Journal of Rajasthan Academy of Physical Sciences
ISSN: 0972-6306; URL: http://raops.org.in
Vol.24, No.1&2, January-June, 2025, 129-139

COUPLED FIXED POINT THEOREMS ON PARAMETRIC METRIC SPACE EMPLOYING C-CLASS FUNCTION

Heera Ahirwar[1*] and **Kavita Shrivastava**[2]
[1,2]Department of Mathematics and Statistics
Dr. Harisingh Gour Vishwavidyalaya Sagar, Madhya Pradesh - 470003, India.
Email: heera.maths15@gmail.com[1*]

Abstract: In this work, we utilize the C-Class function to derive several fixed point results within the framework of parametric metric spaces. Our findings extend and refine the contributions of Heera Ahirwar and Kavita Shrivastava [1], particularly under new rational contractive conditions. To emphasize the significance of our results, we provide examples for better illustration. This development enriches the understanding of fixed point theory and paves the way for its application in more intricate and varied mathematical contexts. As a result, our research propels the field forward, offering a strong basis for future studies and broad applications in mathematical science and engineering.

Keywords: Parametric metric spaces, fixed point Theorem, C-class function.

1. Introduction

Fixed point theory addresses the existence and uniqueness of solutions to various problems, including partial differential equations, integral equations, variational inequalities, approximation theory, and game theory. It has widespread applications across mathematics, engineering, economics, and medical sciences.

In 2014, Hussain et al. introduced the concept of parametric metric spaces as a generalization of traditional metric spaces and proved several fixed point results. Later that year, Rao et al. proposed parametric S-metric spaces, a new type of generalized metric space, and established several widely recognized fixed point theorems under various expansion conditions. Krishnakumar and Sanaatammappa [8] extended these results by introducing complete parametric b-metric spaces and providing supporting examples for their fixed point findings.

In 2016, Daheriya et al. [5] proved fixed point theorems for continuous and surjective mappings involving expansion types. Ansari et al. [2] discussed C-class function on common fixed point theorem of weakly compatible maps in partial metric space.

In 2017, Ege and Karaca [6] expanded on Hussain et al.'s findings using C-class functions, deriving new fixed point theorems for parametric metric spaces. Building on this, Singh and Singh [10] applied the C-class function to establish fixed point and common linked fixed point results.

In 2024, Ahirwar and Shrivastava [1] extended the C-class function to establish fixed point and common linked fixed point results, building upon the work of Singh and Singh [10].

In this work, we expand upon and extend the results of Mohammod et al. [9] and Ahirwar and Srivastava [1] by establishing coupled fixed point theorems under new rational contractive conditions. Additionally, we demonstrate various fixed point results in parametric metric spaces using C-class functions.

2. Preliminaries

To support our main results, we first revisit some key definitions and concepts related to parametric metric spaces.

Definition 2.1 Garg and Ahirwar [7] Let X be a non-empty set and a $Ŧ_p: X \times X \times (0,\infty) \to [0,\infty)$ be a map on X , is said to be parametric metric on X if

(a) $Ꞙ_p(x,y,t) = 0 \Rightarrow x = y$, for all $t > 0$

(b) $Ꞙ_p(x,y,t) = Ꞙ_p(y,x,t)$, for all $t > 0$

(c) $Ꞙ_p(x,y,t) \le Ꞙ_p(x,z,t) + Ꞙ_p(z,y,t)$. for all $x,y,z \in X$ and $t > 0$

Then $Ꞙ_p$ is called parametric metric and the pair $(X, Ŧ_p)$ is called parametric metric space.

Definition 2.2 Let $\{x_n\}_{n=1}^{\infty}$is a sequence in parametric metric space $(X, Ŧ_p)$ $then$

(i) $\{x_n\}_{n=1}^{\infty}$is called convergent to $x \in X$ if $\lim_{n\to\infty} Ꞙ_p(x, x_n, t) = 0$ written as $\lim_{n\to\infty} x_n = x$ for all $t > 0$

(ii) $\{x_n\}_{n=1}^{\infty}$is called Cauchy sequence in X if $\lim_{n,m\to\infty} Ꞙ_p(x_n, x_m, t) = 0$ for all $t > 0$

(iii) A parametric metric space $(X, Ꞙ_p)$ is called complete iff every Cauchy sequence is convergent to $x \in X$.

Definition 2.3 Let (X, ∂) be a complete parametric metric space and a function $Ŧ_p: X \to X$ is a called continuous in $x \in X$ if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = x$ for all $t > 0$ then $\lim_{n\to\infty} Ŧ_p\, x_n = Ŧ_p x$.

Example: 1 Let $X = \{f/f:(0,+\infty) \to R\}$ and defined a function $Ŧ_p: X \times X \times (0,\infty) \to [0,\infty)$ by $Ꞙ(K_1, K_2, t) = |K_1(t) - K_2(t)|, for\ all\ K_1, K_2 \in X$ and $t > 0$ then $Ꞙ_p$ is a parametric metric in X and the pair $(X, Ꞙ_p)$ is called parametric metric space in X

Definition 2.4 Choudhary and Garg [4] In a mapping $Ŧ_p: X \times X \to X$, an element $(x,y) \in X^2$ is called coupled fixed point of the mapping if $Ŧ_p(x,y) = x$ and $Ŧ_p(y,x) = y$, for $x,y \in X$.

Example: 2. Let $X = R$ and $Ŧ_p : X \times X \to X$ is defined by $Ŧ_p(a, b) = \frac{ab}{2}$

Since (0,0) is Coupled fixed point of $Ŧ_p$.

Definition 2.5 Choudhary and Maity [3]**:** A function $\varphi: [0, \infty) \to [0, \infty)$ is called an altering distance function if φ is continuous, non-decreasing, $and\ \varphi(t) = 0 \iff t = 0$.

Definition 2.6: A function $\varphi: [0, \infty) \to [0, \infty)$ is called an ultra-altering distance function if φ is continuous, and $\varphi(0) \geq 0, \varphi(t) > 0, t \neq 0$.

Definition 2.1 Ahirwar and Shrivastava [1] A continuous mapping $Ƒ: (0, \infty) \times [0, \infty) \to R$ is said to be **C-class function** if it satisfies the following conditions:

$\{C_a\}$ $Ƒ(\eta, \varrho) \leq \eta$, For all $\eta, \varrho \in [0, \infty)$.

$\{C_b\}$ $Ƒ(\eta, \varrho) \leq \eta$ ⇒Either $\eta = 0\ or\ \varrho = 0$.

Example: 3 The following functions $Ƒ: (0, \infty) \times [0, \infty) \to R$ are elements of C for all $\eta, \varrho \in [0, \infty)$;

(I) $Ƒ(\eta, \varrho) = \eta - \varrho$, $Ƒ(\eta, \varrho) = \eta \Rightarrow \varrho = 0$;

(II) $Ƒ(\eta, \varrho) = Ƒ(\eta, \varrho) = \eta \Rightarrow \eta = 0\ or\ \ \varrho = 0$;

(III) $Ƒ(\eta, \varrho) = \beta\eta$, $0 < \beta < 1, Ƒ(\eta, \varrho) = \eta \Rightarrow \eta = 0$;

(IV) $Ƒ(\eta, \varrho) = \eta - \frac{t}{k+t}, Ƒ(\eta, \varrho) = \eta = \varrho = 0$;

(V) $Ƒ(\eta, \varrho) = \eta - \frac{2+t}{k+t}, Ƒ(\eta, \varrho) = \eta \Rightarrow \varrho = 0$;

(VI) $Ƒ(\eta, \varrho) = \eta - \varphi(\eta), Ƒ(\eta, \varrho) = \eta \Rightarrow \eta = 0$, here $\varphi: [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;

(VII) $Ƒ(\eta, \varrho) = \eta\beta(\eta), \beta: [0, \infty) \to [0,1)$, and is a continuous function ,

$Ƒ(\eta, \varrho) = \eta \Rightarrow \eta = 0$;

Let Ψ denote the set of all continuous and monotone non-decreasing function $\phi: [0, \infty) \to [0, \infty)$ such that $\phi(\varrho) = 0$ if and only if $\varrho = 0$, $\phi(\eta + \varrho) \leq \phi(\eta) + \phi(\varrho)$ for all $\eta, \varrho \in [0, \infty)$.

Let Φ_1 denote the all continuous function $\phi: [0, \infty) \to [0, \infty)$ such that $\phi(Ҡ) = 0$ if and only if $\varrho = 0$ and Φ_u denote the set of all continuous function $\phi: [0, \infty) \to [0, \infty)$ such that $\phi(0) \geq 0$, note that $\Phi_1 \subset \Phi_u$.

Theorem 2.5: Let *T* be a self-mapping defined on a complete metric space *(X,d)* satisfying the condition

$h(\psi(d(Tx, Ty))) \leq f(\psi(d(x, y)), \phi(d(x, y)))$

for $x, y \in F \subset X$ where *F* is subclosed of *X* and invariant under *T*, ψ and ϕ are the earlier described altering distance functions (or ϕ an ultra-altering distance function), *f* is a function of *C*-class, and *h* is a function of *A*-class, Then *T* has a unique fixed point in *F*.

Now we present our main results with new rational contractive conditions.

3. Main Results

Theorem (3.1) Let (X, ∂) be a complete parametric metric space and $Ŧ_p : X \times X \to X$ is a continuous

Mapping, satisfying the following rational contractive condition:

$$\phi\left(\partial\left(Ŧ_p(x,y), Ŧ_p(r,s), t\right)\right) \leq \mathbb{F}\left(\phi\left(\Omega(x,y)\right), \varphi\left(\Omega(x,y)\right)\right)$$

$$for\ all\ x, y, r, s \in X\ , for\ all\ t > 0\ and\ \mathbb{F} \in \mathbb{C}\ , \phi \in \Psi\ , \varphi \in \Phi_u\ \ and$$

$$\begin{aligned} &\Omega(x,y) \\ &= A[\partial(x,r,t) + \partial(y,s,t)] + B\ Max\left[\partial\left(x, Ŧ_p(x,y), t\right), \partial\left(r, Ŧ_p(r,s), t\right), \partial\left(r, Ŧ_p(x,y), t\right)\right] \\ &+ C\ Min\left[\partial\left(Ŧ_p(x,y), s, t\right)\partial\left(Ŧ_p(r,s), r, t\right), \frac{\partial\left(Ŧ_p(x,y), r, t\right)\partial\left(x, Ŧ_p(r,s), t\right)}{1 + \partial(x,r,t) + \partial(y,s,t)}\right] \end{aligned}$$

Where, $2A + B < 1$, and $a, b, c, r, s \in \left[0, \frac{1}{2}\right)$ then $Ŧ_p$ has a unique coupled Fixed point in $X \times X$.

Proof : Choose $x_0, y_0 \in X$ is arbitrary and define the sequence $\{x_n\}_{n \in N}$ and $\{y_n\}_{n \in N}$ as follows $Ŧ_p(x_n, y_n) = x_{n+1}\ \ and\ Ŧ_p(y_n, x_n) = y_{n+1}\ \ for\ n = 1,2,3 \ldots\ldots$ then consider the inequality:

$$\phi\left(\partial(x_{n+1}, x_{n+2}, t)\right) = \phi\left(\partial\left(Ŧ_p(x_n, y_n), Ŧ_p(x_{n+1}, y_{n+1}), t\right)\right)$$

$$\leq \mathbb{F}\left[\begin{array}{c} \phi\left\{\begin{array}{c} A[\partial(x_n, x_{n+1}, t) + \partial(y_n, y_{n+1}, t)] \\ +B\ Max\left[\partial\left(x_n, Ŧ_p(x_n, y_n), t\right), \partial\left(x_{n+1}, Ŧ_p(x_{n+1}, y_{n+1}), t\right), \partial\left(x_{n+1}, Ŧ_p(x_n, y_n), t\right)\right] \\ +C\ Min\left[\begin{array}{c} \partial\left(Ŧ_p(x_n, y_n), y_{n+1}, t\right)\partial\left(Ŧ_p(x_{n+1}, y_{n+1}), x_{n+1}, t\right), \\ \frac{\partial\left(Ŧ_p(x_n, y_n), x_{n+1}, t\right)\partial\left(x_n, Ŧ_p(x_{n+1}, y_{n+1}), t\right)}{1 + \partial(x_n, x_{n+1}, t) + \partial(y_n, y_{n+1}, t)} \end{array}\right] \end{array}\right\}, \\ \varphi\left\{\begin{array}{c} A[\partial(x_n, x_{n+1}, t) + \partial(y_n, y_{n+1}, t)] \\ +B\ Max\left[\partial\left(x_n, Ŧ_p(x_n, y_n), t\right), \partial\left(x_{n+1}, Ŧ_p(x_{n+1}, y_{n+1}), t\right), \partial\left(x_{n+1}, Ŧ_p(x_n, y_n), t\right)\right] \\ +C\ Min\left[\begin{array}{c} \partial\left(Ŧ_p(x_n, y_n), y_{n+1}, t\right)\partial\left(Ŧ_p(x_{n+1}, y_{n+1}), x_{n+1}, t\right), \\ \frac{\partial\left(Ŧ_p(x_n, y_n), x_{n+1}, t\right)\partial\left(x_n, Ŧ_p(x_{n+1}, y_{n+1}), t\right)}{1 + \partial(x_n, x_{n+1}, t) + \partial(y_n, y_{n+1}, t)} \end{array}\right] \end{array}\right\} \end{array}\right]$$

$$\leq \mathbb{F}\left[\begin{array}{l}\phi\left\{\begin{array}{c}A[\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]\\ +B\ Max[\partial(x_n,x_{n+1},t),\partial(x_{n+1},x_{n+2},t),\partial(x_{n+1},x_n,t)]\\ +C\ Min\left[\partial(x_{n+1},y_{n+1},t)\partial(x_{n+2},x_{n+1},t),\dfrac{\partial(x_{n+1},x_{n+1},t)\partial(x_n,x_{n+2},t)}{1+\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)}\right]\end{array}\right\},\\ \varphi\left\{\begin{array}{c}A[\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]\\ +B\ Max[\partial(x_n,x_{n+1},t),\partial(x_{n+1},x_{n+2},t),\partial(x_{n+1},x_n,t)]\\ +C\ Min\left[\partial(x_{n+1},y_{n+1},t)\partial(x_{n+2},x_{n+1},t),\dfrac{\partial(x_{n+1},x_{n+1},t)\partial(x_n,x_{n+2},t)}{1+\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)}\right]\end{array}\right\}\end{array}\right]$$

$$\leq \phi\left\{\begin{array}{c}A[\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]+B\ Max[\partial(x_n,x_{n+1},t),\partial(x_{n+1},x_{n+2},t),\partial(x_{n+1},x_n,t)]\\ +C\ Min\left[\partial(x_{n+1},y_{n+1},t)\partial(x_{n+2},x_{n+1},t),\dfrac{\partial(x_{n+1},x_{n+1},t)\partial(x_n,x_{n+2},t)}{1+\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)}\right]\end{array}\right\}$$

$$\leq \phi\left\{\begin{array}{c}A[\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]+B\ Max[\partial(x_n,x_{n+1},t),\partial(x_{n+1},x_{n+2},t)]\\ +C\ Min[\partial(x_{n+1},y_{n+1},t)\partial(x_{n+2},x_{n+1},t),0]\end{array}\right\}$$

There are two cases may be possible:

Case (I) if $Max\{\partial(x_n,x_{n+1},t),\partial(x_{n+1},x_{n+2},t)\}=\partial(x_n,x_{n+1},t)$ then we get,

$$\phi\big(\partial(x_{n+1},x_{n+2},t)\big)\leq\phi\{A[\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]+B\ \partial(x_n,x_{n+1},t)\}$$

$$\partial(x_{n+1},x_{n+2},t)\leq A[\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]+B\ \partial(x_n,x_{n+1},t)$$

$$\partial(x_{n+1},x_{n+2},t)\leq(A+B)\partial(x_n,x_{n+1},t)+A\ \partial(y_n,y_{n+1},t)$$

Similarly we can show that,

$$\partial(y_{n+1},y_{n+2},t)\leq(A+B)\ \partial(y_n,y_{n+1},t)+A\ \partial(x_n,x_{n+1},t)$$

By adding,

$$[\partial(x_{n+1},x_{n+2},t)+\partial(y_{n+1},y_{n+2},t)]\leq(2A+B)\ [\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]$$

$$[\partial(x_{n+1},x_{n+2},t)+\partial(y_{n+1},y_{n+2},t)]\leq k[\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]\ ,$$

Where, $k=(2A+B)<1$

By mathematical induction, we get

$$[\partial(x_{n+1},x_{n+2},t)+\partial(y_{n+1},y_{n+2},t)]\leq k^{n+1}[\partial(x_0,x_1,t)+\partial(y_0,y_1,t)]$$

Taking limit as $n\to\infty\ , since\ k<1$ then

$$[\partial(x_{n+1},x_{n+2},t)+\partial(y_{n+1},y_{n+2},t)]\to 0\ \Rightarrow\partial(x_{n+1},x_{n+2},t)\to 0\ \ and\ \partial(y_{n+1},y_{n+2},t)\to 0$$

Hence the sequence $\{x_n\}$ and $\{y_n\}$ is a Cauchy sequence.

The completeness of $(X,\partial)\ \Rightarrow\{x_n\}\ and\ \{y_n\}\ is\ convergent$. Call the limit $\eta_1,\eta_2\in X$, then $x_n\to\eta_1$ and $y_n\to\eta_2$ as $n\to\infty$ and Ŧ is Continuous and $Ŧ_p(x_n,y_n)=x_{n+1}$

then $Ŧ_p(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = \lim_{n\to\infty} Ŧ_p(x_n, y_n) = \lim_{n\to\infty} x_{n+1} = \eta_1$

$$Ŧ_p(\eta_1, \eta_2) = \eta_1$$

Similarly from $Ŧ_p(y_n, x_n) = y_{n+1}$

then $Ŧ_p(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = \lim_{n\to\infty} Ŧ_p(y_n, x_n) = \lim_{n\to\infty} y_{n+1} = \eta_2$

$$Ŧ_p(\eta_2, \eta_1) = \eta_2$$

Thus $\eta_1, \eta_2 \in X \times X$ is a Couple fixed point of $Ŧ_p$ in X.

Case (II) if $Max\{\partial(x_n, x_{n+1}, t), \partial(x_{n+1}, x_{n+2}, t)\} = \partial(x_{n+1}, x_{n+2}, t)$ *then we get,*

$\phi(\partial(x_{n+1}, x_{n+2}, t)) \leq \phi\{A[\partial(x_n, x_{n+1}, t) + \partial(y_n, y_{n+1}, t)] + B\,\partial(x_{n+1}, x_{n+2}, t)\}$

$\partial(x_{n+1}, x_{n+2}, t) \leq A[\partial(x_n, x_{n+1}, t) + \partial(y_n, y_{n+1}, t)] + B\,\partial(x_{n+1}, x_{n+2}, t)$

$(1 - B)\partial(x_{n+1}, x_{n+2}, t) \leq A[\partial(x_n, x_{n+1}, t) + \partial(y_n, y_{n+1}, t)]$

$$\partial(x_{n+1}, x_{n+2}, t) \leq \frac{A}{(1-B)}\,\partial(x_n, x_{n+1}, t) + \frac{A}{(1-B)}\,\partial(y_n, y_{n+1}, t)$$

Similarly we can show that,

$$\partial(y_{n+1}, y_{n+2}, t) \leq \frac{A}{(1-B)}\,\partial(y_n, y_{n+1}, t) + \frac{A}{(1-B)}\,\partial(x_n, x_{n+1}, t)$$

Adding

$$[\partial(x_{n+1}, x_{n+2}, t) + \partial(y_{n+1}, y_{n+2}, t)] \leq \frac{2A}{(1-B)}[\partial(x_n, x_{n+1}, t) + \partial(y_n, y_{n+1}, t)]$$

$$[\partial(x_{n+1}, x_{n+2}, t) + \partial(y_{n+1}, y_{n+2}, t)] \leq \mathcal{L}[\partial(x_n, x_{n+1}, t) + \partial(y_n, y_{n+1}, t)]$$

Where, $\mathcal{L} = \frac{2A}{(1-B)} < 1$

By mathematical induction, we get

$$[\partial(x_{n+1}, x_{n+2}, t) + \partial(y_{n+1}, y_{n+2}, t)] \leq \mathcal{L}^{n+1}[\partial(x_0, x_1, t) + \partial(y_0, y_1, t)]$$

Taking limit as $n \to \infty$, *since* $\mathcal{L} < 1$ then

$$[\partial(x_{n+1}, x_{n+2}, t) + \partial(y_{n+1}, y_{n+2}, t)] \to 0 \;\Rightarrow \partial(x_{n+1}, x_{n+2}, t) \to 0 \text{ and } \partial(y_{n+1}, y_{n+2}, t) \to 0$$

Hence the sequence $\{x_n\}$ and $\{y_n\}$ is a Cauchy sequence. The completeness of $(X, \partial) \Rightarrow \{x_n\}$ *and* $\{y_n\}$ is convergent. Call the limit $ʒ_1, ʒ_2 \in X$,

then $x_n \to \eta_1$ and $y_n \to \eta_2$ as $n \to \infty$ and $Ŧ_p$ is Continuous and $Ŧ_p(x_n, y_n) = x_{n+1}$ then

$Ŧ_p(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = \lim_{n\to\infty} Ŧ_p(x_n, y_n) = \lim_{n\to\infty} x_{n+1} = ʒ_1$

$Ŧ_p(ʒ_1, ʒ_2) = ʒ_1$

Similarly, from $Ŧ_p(y_n, x_n) = y_{n+1}$ *then*

$$Ŧ_p(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = \lim_{n\to\infty} Ŧ_p(y_n, x_n) = \lim_{n\to\infty} y_{n+1} = ʒ_2$$

$$Ŧ_p(ʒ_2, ʒ_1) = ʒ_2$$

Thus $ʒ_1, ʒ_2 \in X \times X$ is a Couple fixed point of $Ŧ_p$ in X.

Uniqueness: Let us consider (λ_1, λ_2) and $(Ɛ_1, Ɛ_2)$ are two distinct coupled fixed point of $Ŧ_p$ $in\, X \times X$. such that $\lambda_1 = Ŧ_p(\lambda_1, \lambda_2)$, $\lambda_2 = Ŧ_p(\lambda_2, \lambda_1)$ and $Ɛ_1 = Ŧ_p(Ɛ_1, Ɛ_2)$, $Ɛ_2 = Ŧ_p(Ɛ_2, Ɛ_1)$

then by above inequality,

$$\phi\big(\partial(\lambda_1, Ɛ_1, t)\big) = \phi\left(\partial\big(Ŧ_p(\lambda_1, \lambda_2), Ŧ_p(Ɛ_1, Ɛ_2), t\big)\right)$$

$$\leq \mathbb{F}\left[\begin{array}{c} \phi\left\{\begin{array}{c} A[\partial(\lambda_1, Ɛ_1, t) + \partial(\lambda_2, Ɛ_2, t)] \\ +B\, Max[\partial(\lambda_1, \lambda_1, t), \partial(Ɛ_1, Ɛ_1, t), \partial(Ɛ_1, \lambda_1, t)] \\ +C\, Min\left[\partial(\lambda_1, Ɛ_1, t)\partial(Ɛ_1, Ɛ_1, t), \dfrac{\partial(\lambda_1, Ɛ_1, t)\partial(\lambda_1, Ɛ_1, t)}{1 + \partial(\lambda_1, Ɛ_1, t) + \partial(\lambda_2, Ɛ_2, t)}\right] \end{array}\right\}, \\ \varphi\left\{\begin{array}{c} A[\partial(\lambda_1, Ɛ_1, t) + \partial(\lambda_2, Ɛ_2, t)] \\ +B\, Max\big[\partial\big(\lambda_1, Ŧ_p(\lambda_1, \lambda_2), t\big), \partial\big(Ɛ_1, Ŧ_p(Ɛ_1, Ɛ_2), t\big), \partial\big(Ɛ_1, Ŧ_p(\lambda_1, \lambda_2), t\big)\big] \\ +C\, Min\left[\partial\big(Ŧ_p(\lambda_1, \lambda_2), Ɛ_1, t\big)\partial\big(Ŧ_p(Ɛ_1, Ɛ_2), Ɛ_1, t\big), \dfrac{\partial\big(Ŧ_p(\lambda_1, \lambda_2), Ɛ_1, t\big)\partial\big(\lambda_1, Ŧ_p(Ɛ_1, Ɛ_2), t\big)}{1 + \partial(\lambda_1, Ɛ_1, t) + \partial(\lambda_2, Ɛ_2, t)}\right] \end{array}\right\} \end{array}\right]$$

$$\leq \mathbb{F}\left[\begin{array}{c} \phi\left\{\begin{array}{c} A[\partial(\lambda_1, Ɛ_1, t) + \partial(\lambda_2, Ɛ_2, t)] + B\, Max[\partial(\lambda_1, \lambda_1, t), \partial(Ɛ_1, Ɛ_1, t), \partial(Ɛ_1, \lambda_1, t)] \\ +C\, Min\left[\partial(\lambda_1, Ɛ_1, t)\partial(Ɛ_1, Ɛ_1, t), \dfrac{\partial(\lambda_1, Ɛ_1, t)\partial(\lambda_1, Ɛ_1, t)}{1 + \partial(\lambda_1, Ɛ_1, t) + \partial(\lambda_2, Ɛ_2, t)}\right] \end{array}\right\}, \\ \varphi\left\{\begin{array}{c} A[\partial(\lambda_1, Ɛ_1, t) + \partial(\lambda_2, Ɛ_2, t)] + B\, Max[\partial(\lambda_1, \lambda_1, t), \partial(Ɛ_1, Ɛ_1, t), \partial(Ɛ_1, \lambda_1, t)] \\ +C\, Min\left[\partial(\lambda_1, Ɛ_1, t)\partial(Ɛ_1, Ɛ_1, t), \dfrac{\partial(\lambda_1, Ɛ_1, t)\partial(\lambda_1, Ɛ_1, t)}{1 + \partial(\lambda_1, Ɛ_1, t) + \partial(\lambda_2, Ɛ_2, t)}\right] \end{array}\right\} \end{array}\right]$$

$$\leq \phi\left\{\begin{array}{c} A[\partial(\lambda_1, Ɛ_1, t) + \partial(\lambda_2, Ɛ_2, t)] + B\, Max[\partial(\lambda_1, \lambda_1, t), \partial(Ɛ_1, Ɛ_1, t), \partial(Ɛ_1, \lambda_1, t)] \\ +C\, Min\left[\partial(\lambda_1, Ɛ_1, t)\partial(Ɛ_1, Ɛ_1, t), \dfrac{\partial(\lambda_1, Ɛ_1, t)\partial(\lambda_1, Ɛ_1, t)}{1 + \partial(\lambda_1, Ɛ_1, t) + \partial(\lambda_2, Ɛ_2, t)}\right] \end{array}\right\}$$

$$\phi\big(\partial(\lambda_1, Ɛ_1, t)\big) \leq \phi\{A[\partial(\lambda_1, Ɛ_1, t) + \partial(\lambda_2, Ɛ_2, t)] + B\partial(Ɛ_1, \lambda_1, t)\}$$

$$\partial(\lambda_1, Ɛ_1, t) \leq \{A[\partial(\lambda_1, Ɛ_1, t) + \partial(\lambda_2, Ɛ_2, t)] + B\partial(Ɛ_1, \lambda_1, t)\}$$

$$\partial(\lambda_1, Ɛ_1, t) \leq (A + B)\partial(\lambda_1, Ɛ_1, t) + B\partial(\lambda_2, Ɛ_2, t)$$

Similarly we can show that

$\partial(\lambda_2, \mathcal{E}_2, t) \leq (A+B)\Gamma(\lambda_2, \mathcal{E}_2, t) + B\partial(\lambda_1, \mathcal{E}_1, t)$

Adding

$[\partial(\lambda_1, \mathcal{E}_1, t) + \partial(\lambda_2, \mathcal{E}_2, t)] \leq (2A+B)[\partial(\lambda_1, \mathcal{E}_1, t) + \partial(\lambda_2, \mathcal{E}_2, t)]$

Since, $(2A+B) < 1$ so that the above inequality is only true when

$[\partial(\lambda_1, \mathcal{E}_1, t) + \partial(\lambda_2, \mathcal{E}_2, t)] = 0 \Rightarrow \partial(\lambda_1, \mathcal{E}_1, t) = \partial(\lambda_2, \mathcal{E}_2, t) = 0$

This implies that, $\lambda_1 = \mathcal{E}_1 \ \ and\ \lambda_2 = \mathcal{E}_2$

Thus, $(\lambda_1, \lambda_2) = (\mathcal{E}_1, \mathcal{E}_2)$

$Hence\ \text{Ŧ}_p\ has\ a\ unique\ coupled\ Fixed\ point\ in\ X \times X.$

Definition 2.8 Let us we introduced dominate function used in our main theorem.

Let dominate function $\boldsymbol{\Psi}$ denotes the set of all function $\Omega: [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(a) $\boldsymbol{\Psi}_{(\boldsymbol{a})}: \Omega\ is\ continuous\ on\ [0, +\infty).$

(b) $\boldsymbol{\Psi}_{(\boldsymbol{b})}:\ \Omega(u) < u\ \ for\ all\ \alpha > 0.\ In\,fact\ \ \Omega(0) = 0\ \ and\ \ \Omega(u) \leq u\ \ for\ all\ u \geq 0.$

Now we draw upon rational type contractive conditions by using dominate function and prove unique coupled fixed point theorem.

Theorem (3.2) Let (X, ∂) be a complete parametric metric space and $\text{Ŧ}_p: X \times X \to X$ is a continuous mapping, satisfying the following rational contractive condition:

$$\phi\left(\partial\big(\text{Ŧ}_p(x,y), \text{Ŧ}_p(r,s), t\big)\right) \leq\ \mathbb{F}\left(\phi\big(\Delta(x,y)\big), \varphi\big(\Delta(x,y)\big)\right)$$

$$for\ all\ x, y, r, s \in X\ , for\ all\ t > 0\ and\ \mathbb{F} \in \mathcal{C}\ , \phi \in \Psi\ , \varphi \in \Phi_u\ and$$

$$\Delta(x,y) = A[\Omega\{\partial(x,r,t) + \partial(y,s,t)\}] + BMax\big[\Omega\big\{\partial\big(x, \text{Ŧ}_p(x,y), t\big)\big\}, \Omega\big\{\partial\big(r, \text{Ŧ}_p(r,s), t\big)\big\}, \Omega\big\{\partial\big(r, \text{Ŧ}_p(x,y), t\big)\big\}\big]$$

$$+C\ Min\left[\Omega\big\{\partial\big(\text{Ŧ}_p(x,y), s, t\big)\big\}\ \Omega\big\{\partial\big(\text{Ŧ}_p(r,s), r, t\big)\big\}, \frac{\Omega\big\{\partial\big(\text{Ŧ}_p(x,y), r, t\big)\big\}\Omega\big\{\partial\big(x, \text{Ŧ}_p(r,s), t\big)\big\}}{1 + \Omega\{\partial(x,r,t)\} + \Omega\{\partial(y,s,t)\}}\right]$$

Where, $2A + B < 1$, and $a, b, c, r, s \in \left[0, \frac{1}{2}\right)$ then Ŧ_p has a unique coupled Fixed point in $X \times X$.

Proof: Choose $x_0, y_0 \in X$ is arbitrary and define the sequence $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ as follows $\text{Ŧ}_p(x_n, y_n) = x_{n+1}\ and\ \text{Ŧ}_p(y_n, x_n) = y_{n+1}\ for\ n = 1,2,3 \ldots\ldots$ then consider the inequality:

$$\phi\big(\partial(x_{n+1}, x_{n+2}, t)\big) = \phi\left(\partial\big(\text{Ŧ}_p(x_n, y_n), \text{Ŧ}_p(x_{n+1}, y_{n+1}), t\big)\right)$$

$$\leq \mathbb{F}\left[\begin{array}{c}\phi\left\{\begin{array}{c}A\Omega\{[\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]\}\\ +B\,Max\left[\Omega\{\partial(x_n,Ŧ_p(x_n,y_n),t)\},\Omega\{\partial(x_{n+1},Ŧ_p(x_{n+1},y_{n+1}),t)\},\Omega\{\partial(x_{n+1},Ŧ_p(x_n,y_n),t)\}\right]\\ +C\,Min\left[\begin{array}{c}\Omega\{\partial(Ŧ_p(x_n,y_n),y_{n+1},t)\}\Omega\{\partial(Ŧ_p(x_{n+1},y_{n+1}),x_{n+1},t)\},\\ \dfrac{\Omega\{\partial(Ŧ_p(x_n,y_n),x_{n+1},t)\}\Omega\{\partial(x_n,Ŧ_p(x_{n+1},y_{n+1}),t)\}}{1+\Omega\{\partial(x_n,x_{n+1},t)\}+\Omega\{\partial(y_n,y_{n+1},t)\}}\end{array}\right]\end{array}\right\},\\ \varphi\left\{\begin{array}{c}A\Omega\{[\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]\}\\ +B\,Max\left[\Omega\{\partial(x_n,Ŧ_p(x_n,y_n),t)\},\Omega\{\partial(x_{n+1},Ŧ_p(x_{n+1},y_{n+1}),t)\},\Omega\{\partial(x_{n+1},Ŧ_p(x_n,y_n),t)\}\right]\\ +C\,Min\left[\begin{array}{c}\Omega\{\partial(Ŧ_p(x_n,y_n),y_{n+1},t)\}\Omega\{\partial(Ŧ_p(x_{n+1},y_{n+1}),x_{n+1},t)\},\\ \dfrac{\Omega\{\partial(Ŧ_p(x_n,y_n),x_{n+1},t)\}\Omega\{\partial(x_n,Ŧ_p(x_{n+1},y_{n+1}),t)\}}{1+\Omega\{\partial(x_n,x_{n+1},t)\}+\Omega\{\partial(y_n,y_{n+1},t)\}}\end{array}\right]\end{array}\right\}\end{array}\right]$$

$$\leq \mathbb{F}\left[\begin{array}{c}\phi\left\{\begin{array}{c}A\Omega\{[\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]\}\\ +B\,Max[\Omega\{\partial(x_n,x_{n+1},t)\},\Omega\{\partial(x_{n+1},x_{n+2},t)\},\Omega\{\partial(x_{n+1},x_n,t)\}]\\ +C\,Min\left[\Omega\{\partial(x_{n+1},y_{n+1},t)\}\Omega\{\partial(x_{n+2},x_{n+1},t)\},\dfrac{\Omega\{\partial(x_{n+1},x_{n+1},t)\}\Omega\{\partial(x_n,x_{n+2},t)\}}{1+\Omega\{\partial(x_n,x_{n+1},t)\}+\Omega\{\partial(y_n,y_{n+1},t)\}}\right]\end{array}\right\},\\ \varphi\left\{\begin{array}{c}A\Omega\{[\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]\}\\ +B\,Max[\Omega\{\partial(x_n,x_{n+1},t)\},\Omega\{\partial(x_{n+1},x_{n+2},t)\},\Omega\{\partial(x_{n+1},x_n,t)\}]\\ +C\,Min\left[\Omega\{\partial(x_{n+1},y_{n+1},t)\}\Omega\{\partial(x_{n+2},x_{n+1},t)\},\dfrac{\Omega\{\partial(x_{n+1},x_{n+1},t)\}\Omega\{\partial(x_n,x_{n+2},t)\}}{1+\Omega\{\partial(x_n,x_{n+1},t)\}+\Omega\{\partial(y_n,y_{n+1},t)\}}\right]\end{array}\right\}\end{array}\right]$$

$$\leq \phi\left\{\begin{array}{c}A\Omega\{[\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]\}\\ +B\,Max[\Omega\{\partial(x_n,x_{n+1},t)\},\Omega\{\partial(x_{n+1},x_{n+2},t)\},\Omega\{\partial(x_{n+1},x_n,t)\}]\\ +C\,Min\left[\Omega\{\partial(x_{n+1},y_{n+1},t)\}\Omega\{\partial(x_{n+2},x_{n+1},t)\},\dfrac{\Omega\{\partial(x_{n+1},x_{n+1},t)\}\Omega\{\partial(x_n,x_{n+2},t)\}}{1+\Omega\{\partial(x_n,x_{n+1},t)\}+\Omega\{\partial(y_n,y_{n+1},t)\}}\right]\end{array}\right\}$$

$$\leq A\Omega\{[\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]\}+B\,Max[\Omega\{\partial(x_n,x_{n+1},t)\},\Omega\{\partial(x_{n+1},x_{n+2},t)\}]$$
$$+C\,Min\left[\Omega\{\partial(x_{n+1},y_{n+1},t)\}\Omega\{\partial(x_{n+2},x_{n+1},t)\},\frac{\Omega\{0\}\Omega\{\partial(x_n,x_{n+2},t)\}}{1+\Omega\{\partial(x_n,x_{n+1},t)\}+\Omega\{\partial(y_n,y_{n+1},t)\}}\right]$$

$$\leq \phi\left\{\begin{array}{c}A[\partial(x_n,x_{n+1},t)+\partial(y_n,y_{n+1},t)]+B\,Max[\partial(x_n,x_{n+1},t),\partial(x_{n+1},x_{n+2},t)]\\ +C\,Min[\partial(x_{n+1},y_{n+1},t)\partial(x_{n+2},x_{n+1},t),0]\end{array}\right\}$$

Now follows same proof of theorem (3.1)

Corollary (1) If we take $C=0$ in theorem (3.1) then

let (X,∂) be a complete parametric metric space and $Ŧ_p: X\times X\to X$ is a continuous mapping, satisfying the following rational contractive condition:

$$\phi\left(\partial(Ŧ_p(x,y),Ŧ_p(r,s),t)\right)\leq \mathbb{F}\left(\phi(\Omega(x,y)),\varphi(\Omega(x,y))\right)$$

$for\ all\ x,y,r,s\in X\ ,for\ all\ t>0\ and\ \mathbb{F}\in\mathcal{C}\ ,\phi\in\Psi\ ,\varphi\in\Phi_u\ and$

$$\Omega(x,y) = A[\partial(x,r,t) + \partial(y,s,t)] + B\ Max\big[\partial\big(x, Ŧ_p(x,y),t\big), \partial\big(r, Ŧ_p(r,s),t\big), \partial\big(r, Ŧ_p(x,y),t\big)\big]$$

Where, $2A + B < 1$, and $a, b, r, s \in \left[0, \frac{1}{2}\right)$ then $Ŧ_p$ has a unique coupled Fixed point in $X \times X$.

Corollary (2) If we take $B = 0$ in theorem (3.1) then

let (X, ∂) be a complete parametric metric space and $Ŧ_p: X \times X \to X$ is a continuous mapping, satisfying the following rational contractive condition:

$$\phi\left(\partial\big(Ŧ_p(x,y), Ŧ_p(r,s), t\big)\right) \le \mathbb{F}\left(\phi\big(\Omega(x,y)\big), \varphi\big(\Omega(x,y)\big)\right)$$

$for\ all\ x, y, r, s \in X\ , for\ all\ t > 0\ and\ \mathbb{F} \in \mathbb{C}\ , \phi \in \Psi\ , \varphi \in \Phi_u\ and$

$$\Omega(x,y) = A[\partial(x,r,t) + \partial(y,s,t)] + C\ Min\left[\partial\big(Ŧ_p(x,y),s,t\big)\partial\big(Ŧ_p(r,s),r,t\big), \frac{\partial\big(Ŧ_p(x,y),r,t\big)\partial\big(x, Ŧ_p(r,s),t\big)}{1 + \partial(x,r,t) + \partial(y,s,t)}\right]$$

Where, $2A < 1$, and $a, c, r, s \in \left[0, \frac{1}{2}\right)$ then $Ŧ_p$ has a unique coupled Fixed point in $X \times X$.

Example 4. Let $X = [0,1]$ define $\partial: X \times X \to R^+\ by\ \partial(x,y),t) = |x - y| + |x|\ for\ all\ x, y \in X$. then (X, ∂) be a complete parametric metric space and define $Ŧ_p: X \times X \to X\ by\ Ŧ_p(x,y) = \frac{1}{8}xy\ \ for\ all\ x, y\ \in X$. since $|xy - rs| \le |x - r| + |y - s|$ and $|xy| \le |x| + |y|$. For all $x, y \in X$ then, $\partial\big(Ŧ_p(x,y), Ŧ_p(r,s), t\big) = \left|\frac{1}{8}xy - \frac{1}{8}rs\right| + \left|\frac{1}{8}xy\right|$

$$\le \frac{1}{8}(|x - r| + |y - s|) + \frac{1}{8}(|x| + |y|)$$

$$\le \frac{1}{8}(|x - r| + |y - s| + |x| + |y|)$$

$$\le \frac{1}{4}[(|x - r| + |x|) + \frac{1}{4}(|y - s| + |y|)$$

$$\Gamma(Ŧ(x,y), Ŧ(r,s), t) \le \frac{1}{4}[\partial(x,r,t) + \partial(y,s,t)].$$

So, $A = \frac{1}{4}\ and\ B = C = 0\ all\ the\ conditions\ of\ above\ theorem\ are\ satisfied.$

so (0,0) is the coupled fixed point of $Ŧ_p$ in $X \times X$.

4. Conclusion

In this research paper, we have improved, extended and generalized the many literatures theorem for new rational contractive conditions and prove fixed point theorem and coupled fixed point theorem for Parametric metric spaces with the assistance of C-class function. The result of our proposed work can be further extended for other metric spaces and fuzzy metric space.

Acknowledgements: The authors are thankful to the Referee for valuable comments and suggestions.

References

[1] Ahirwar, H. and Shrivastava K. (2024). Unique Fixed Point On Parametric Metric Space Employing C-Class Function. International Journal of Creative Research Thoughts, **12**(11).

[2] Ansari, A.H., Kumar, J. and Vashistha, S. (2019). C-class function on Common Fixed point theorem of Weakly Compatible Maps in Partial Metric space. International Journal of Advances in Mathematics, **3**, 15-23.

[3] Choudhary, B.S. and Maity, P. (2011). Coupled fixed point theorem in generalized metric spaces., Mathematical and Computer Modelling, **54**, 73-79.

[4] Choudhary, R. and Garg, A.K. (2019). Fixed point results in parametric metric space., International Journal on Emerging Technologies, **10**(2b), 100-104.

[5] Daheriya, R.D., Shrivastva, S. and Ughade, M. (2016). Parametric metric space, parametric b-metric space and expansive typing mapping. International Journal of Mathematics And its Applications, **4**, 107-117.

[6] Ege, O. and Karaca, I. (2017). Fixed point theorems and an application in parametric metric space., Azerbaijan Journal of Mathematics, **7**, 27-39.

[7] Garg, A.K. and Ahirwar, S. (2022). Parametric metric space and fixed point theorems, International Journal of research in applied, Natural and social sciences, **10**, 49-56.

[8] Krishnakumar, R. and Sanaatammappa, N.P. (2016). Some fixed point theorem in parametric metric space., International Journal of Mathematics Research, **8**, 213-220.

[9] Mohammod, M. et.al. (2018). Coupled fixed point theorem and dislocated quasi-metric space, Int. J., of pure and app. Math., **19**, 1249-1260.

[10] Singh, U.S. and Singh, N. (2020). A New Presumptive on Parametric Metric Space Via C-Class Function, Sambodhi 43.