a putting $m = 1$, $\mu_1 = \beta$, $\rho_1 = \frac{1}{\delta}$, $l = 1$ and $\alpha = 1$ in equation (28), it is reduced to equation (34) defined by Gehlot [6]

$$\int_0^\infty z^{a-1} e^{-sz} {}_pE_k^{\gamma,q}[(\beta, \delta); xz^\sigma]\, dz$$

$$= \frac{(k) s^{-a} p^{-(\frac{\beta}{k})}}{\Gamma\frac{\gamma}{k}} {}_2\psi_1 \left[\begin{matrix} (\frac{\gamma}{k}, q)(a, \sigma) \\ (\frac{\beta}{k}, \frac{\delta}{k}) \end{matrix} \Big| \frac{xp^{q-\frac{\delta}{k}}}{s^\sigma} \right]. \tag{29}$$

b For $l = 1$ and $\alpha = 1$ then from equation (28), we get classical Laplace transform of $p - k$ multi-index Mittag Leffler function

$$\int_0^\infty z^{a-1} e^{-sz} {}_pE_k^{\gamma,q}[(\mu_1, \frac{1}{\rho_1})(\mu_2, \frac{1}{\rho_2}) \cdots (\mu_m, \frac{1}{\rho_m}); xz^\sigma]\, dz$$

$$= \frac{(k^m) s^{-a} p^{-(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k})}}{\Gamma\frac{\gamma}{k}} {}_2\psi_m \left[\begin{matrix} (\frac{\gamma}{k}, q)(a,\sigma) \\ (\frac{\mu_1}{k}, \frac{1}{\rho_1 k}) \cdots (\frac{\mu_m}{k}, \frac{1}{\rho_m k}) \end{matrix} \bigg| \frac{x p^{(q-\frac{1}{\rho_1 k}-\frac{1}{\rho_2 k}\cdots-\frac{1}{\rho_m k})}}{s^\sigma} \right]. \quad (30)$$

c substituting $p = k,\ q = 1\ m = 1,\ \mu_1 = \beta,\ \rho_1 = \frac{1}{\delta},\ l = 1$ and $\alpha = 1$ in the equation (28), it is reduced to equation (35) defined by Gehlot [7]

$$\int_0^\infty z^{a-1} e^{-sz} {}_kE_k^{\gamma,1}[(\beta,\delta); xz^\sigma]\, dz$$

$$= \frac{s^{-a} k^{1-(\frac{\beta}{k})}}{\Gamma\frac{\gamma}{k}} {}_2\psi_1 \left[\begin{matrix} (\frac{\gamma}{k}, 1)(a,\sigma) \\ (\frac{\beta}{k}, \frac{\delta}{k}) \end{matrix} \bigg| \frac{x k^{1-\frac{\delta}{k}}}{s^\sigma} \right]. \quad (31)$$

Theorem 3.4

Let $m \geq 1$ and $m \in \mathbb{Z}$ and $\mu_1, \mu_2, \cdots \mu_m$ are arbitrary real (complex) numbers and $Re(\mu_i) > 0\ \ \forall\, i = 1,2 \cdots m$; $q \in \mathbb{N}$; $k, p \in \mathbb{R}^+ - (0)$, $\rho_1, \rho_2, \cdots \rho_m > 0$ and $\frac{1}{\rho_1}, \frac{1}{\rho_2}, \cdots \frac{1}{\rho_m} \in \mathbb{R},\ \gamma \in \mathbb{C}/\mathbb{Z}^-$ and $s > 0$ then **Mellin transform** of $p - k$ multi-index Mittag Leffler function is,

$$\int_0^\infty t^{s-1} {}_pE_k^{\gamma,q}\left[\left(\mu_1, \frac{1}{\rho_1}\right)\left(\mu_2, \frac{1}{\rho_2}\right)\cdots\left(\mu_m, \frac{1}{\rho_m}\right)\right](-t)dt =$$

$$\frac{(k^m)\, p^{-(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k})}}{\Gamma\frac{\gamma}{k}} \frac{\Gamma s\, \Gamma\left(\frac{\gamma}{k} - qs\right) p^{(\frac{1}{\rho_1 k}-\frac{1}{\rho_2 k}\cdots-\frac{1}{\rho_m k}-q)^s}}{\Gamma(\frac{\mu_1}{k}-\frac{s}{\rho_1 k})\cdots\Gamma(\frac{\mu_m}{k}-\frac{s}{\rho_m k})}. \quad (32)$$

Proof: Substituting $z = -t$ in the equation (22), we get

$${}_pE_k^{\gamma,q}\left[\left(\mu_1, \frac{1}{\rho_1}\right)\cdots\left(\mu_m, \frac{1}{\rho_m}\right); (-t)\right] =$$

$$\frac{(k^m) p^{-(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k})}}{2\pi i \Gamma\frac{\gamma}{k}} \int_L \frac{\Gamma s \Gamma(\frac{\gamma}{k} - qs) p^{(q-\frac{1}{\rho_1 k}-\frac{1}{\rho_2 k}\cdots-\frac{1}{\rho_m k})^{-s}}}{\Gamma(\frac{\mu_1}{k} - \frac{s}{\rho_1 k})\Gamma(\frac{\mu_2}{k} - \frac{s}{\rho_2 k})\cdots\Gamma(\frac{\mu_m}{k} - \frac{s}{\rho_m k})} (t)^{-s}\, ds.$$

By using equation (15), we find

$${}_pE_k^{\gamma,q}[(\mu_1, \frac{1}{\rho_1})\cdots(\mu_m, \frac{1}{\rho_m}); (-t)] == \frac{1}{2\pi i}\int_L f^*(s)(t)^{-s}\, ds. \quad (33)$$

where,

$$f^*(s) = \frac{(k^m)p^{-\left(\frac{\mu_1}{k}+\cdots+\frac{\mu_m}{k}\right)}}{\Gamma\frac{\gamma}{k}} \frac{\Gamma s\, \Gamma\left(\frac{\gamma}{k} - qs\right)\, p^{\left(\frac{1}{\rho_1 k} - \frac{1}{\rho_2 k} \cdots - \frac{1}{\rho_m k} - q\right)^s}}{\Gamma(\frac{\mu_1}{k} - \frac{s}{\rho_1 k}) \cdots \Gamma(\frac{\mu_m}{k} - \frac{s}{\rho_m k})}.$$

By using the equations (14) and (33), we get the desired result the equation (32).

3.5 Basic properties of p-k multi index Mittag Leffler function

Theorem 3.5

Let $m \geq 1$ and $m \in \mathbb{Z}$ and $\mu_1, \mu_2, \cdots \mu_m$ are arbitrary real (complex) numbers and $Re(\mu_i) > 0 \quad \forall\, i = 1,2 \cdots m$; $q \in \mathbb{N}$; $k, p \in \mathbb{R}^+ - (0)$, $\rho_1, \rho_2, \cdots \rho_m > 0$ and $\frac{1}{\rho_1}, \frac{1}{\rho_2}, \cdots \frac{1}{\rho_m} \in \mathbb{R}$, $\gamma \in \mathbb{C}/\mathbb{Z}^-$, then

$$k \quad {}_pE_k^{\gamma,q}[(\mu_1, \frac{1}{\rho_1})(\mu_2, \frac{1}{\rho_2}) \cdots (\mu_m, \frac{1}{\rho_m}); z]$$

$$= p\mu_1{}_pE_k^{\gamma,q}\left[\left(\mu_1 + k, \frac{1}{\rho_1}\right)\left(\mu_2, \frac{1}{\rho_2}\right) \cdots \left(\mu_m, \frac{1}{\rho_m}\right); z\right] + \frac{zp}{\rho_1}\left(\frac{d}{dz}\right) \quad {}_pE_k^{\gamma,q}\left[\left(\mu_1 + k, \frac{1}{\rho_1}\right)\left(\mu_2, \frac{1}{\rho_2}\right) \cdots \left(\mu_m, \frac{1}{\rho_m}\right); z\right] \tag{34}$$

Proof: Using the equation (16) in the RHS of the equation (34), we get

$$= \sum_{n=0}^{\infty} \frac{(p\mu_1)\ {}_p(\gamma)_{nq,k} z^n}{{}_p\Gamma_k(\mu_1 + k + \frac{n}{\rho_1}) \cdots {}_p\Gamma_k(\mu_m + \frac{n}{\rho_m})(n!)}$$

$$+ \frac{zp}{\rho_1} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}\ nz^{n-1}}{{}_p\Gamma_k(\mu_1 + k + \frac{n}{\rho_1}) \cdots {}_p\Gamma_k(\mu_m + \frac{n}{\rho_m})(n!)}$$

$$= p \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k} z^n(\mu_1 + \frac{n}{\rho_1})}{{}_p\Gamma_k(\mu_1 + k + \frac{n}{\rho_1}) \cdots {}_p\Gamma_k(\mu_m + \frac{n}{\rho_m})(n!)}$$

Using the property of two parameter gamma function namely ${}_p\Gamma_k(x + k) = \frac{px}{k}\, {}_p\Gamma_k x$ of equation (2.23) of [3], we get

$$= k \quad {}_pE_k^{\gamma,q}[(\mu_1, \frac{1}{\rho_1})(\mu_2, \frac{1}{\rho_2}) \cdots (\mu_m, \frac{1}{\rho_m}); z].$$

Hence the result.

Theorem 3.6

Let $m \geq 1$ and $m \in \mathbb{Z}$ and $\mu_1, \mu_2, \cdots \mu_m$ are arbitrary real (complex) numbers and $Re(\mu_i) > 0 \quad \forall\, i = 1,2 \cdots m$; $q \in \mathbb{N}$; $k, p \in \mathbb{R}^+ - (0)$, $\rho_1, \rho_2, \cdots \rho_m > 0$ and

$\frac{1}{\rho_1}, \frac{1}{\rho_2}, \cdots \frac{1}{\rho_m} \in \mathbb{R}$, $\gamma \in \mathbb{C}/\mathbb{Z}^-$, then **derivative** of the $p-k$ multi-index Mittag Leffler function is given by,

$$\left(\frac{d}{dz}\right)^j {}_pE_k^{\gamma,q}[(\mu_1, \frac{1}{\rho_1})(\mu_2, \frac{1}{\rho_2}) \cdots (\mu_m, \frac{1}{\rho_m}); z]$$

$$= {}_p(\gamma)_{jq,k} \quad {}_pE_k^{\gamma+jqk,q}[(\mu_1 + \frac{j}{\rho_1}, \frac{1}{\rho_1})(\mu_2 + \frac{j}{\rho_2}, \frac{1}{\rho_2}) \cdots (\mu_m + \frac{j}{\rho_m}, \frac{1}{\rho_m}); z]. \tag{35}$$

Proof :Taking LHS of equation (35)

$$\left(\frac{d}{dz}\right)^j {}_pE_k^{\gamma,q}[(\mu_1, \frac{1}{\rho_1})(\mu_2, \frac{1}{\rho_2}) \cdots (\mu_m, \frac{1}{\rho_m}); z]$$

Using the equation (3.1), we have

$$= \left(\frac{d}{dz}\right)^j \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k} z^n}{{}_p\Gamma_k(\mu_1 + \frac{n}{\rho_1}) \cdots {}_p\Gamma_k(\mu_m + \frac{n}{\rho_m})(n!)}$$

Differentiating j times, we get

$$= \sum_{n=j}^{\infty} \frac{{}_p(\gamma)_{nq,k} z^{n-j}}{{}_p\Gamma_k(\mu_1 + \frac{n}{\rho_1}) \cdots {}_p\Gamma_k(\mu_m + \frac{n}{\rho_m})((n-j)!)}$$

Using equation (2.34) of [3], we get,

$$= {}_p(\gamma)_{jq,k} \quad {}_pE_k^{\gamma+jqk,q}[(\mu_1 + \frac{j}{\rho_1}, \frac{1}{\rho_1})(\mu_2 + \frac{j}{\rho_2}, \frac{1}{\rho_2}) \cdots (\mu_m + \frac{j}{\rho_m}, \frac{1}{\rho_m}); z].$$

Hence the result.

Theorem 3.7

Let $m \geq 1$ and $m \in \mathbb{Z}$ and $\mu_1, \mu_2, \cdots \mu_m$ are arbitrary real (complex) numbers and $Re(\mu_i) > 0 \quad \forall\, i = 1,2 \cdots m$; $q \in \mathbb{N}$; $k, p \in \mathbb{R}^+ - (0)$, $\gamma \in C/Z^-$, $\rho_1, \rho_2, \cdots \rho_m > 0$ and $\frac{1}{\rho_1}, \frac{1}{\rho_2}, \cdots \frac{1}{\rho_m} \in \mathbb{R}$ then the **relation between $p-k$ multi index Mittag Leffler function and Mittag Leffler function** defined by equation (20) is given by

$${}_pE_k^{\gamma,q}\left[\left(\mu_1, \frac{1}{\rho_1}\right)\left(\mu_2, \frac{1}{\rho_2}\right) \cdots \left(\mu_m, \frac{1}{\rho_m}\right); z\right] =$$

$$k^m p^{-(\frac{\mu_1}{k} \cdots + \frac{\mu_m}{k})} E^{(\frac{\gamma}{k}),q}[(\frac{\mu_1}{k}, \frac{1}{k\rho_1}) \cdots (\frac{\mu_m}{k}, \frac{1}{k\rho_m}); zp^{q - \frac{1}{\rho_1 k} \cdots - \frac{1}{\rho_m k}}]. \tag{36}$$

Proof: Using the equation (16) and the equations (2.19) and (2.20) of [3] in the LHS of the equation (35), we get

$$ {}_pE_k^{\gamma,q}[(\mu_1,\frac{1}{\rho_1})(\mu_2,\frac{1}{\rho_2})\cdots(\mu_m,\frac{1}{\rho_m});z] $$

$$ =\sum_{n=0}^{\infty}\frac{p^{nq}(\gamma/k)_{nq}}{(\frac{p^{(\frac{\mu_1}{k}+\frac{n}{\rho_1 k})}}{k}\Gamma(\frac{\mu_1}{k}+\frac{n}{\rho_1 k}))\cdots(\frac{p^{(\frac{\mu_1}{k}+\frac{n}{\rho_1 k})}}{k}\Gamma(\frac{\mu_m}{k}+\frac{n}{\rho_m k}))}\frac{z^n}{n!}. $$

$$ =k^m p^{-(\frac{\mu_1}{k}\cdots+\frac{\mu_m}{k})}\sum_{n=0}^{\infty}\frac{(\frac{\gamma}{k})_{nq}\,p^{nq-\frac{n}{\rho_1 k}\cdots-\frac{n}{\rho_m k}}}{\Gamma(\frac{\mu_1}{k}+\frac{n}{\rho_1 k})+\cdots+\Gamma(\frac{\mu_m}{k}+\frac{n}{\rho_m k})}\frac{z^n}{n!}. $$

Using equation (20), we get

$$ =k^m p^{-(\frac{\mu_1}{k}\cdots+\frac{\mu_m}{k})}E^{(\frac{\gamma}{k}),q}[(\frac{\mu_1}{k},\frac{1}{k\rho_1})\cdots(\frac{\mu_m}{k},\frac{1}{k\rho_m});zp^{q-\frac{1}{\rho_1 k}\cdots-\frac{1}{\rho_m k}}]. $$

Hence the result.

And counter part of equation (34) is

$$ E^{(\frac{\gamma}{k}),q}\left[\left(\frac{\mu_1}{k},\frac{1}{k\rho_1}\right)\cdots\left(\frac{\mu_m}{k},\frac{1}{k\rho_m}\right);z\right]= $$

$$ \frac{p^{(\frac{\mu_1}{k}\cdots+\frac{\mu_m}{k})}}{k^m}\,{}_pE_k^{\gamma,q}[(\mu_1,\frac{1}{\rho_1})(\mu_2,\frac{1}{\rho_2})\cdots(\mu_m,\frac{1}{\rho_m});zp^{\frac{1}{\rho_1 k}\cdots+\frac{1}{\rho_m k}-q}]. \quad (37) $$

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