Journal of Rajasthan Academy of Physical Sciences
ISSN: 0972-6306; URL: http://raops.org.in
Vol.22, No.1&2, January-June, 2023, 97-113

ON A WEAKLY-BERWALD SPACE WITH SPECIAL EXPONENTIAL (α, β)-METRIC

Brijesh Kumar Tripathi[1], Dhruvisha Patel[2] and T.N. Pandey[3]
[1]Department of Mathematics, L.D. College of Engineering Ahmedabad-380015 (Gujarat), India.
[2]Research scholar, Science Mathematics Branch, Gujarat Technological University, Ahmedabad, Gujarat-382424, India.
[3]Ex-Professor in Mathematics, Department of Mathematics and Statistics, D D U Gorakhpur University, Gorakhpur, U P, India.
E-mail: [1]brijeshkumartripathi4@gmail.com,[2] dhruvisha14299@gmail.com, [3]tnp1952@gmail.com

Abstract: In the field of theoretical physics, Riemannian metric 'α' and 1-form metric 'β' plays very important role and a homogeneous function of degree one with Riemannian and differentiable 1-form metric is known as (α, β)-metric which is a special kind Finsler metric and it is very interesting for the development of the special Finsler spaces. In the present paper, we have studied Weakly-Berwald space of a Finsler space with exponential form of (α, β)-metric, and examined the nature of Weakly-Berwald space for a special exponential form of (α, β)-metric with various constraints.

Keywords: Finsler space, Weakly Berwald space, (α, β)-metric, Exponential (α, β)-metric.

AMS Subject Classification: 53B40, 53C60.

1. Introduction

Let M^n be an n-dimensional C^∞ related manifold and T_xM is the tangent space of M^n at x then the tangent bundle of M^n is the union of tangent spaces $TM = \cup_{x\in M^n} T_xM$. Further if the elements of TM is given by (x, y), whereas $y\in T_xM^n$. Further let $TM_0=TM- \{0\}$, then the definition of Finsler space is given by

Definition 1.1. Let us consider a metric function L: TM → [0, ∞] on a n-dimensional smooth manifold M^n which satisfying following properties:

1. L is C^∞ on TM_0
2. L is positively 1-homogeneous on the fibers of tangent bundle TM, and
3. the Hessian of L^2 with element $g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\, \partial y^j}$ is regular on TM_0, i.e., $\det(g_{ij}) \neq 0$

then the space equipped with (M^n, L) is known as Finsler space and the metric L is known as Finsler metric.

Now if the metric L is a function α and β then the metric is known as (α, β)-metrics and the interesting and significant examples of an (α, β)-metrics are Randers metric (α + β), Kropina metric $\frac{\alpha^2}{\beta}$ and Matsumoto metric [7] $\frac{\alpha^2}{(\alpha-\beta)}$. The notion of an (α, β)-metric was introduced by M. Matsumoto [6] and has been studied by many authors [9, 10, 5, 6, 7, 12] and obtained various important results which are the back bones of the development of the Finsler geometry.

In 1929, L Berwald studied [2] an affinely connected Finsler space, if connection coefficients $G^i_{jk}$of BΓ are functions of position x^i alone which is same as the case of linear connection and in this case the equation of geodesics is given by

$$\frac{d^2x^i}{ds^2} + G^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0,$$

This is similar to the case of Riemannian space. Now a day Berwald space is defined as

Definition 1.2. A Finsler space is called a Berwald space, if the connection coefficients $G^i_{jk}$of BΓ are functions of position x^i alone, in any coordinate system.

The concept of Weakly-Berwald space introduced by Matsumoto [8] which is the generalization of Berwald space i.e. A Finsler space with an (α, β)-metric is a Weakly-Berwald space if $B^m_m = \frac{\partial B^m}{\partial y^m}$ is a one form [4], i.e., B^m_m is a homogeneous polynomial in (y^i) of degree one. He also investigated that a Finsler space with an (α, β)-metric is a Weakly-Berwald space, if B^m are homogeneous polynomials in (y^i) of degree two. Further several authors [1, 4, 15] studied various important and interesting properties for Weakly-Berwald space.

In 2020, Tripathi [13,14] considered a special type Finsler space with exponential (α, β)-metric and obtained the basic properties of Finsler space with exponential (α, β)-metric and obtained conditions for Finslerian hypersurfaces with this metric. In the present paper we have consider an n-dimensional Finsler space $F^n = \{M^n, L(\alpha, \beta)\}$, that is, a pair consisting of an n-dimensional differentiable manifold M^n equipped with a Fundamental function L as a special Finsler space with the metric

$$L(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{\frac{-\beta}{\alpha}}, \tag{1}$$

where $\alpha = \sqrt{\alpha_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ a differential one form β.

The metric given by equation (1) will be termed as special exponential (α, β)-metric. Pandey et al. [114] studied Berwald connection and geodesic of a Finsler space with generalized (α, β)-metric.

Further we obtained the condition for a Finsler space with special exponential (α, β)-metric which is defined in equation (1) will be a Weakly-Berwald space.

2. Preliminaries

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with an (α, β)-metric L (α, β). In this paper, the symbol (;) stands for h-covariant derivation with respect to the Riemannian connection in the associated Riemannian space (M^n, α) and $\gamma^i_{jk}(x)$ stands for the Christoffel symbols in the space (M^n, α).

We use the following notations [4].

$$r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}),\ s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}),\ r^i_j = a^{ir} r_{rj},\ s^i_j = a^{ir} s_{rj},\ r_j = b_r r^r_j,$$

$$s_j = b_r s^r_j,\ b^i = a^{ir} b_r,\ \ b^2 = a^{rs} b_r b_s.$$

Now we consider the functions $G^i(x,y)$ of F^n with an (α,β)-metric. According to [4], they are written in the form

$$2G^m = \gamma^m_{00} + 2B^m,$$

$$B^m = \left(\frac{E^*}{\alpha}\right) y^m + \left(\frac{\alpha L_\beta}{L_\alpha}\right) s^m_0 - \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right) C^* \left\{\left(\frac{y^m}{\alpha}\right) - \left(\frac{\alpha}{\beta}\right) b^m\right\}, \tag{2}$$

Where,

$$E^* = \left(\frac{\beta L_\beta}{L}\right) C^*,\ \ C^* = \frac{\alpha\beta(r_{00} L_\alpha - 2 s_0 \alpha L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})},$$

$$\gamma^2 = b^2\alpha^2 - \beta^2, \tag{3}$$

$$B^m_m = \left\{\dot\partial_m \left(\frac{\beta L_\beta}{\alpha L}\right) y^m + \frac{n\beta L_\beta}{\alpha L} - \dot\partial_m \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right)\left(\frac{\beta y^m - \alpha^2 b^m}{\alpha\beta}\right)\right\} C^* - \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right)\left\{\dot\partial_m \left(\frac{1}{\alpha}\right) y^m + \left(\frac{1}{\alpha}\right)\delta^m_m - \dot\partial_m \left(\frac{\alpha}{\beta}\right) b^m\right\} C^* + \left(\frac{\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha}}{\alpha L L_\alpha}\right)(\dot\partial_m C^*) y^m + \left(\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}\right)(\dot\partial_m C^*) b^m + \dot\partial_m \left(\frac{\alpha L_\beta}{L_\alpha}\right) s^m_0.$$

Since L = L(α, β) is a positively homogeneous function of α and β of degree one, we have

$$L_\alpha \alpha + L_\beta \beta = L,\ L_{\alpha\alpha}\alpha + L_{\alpha\beta}\beta = 0,$$

$$L_{\beta\alpha}\alpha + L_{\beta\beta}\beta = 0,\ \ \ L_{\alpha\alpha\alpha}\alpha + L_{\alpha\alpha\beta}\beta = -L_{\alpha\alpha}.$$

Using the above and the homogeneity of (y^i), we obtain

$$\dot\partial_m \left(\frac{\beta L_\beta}{\alpha L}\right) y^m = \frac{-\beta L_\beta}{\alpha L}, \tag{4}$$

$$\dot\partial_m \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right)\left(\frac{\beta y^m - \alpha^2 b^m}{\alpha\beta}\right) = \frac{\gamma^2}{(\beta L_\alpha)^2}\{L_\alpha L_{\alpha\alpha} + \alpha L_\alpha L_{\alpha\alpha\alpha} - \alpha(L_{\alpha\alpha})^2\}, \tag{5}$$

$$\dot\partial_m \left(\frac{1}{\alpha}\right) y^m + \left(\frac{1}{\alpha}\right)\delta^m_m - \dot\partial_m \left(\frac{\alpha}{\beta}\right) b^m = \frac{1}{\alpha\beta^2}(\gamma^2 + (n-1)\beta^2), \tag{6}$$

$$(\dot\partial_m C^*) y^m = 2C^*, \tag{7}$$

$$(\dot\partial_m C^*) b^m = \frac{1}{2\alpha\beta\Omega^2}\left[\Omega\{\beta(\gamma^2 + 2\beta^2)W + 2\alpha^2\beta^2 L_\alpha r_0 - \alpha\beta\gamma^2 L_{\alpha\alpha} r_{00} - 2\alpha(\beta^3 L_\beta + \alpha^2\gamma^2 L_{\alpha\alpha}) s_0\} - \alpha^2\beta W\{2b^2\beta^2 L_\alpha - \gamma^4 L_{\alpha\alpha\alpha} - b^2\alpha\gamma^2 L_{\alpha\alpha}\}\right], \tag{8}$$

$$\dot{\partial}_m \left(\frac{\alpha L_\beta}{L_\alpha}\right) s_0^m = \frac{\alpha^2 L L_{\alpha\alpha} s_0}{(\beta L_\alpha)^2}, \quad (9)$$

Where,

$$W = \left(r_{00}L_\alpha - 2s_0\alpha L_\beta\right), \Omega = (\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \text{ provided that } \Omega \neq 0. \quad (10)$$

$Y_i = a_{ir}y^r,\ s_{00} = 0,\ b^r s_r = 0,\ a^{ij}s_{ij} = 0.$

Substituting (4), (5), (6), (7), (8) and (9) into (3), we have

$$B_m^m = \frac{1}{2\alpha L(\beta L_\alpha)^2\Omega^2}\{2\Omega^2 AC^* + 2\alpha L\Omega^2 Bs_0 + \alpha^2 LL_\alpha L_{\alpha\alpha}(Cr_{00} + Ds_0 + Er_0)\}, \quad (11)$$

$$A = (n+1)\beta^2 L_\alpha\left(\beta L_\alpha L_\beta - \alpha LL_{\alpha\alpha}\right) + \alpha\gamma^2 L\{\alpha(L_{\alpha\alpha})^2 - 2L_\alpha L_{\alpha\alpha} - \alpha L_\alpha L_{\alpha\alpha\alpha}\}, \quad (12)$$

$B = \alpha^2 LL_{\alpha\alpha}$,

$C = \beta\gamma^2\{-\beta^2(L_\alpha)^2 + 2b^2\alpha^3 L_\alpha L_{\alpha\alpha} - \alpha^2\gamma^2(L_{\alpha\alpha})^2 + \alpha^2\gamma^2 L_\alpha L_{\alpha\alpha\alpha}\}$,

$D = 2\alpha\{\beta^3(\gamma^2 - \beta^2)L_\alpha L_\beta - \alpha^2\beta^2\gamma^2 L_\alpha L_{\alpha\alpha} - 2\alpha\beta\gamma^2(\gamma^2 + 2\beta^2)L_\beta L_{\alpha\alpha} - \alpha^3\gamma^4(L_{\alpha\alpha})^2 - \alpha^2\beta\gamma^4 L_\beta L_{\alpha\alpha\alpha}\}$,

$E = 2\alpha^2\beta^2 L_\alpha\Omega.$

Thus we have [4]

Theorem 2.1. The necessary and sufficient condition for a Finsler space F^n with an (α,β)-metric to be a Weakly-Berwald space is that $G_m^m = \gamma_{0m}^m + B_m^m$ and B_m^m is a homogeneous polynomial in (y^m) of degree one, where B_m^m is given by (11) and (12), provided that $\Omega^2 \neq 0$.

Lemma 2.1.[3] If α^2 contains β as a factor, then the dimension is equal to two and $b^2 \neq 0$. Throughout this paper, we assume that the dimension is more than two and $b^2 \neq 0$, that is, $\alpha^2 \neq 0 (\mathrm{mod}\, \beta)$.

Lemma 2.2.[3] If $\alpha^2 \cong 0$ (mod β), that is, $a_{ij}(x)y^iy^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_ib_i = 2$.

3. The Condition to be a Weakly-Berwald Space

In this section, we have obtained the condition for a F^n Finsler space with an exponential (α, β)-metric (1) is a Weakly Berwald space.

The partial derivative with respect to α and β of (1) are given by

$$L_\alpha = \frac{\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha\beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^2}, \quad L_\beta = \frac{\alpha e^{\frac{\beta}{\alpha}} + \alpha e^{\frac{-\beta}{\alpha}} - \beta e^{\frac{-\beta}{\alpha}}}{\alpha} \quad (13)$$

$$L_{\alpha\alpha} = \frac{\alpha\beta^2 e^{\frac{\beta}{\alpha}} - 2\alpha\beta^2 e^{\frac{-\beta}{\alpha}} + \beta^3 e^{\frac{-\beta}{\alpha}}}{\alpha^4}, \quad L_{\beta\beta} = \frac{\alpha e^{\frac{\beta}{\alpha}} - 2\alpha e^{\frac{-\beta}{\alpha}} + \beta e^{\frac{-\beta}{\alpha}}}{\alpha^2}$$

$$L_{\alpha\beta} = \frac{-\alpha\beta e^{\frac{\beta}{\alpha}} + 2\alpha\beta e^{\frac{-\beta}{\alpha}} - \beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^3}, L_{\beta\alpha} = \frac{-\alpha\beta e^{\frac{\beta}{\alpha}} + 2\alpha\beta e^{\frac{-\beta}{\alpha}} - \beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^3}$$

$$L_{\alpha\alpha\alpha} = \frac{-3\alpha^2\beta^2 e^{\frac{\beta}{\alpha}} - \alpha\beta^3 e^{\frac{\beta}{\alpha}} + 6\alpha^2\beta^2 e^{\frac{-\beta}{\alpha}} - 6\alpha\beta^3 e^{\frac{-\beta}{\alpha}} + \beta^4 e^{\frac{-\beta}{\alpha}}}{\alpha^6},$$

$$L_{\alpha\alpha\beta} = \frac{2\alpha^2\beta e^{\frac{\beta}{\alpha}} + \alpha\beta^2 e^{\frac{\beta}{\alpha}} - 4\alpha^2\beta e^{\frac{-\beta}{\alpha}} + 5\alpha\beta^2 e^{\frac{-\beta}{\alpha}} - \beta^3 e^{\frac{-\beta}{\alpha}}}{\alpha^5},$$

Since $L(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ is a positively homogeneous functionof α and β of degree one, we have

$$L_\alpha \alpha + L_\beta \beta = \left(\frac{\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha\beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^2}\right)\alpha + \left(\frac{\alpha e^{\frac{\beta}{\alpha}} + \alpha e^{\frac{-\beta}{\alpha}} - \beta e^{\frac{-\beta}{\alpha}}}{\alpha}\right)\beta = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}} = L(\alpha, \beta),$$

$$L_{\alpha\alpha}\alpha + L_{\alpha\beta}\beta = \alpha\left(\frac{\beta^2 e^{\frac{\beta}{\alpha}}}{\alpha^3} - \frac{2\beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^3} + \frac{\beta^3 e^{\frac{-\beta}{\alpha}}}{\alpha^4}\right) + \frac{\beta\left(-\alpha\beta e^{\frac{\beta}{\alpha}} + 2\alpha\beta e^{\frac{-\beta}{\alpha}} - \beta^2 e^{\frac{-\beta}{\alpha}}\right)}{\alpha^3} = 0,$$

$$L_{\alpha\alpha\alpha}\alpha + L_{\alpha\alpha\beta}\beta = \left(\frac{-3\alpha^2\beta^2 e^{\frac{\beta}{\alpha}} - \alpha\beta^3 e^{\frac{\beta}{\alpha}} + 6\alpha^2\beta^2 e^{\frac{-\beta}{\alpha}} - 6\alpha\beta^3 e^{\frac{-\beta}{\alpha}} + \beta^4 e^{\frac{-\beta}{\alpha}}}{\alpha^6}\right)\alpha + \left(\frac{2\alpha^2\beta e^{\frac{\beta}{\alpha}} + \alpha\beta^2 e^{\frac{\beta}{\alpha}} - 4\alpha^2\beta e^{\frac{-\beta}{\alpha}} + 5\alpha\beta^2 e^{\frac{-\beta}{\alpha}} - \beta^3 e^{\frac{-\beta}{\alpha}}}{\alpha^5}\right)\beta = -\left(\frac{\beta^2 e^{\frac{\beta}{\alpha}}}{\alpha^3} - \frac{2\beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^3} + \frac{\beta^3 e^{\frac{-\beta}{\alpha}}}{\alpha^4}\right) = -L_{\alpha\alpha},$$

$$\alpha L_{\beta\alpha} + \beta L_{\beta\beta} = \left(\frac{-\alpha\beta e^{\frac{\beta}{\alpha}} + 2\alpha\beta e^{\frac{-\beta}{\alpha}} - \beta^2 e^{\frac{-\beta}{\alpha}}}{\alpha^3}\right)\alpha + \left(\frac{\alpha e^{\frac{\beta}{\alpha}} - 2\alpha e^{\frac{-\beta}{\alpha}} + \beta e^{\frac{-\beta}{\alpha}}}{\alpha^2}\right)\beta = 0.$$

Substituting (13) into (2), (3), (10) and (12), we have

$$C^* = \frac{\alpha^2\beta}{2}\{\frac{M}{N}\}$$

Where,

$$M = r_{00}\left(\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha\beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}}\right) - 2S_0\alpha^2\left(\alpha e^{\frac{\beta}{\alpha}} + \alpha e^{\frac{-\beta}{\alpha}} - \beta e^{\frac{-\beta}{\alpha}}\right),$$

$$N = \alpha\beta^2\left(\alpha^2 e^{\frac{\beta}{\alpha}} - \alpha\beta e^{\frac{\beta}{\alpha}} + \beta^2 e^{\frac{-\beta}{\alpha}}\right) + \alpha^2 b^2\left(\alpha\beta^2 e^{\frac{\beta}{\alpha}} - 2\alpha\beta^2 e^{\frac{-\beta}{\alpha}} + \beta^3 e^{\frac{-\beta}{\alpha}}\right) - \beta^2\left(\alpha\beta^2 e^{\frac{\beta}{\alpha}} - 2\alpha\beta^2 e^{\frac{-\beta}{\alpha}} + \beta^3 e^{\frac{-\beta}{\alpha}}\right),$$

$$E^{*}=\left(\frac{\alpha\beta e^{\frac{\beta}{\alpha}}+\alpha\beta e^{\frac{-\beta}{\alpha}}-\beta^{2}e^{\frac{-\beta}{\alpha}}}{\alpha^{2}e^{\frac{\beta}{\alpha}}+\alpha\beta e^{\frac{-\beta}{\alpha}}}\right)\times C^{*},$$

$$B^{m}=$$
$$C^{*}\times\left\{\left(\frac{\alpha\beta e^{\frac{\beta}{\alpha}}+\alpha\beta e^{-\frac{\beta}{\alpha}}-\beta^{2}e^{\frac{-\beta}{\alpha}}}{\alpha^{2}e^{\frac{\beta}{\alpha}}+\alpha\beta e^{\frac{-\beta}{\alpha}}}-\frac{\alpha\beta^{2}e^{\frac{\beta}{\alpha}}-2\alpha\beta^{2}e^{\frac{-\beta}{\alpha}}+\beta^{3}e^{\frac{-\beta}{\alpha}}}{\alpha^{4}e^{\frac{\beta}{\alpha}}-\alpha^{3}\beta e^{\frac{\beta}{\alpha}}+\alpha^{2}\beta^{2}e^{\frac{-\beta}{\alpha}}}\right)y^{m}+\left(\frac{\alpha\beta e^{\frac{\beta}{\alpha}}-2\alpha\beta e^{\frac{-\beta}{\alpha}}+\beta^{2}e^{\frac{-\beta}{\alpha}}}{\alpha^{2}e^{\frac{\beta}{\alpha}}-\alpha\beta e^{\frac{\beta}{\alpha}}+\beta^{2}e^{\frac{-\beta}{\alpha}}}\right)b^{m}\right\}+$$
$$\left(\frac{\alpha^{3}e^{\frac{\beta}{\alpha}}+\alpha^{3}e^{\frac{-\beta}{\alpha}}-\alpha^{2}\beta e^{\frac{-\beta}{\alpha}}}{\alpha^{2}e^{\frac{\beta}{\alpha}}-\alpha\beta e^{\frac{\beta}{\alpha}}+\beta^{2}e^{\frac{-\beta}{\alpha}}}\right)s_{0}^{m}, \qquad (14)$$

$$W=\frac{r_{00}\left(\alpha^{2}e^{\frac{\beta}{\alpha}}-\alpha\beta e^{\frac{\beta}{\alpha}}+\beta^{2}e^{\frac{-\beta}{\alpha}}\right)-2S_{0}\alpha^{2}\left(\alpha e^{\frac{\beta}{\alpha}}+\alpha e^{\frac{-\beta}{\alpha}}-\beta e^{\frac{-\beta}{\alpha}}\right)}{\alpha^{2}}, \qquad (15)$$

$$\Omega=\frac{P}{Q} \qquad (16)$$

Where

$$P=\alpha^{3}\beta^{2}e^{\frac{\beta}{\alpha}}-\alpha^{2}\beta^{3}e^{\frac{\beta}{\alpha}}+\alpha\beta^{4}e^{-\frac{\beta}{\alpha}}+\alpha^{3}\beta^{2}b^{2}e^{\frac{\beta}{\alpha}}-2\alpha^{3}\beta^{2}e^{\frac{-\beta}{\alpha}}b^{2}+\alpha^{2}\beta^{3}e^{\frac{-\beta}{\alpha}}b^{2}-\alpha\beta^{4}e^{\frac{\beta}{\alpha}}+$$
$$2\alpha\beta^{4}e^{\frac{-\beta}{\alpha}}-\beta^{5}e^{\frac{-\beta}{\alpha}},$$

$$Q=\alpha^{3},$$

$$\boldsymbol{A}$$
$$=\left(\frac{n+1}{\alpha^{5}}\right)\left\{\left(-2\alpha^{4}\beta^{4}e^{\frac{3\beta}{\alpha}}+\alpha^{5}\beta^{3}e^{\frac{3\beta}{\alpha}}+\alpha^{5}\beta^{3}e^{\frac{\beta}{\alpha}}+4\alpha^{3}\beta^{5}e^{\frac{-\beta}{\alpha}}-5\alpha^{2}\beta^{6}e^{\frac{-\beta}{\alpha}}+2\alpha^{3}\beta^{5}e^{\frac{3\beta}{\alpha}}-\alpha^{4}\beta^{4}e^{\frac{\beta}{\alpha}}-\alpha^{4}\beta^{4}e^{\frac{3\beta}{\alpha}}+2\alpha\beta^{7}e^{\frac{-\beta}{\alpha}}-2\alpha^{2}\beta^{6}e^{\frac{\beta}{\alpha}}+\alpha^{3}\beta^{5}e^{\frac{\beta}{\alpha}}\right.\right.$$
$$\left.+\;3\alpha\beta^{7}e^{\frac{-3\beta}{\alpha}}-2\beta^{8}e^{\frac{-3\beta}{\alpha}}\right)$$
$$+\;\frac{1}{\alpha^{5}}\left(4b^{2}\alpha^{3}\beta^{5}e^{\frac{\beta}{\alpha}}-8b^{2}\alpha^{4}\beta^{4}e^{\frac{\beta}{\alpha}}+8b^{2}\alpha^{4}\beta^{4}e^{\frac{-\beta}{\alpha}}+b^{2}\alpha^{6}\beta^{2}e^{\frac{3\beta}{\alpha}}-2b^{2}\alpha^{6}\beta^{2}e^{\frac{\beta}{\alpha}}+6b^{2}\alpha^{5}\beta^{3}e^{\frac{\beta}{\alpha}}+4b^{2}\alpha^{2}\beta^{6}e^{\frac{-\beta}{\alpha}}-8b^{2}\alpha^{3}\beta^{5}e^{\frac{-\beta}{\alpha}}\right.$$
$$\left.\left.+\;2b^{2}\alpha^{3}\beta^{5}e^{\frac{-3\beta}{\alpha}}+b^{2}\alpha^{5}\beta^{3}e^{\frac{\beta}{\alpha}}-2b^{2}\alpha^{5}\beta^{3}e^{\frac{-\beta}{\alpha}}-4\beta^{8}e^{\frac{-\beta}{\alpha}}+8\alpha\beta^{7}e^{\frac{-\beta}{\alpha}}-2\alpha\beta^{7}e^{\frac{-3\beta}{\alpha}}-\alpha^{3}\beta^{5}e^{\frac{\beta}{\alpha}}+2\alpha^{3}\beta^{5}e^{\frac{-\beta}{\alpha}}-8\alpha^{2}\beta^{6}e^{\frac{-\beta}{\alpha}}-4\alpha\beta^{7}e^{\frac{\beta}{\alpha}}+8\alpha^{2}\beta^{6}e^{\frac{\beta}{\alpha}}-\alpha^{4}\beta^{4}e^{\frac{3\beta}{\alpha}}+2\alpha^{4}\beta^{4}e^{\frac{\beta}{\alpha}}-6\alpha^{3}\beta^{5}e^{\frac{\beta}{\alpha}}\right)\right\}$$
$$(17)$$

$$B=\frac{\alpha^{2}\beta^{2}e^{\frac{2\beta}{\alpha}}-2\alpha^{2}\beta^{2}+2\alpha\beta^{3}-2\alpha\beta^{3}e^{\frac{-2\beta}{\alpha}}+\beta^{4}e^{\frac{-2\beta}{\alpha}}}{\alpha^{2}}, \qquad (18)$$

$$C=\frac{\beta(b^{2}\alpha^{2}-\beta^{2})}{\alpha^{5}}\times\left(-\alpha^{5}\beta^{2}e^{\frac{2\beta}{\alpha}}-8\alpha^{3}\beta^{4}-3\alpha\beta^{6}e^{\frac{-2\beta}{\alpha}}+2\alpha^{4}\beta^{3}e^{\frac{2\beta}{\alpha}}+14\alpha^{2}\beta^{5}+\right.$$
$$2\alpha^{3}\beta^{4}e^{\frac{2\beta}{\alpha}}-b^{2}\alpha^{5}\beta^{2}e^{\frac{2\beta}{\alpha}}+2\alpha^{5}\beta^{2}b^{2}-6b^{2}\alpha^{4}\beta^{3}-8b^{2}\alpha^{3}\beta^{4}e^{\frac{-2\beta}{\alpha}}+6b^{2}\alpha^{2}\beta^{5}e^{\frac{-2\beta}{\alpha}}-$$
$$\left.4b^{2}\alpha^{2}\beta^{5}-b^{2}\alpha\beta^{6}e^{\frac{-2\beta}{\alpha}}+8b^{2}\alpha^{3}\beta^{4}+4\beta^{7}-8\alpha\beta^{6}+2\beta^{7}e^{\frac{-2\beta}{\alpha}}-2\alpha^{2}\beta^{5}e^{\frac{2\beta}{\alpha}}\right), \qquad (19)$$

$$D=\frac{1}{\alpha^{3}}\times\left(-4b^{2}\alpha^{5}\beta^{4}e^{\frac{2\beta}{\alpha}}+10b^{2}\alpha^{4}\beta^{5}+2b^{2}\alpha^{6}\beta^{3}e^{\frac{2\beta}{\alpha}}+2b^{2}\alpha^{6}\beta^{3}+26b^{2}\alpha^{4}\beta^{5}e^{\frac{-2\beta}{\alpha}}-\right.$$
$$30b^{2}\alpha^{3}\beta^{6}e^{\frac{-2\beta}{\alpha}}+6\alpha^{3}\beta^{6}e^{\frac{2\beta}{\alpha}}+4\alpha^{3}\beta^{6}-12\alpha^{2}\beta^{7}-4\alpha^{4}\beta^{5}e^{\frac{2\beta}{\alpha}}-4\alpha^{4}\beta^{5}-28\alpha^{2}\beta^{7}e^{\frac{-2\beta}{\alpha}}+$$
$$4\alpha\beta^{8}e^{\frac{-2\beta}{\alpha}}-18b^{2}\alpha^{4}\beta^{5}e^{\frac{2\beta}{\alpha}}+10b^{2}\alpha^{2}\beta^{7}e^{\frac{-2\beta}{\alpha}}+8\alpha^{2}\beta^{7}e^{\frac{2\beta}{\alpha}}+2b^{4}\alpha^{6}\beta^{3}e^{\frac{2\beta}{\alpha}}-2b^{4}\alpha^{6}\beta^{3}+$$

$$b^4\alpha^5\beta^4 - 4b^4\alpha^6\beta^3 e^{\frac{-2\beta}{\alpha}} + 20b^4\alpha^5\beta^4 e^{\frac{-2\beta}{\alpha}} + 4b^4\alpha^4\beta^5 e^{\frac{-2\beta}{\alpha}} + 8b^2\alpha^4\beta^5 e^{\frac{2\beta}{\alpha}} - 32b^2\alpha^3\beta^6 + 24\alpha\beta^8 - 8b^4\alpha^4\beta^5 + 16b^2\alpha^2\beta^7 - 8\beta^9 - 14b^4\alpha^4\beta^5 e^{\frac{-2\beta}{\alpha}}\Big), \quad (20)$$

$$E = \frac{1}{\alpha^3} \times \Big(2\alpha^5\beta^4 e^{\frac{2\beta}{\alpha}} - 4\alpha^4\beta^5 e^{\frac{2\beta}{\alpha}} + 8\alpha^3\beta^6 + 2b^2\alpha^5\beta^4 e^{\frac{2\beta}{\alpha}} - 4\alpha^5\beta^4 b^2 + 6b^2\alpha^4\beta^5 - 10\alpha^2\beta^7 - 2b^2\alpha^4\beta^5 e^{\frac{2\beta}{\alpha}} + 2\alpha^2\beta^7 e^{\frac{2\beta}{\alpha}} + 6\alpha\beta^8 e^{\frac{-2\beta}{\alpha}} - 4b^2\alpha^3\beta^6 e^{\frac{-2\beta}{\alpha}} + 2b^2\alpha^2\beta^7 e^{\frac{-2\beta}{\alpha}} - 2\beta^9 e^{\frac{-2\beta}{\alpha}}\Big). \quad (21)$$

Substituting (17), (18), (19), (20), (21) into (11), we have

$$B^m_m\{a_0\alpha^{11}\beta^2 + a_1\alpha^{10}\beta^3 + a_2\alpha^9\beta^4 + a_3\alpha^8\beta^5 + a_4\alpha^7\beta^6 + a_5\alpha^6\beta^7 + a_6\alpha^5\beta^8 + a_7\alpha^4\beta^9 + a_8\alpha^3\beta^{10} + a_9\alpha^2\beta^{11} + a_{10}\alpha\beta^{12} + a_{11}\beta^{13}\} + r_{00}\{a_{12}\alpha^{10}\beta^2 + a_{13}\alpha^9\beta^3 + a_{14}\alpha^8\beta^4 + a_{15}\alpha^7\beta^5 + a_{16}\alpha^6\beta^6 + a_{17}\alpha^5\beta^7 + a_{18}\alpha^4\beta^8 + a_{19}\alpha^3\beta^9 + a_{20}\alpha^2\beta^{10} + a_{21}\alpha\beta^{11} + a_{22}\beta^{12}\} + s_0\{a_{23}\alpha^{13} + a_{24}\alpha^{12}\beta + a_{25}\alpha^{11}\beta^2 + a_{26}\alpha^{10}\beta^3 + a_{27}\alpha^9\beta^4 + a_{28}\alpha^8\beta^5 + a_{29}\alpha^7\beta^6 + a_{30}\alpha^6\beta^7 + a_{31}\alpha^5\beta^8 + a_{32}\alpha^4\beta^9 + a_{33}\alpha^3\beta^{10} + a_{34}\alpha^2\beta^{11}\} + r_0\{a_{35}\alpha^{11}\beta^2 + a_{36}\alpha^{10}\beta^3 + a_{37}\alpha^9\beta^4 + a_{38}\alpha^8\beta^5 + a_{39}\alpha^7\beta^6 + a_{40}\alpha^6\beta^7 + a_{41}\alpha^5\beta^8 + a_{42}\alpha^4\beta^9 + a_{43}\alpha^3\beta^{10} + a_{44}\alpha^2\beta^{11}\} \quad (22)$$

Where,

$$a_0 = \Big(2e^{\frac{5\beta}{\alpha}} + 4b^2 e^{\frac{5\beta}{\alpha}} - 8b^2 e^{\frac{3\beta}{\alpha}} + 2b^4 e^{\frac{5\beta}{\alpha}} - 8b^4 e^{\frac{3\beta}{\alpha}} + 8b^4 e^{\frac{\beta}{\alpha}}\Big),$$

$$a_1 = \Big(-8e^{\frac{5\beta}{\alpha}} + 32b^2 e^{\frac{3\beta}{\alpha}} - 12b^2 e^{\frac{5\beta}{\alpha}} + 22b^4 e^{\frac{3\beta}{\alpha}} - 32b^4 e^{\frac{\beta}{\alpha}} - 4b^4 e^{\frac{5\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}} - 8b^2 e^{\frac{\beta}{\alpha}} + 8b^4 e^{\frac{-\beta}{\alpha}}\Big),$$

$$a_2 = \Big(8e^{\frac{5\beta}{\alpha}} + 8e^{\frac{3\beta}{\alpha}} - 20b^2 e^{\frac{3\beta}{\alpha}} + 8b^2 e^{\frac{5\beta}{\alpha}} + 30b^4 e^{\frac{\beta}{\alpha}} - 16b^4 e^{\frac{3\beta}{\alpha}} + 2b^4 e^{\frac{5\beta}{\alpha}} - 8b^4 e^{\frac{-\beta}{\alpha}} - 12b^2 e^{\frac{\beta}{\alpha}}\Big),$$

$$a_3 = \Big(-44e^{\frac{3\beta}{\alpha}} + 4e^{\frac{5\beta}{\alpha}} + 100b^2 e^{\frac{\beta}{\alpha}} - 36b^2 e^{\frac{3\beta}{\alpha}} + 4b^2 e^{\frac{5\beta}{\alpha}} + 16e^{\frac{\beta}{\alpha}} - 40b^2 e^{\frac{-\beta}{\alpha}} + 16b^4 e^{\frac{-3\beta}{\alpha}} + 2b^4 e^{\frac{3\beta}{\alpha}} - 22b^4 e^{\frac{-\beta}{\alpha}}\Big),$$

$$a_4 = \Big(44e^{\frac{3\beta}{\alpha}} - 8e^{\frac{\beta}{\alpha}} - 100b^2 e^{\frac{\beta}{\alpha}} - 8e^{\frac{5\beta}{\alpha}} + 52b^2 e^{\frac{-\beta}{\alpha}} - 4b^4 e^{\frac{\beta}{\alpha}} + 24b^4 e^{\frac{-\beta}{\alpha}} - 24b^4 e^{\frac{-3\beta}{\alpha}} + 40b^2 e^{\frac{3\beta}{\alpha}} - 4b^2 e^{\frac{5\beta}{\alpha}} - 4b^2 e^{\frac{3\beta}{\alpha}}\Big),$$

$$a_5 = \left(-68e^{\frac{\beta}{\alpha}} + 8e^{\frac{3\beta}{\alpha}} + 44b^2e^{\frac{-\beta}{\alpha}} + 8b^2e^{\frac{\beta}{\alpha}} - 4b^4e^{\frac{-\beta}{\alpha}} + 4b^4e^{\frac{-3\beta}{\alpha}} + 44e^{\frac{-\beta}{\alpha}} - 56b^2e^{\frac{-3\beta}{\alpha}} + 8b^4e^{\frac{-5\beta}{\alpha}} - 4b^2e^{\frac{3\beta}{\alpha}}\right),$$

$$a_6 = \left(74e^{\frac{\beta}{\alpha}} - 60e^{\frac{-\beta}{\alpha}} - 72b^2e^{\frac{-\beta}{\alpha}} + 76b^2e^{\frac{-3\beta}{\alpha}} + 8b^2e^{\frac{\beta}{\alpha}} + 2b^4e^{\frac{-3\beta}{\alpha}} - 20e^{\frac{3\beta}{\alpha}} + 2e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{-5\beta}{\alpha}}\right),$$

$$a_7 = \left(-18e^{\frac{-\beta}{\alpha}} - 8e^{\frac{\beta}{\alpha}} - 12b^2e^{\frac{-3\beta}{\alpha}} + 12b^2e^{\frac{-\beta}{\alpha}} + 48e^{\frac{-3\beta}{\alpha}} - 23b^2e^{\frac{-5\beta}{\alpha}} - 4b^2e^{\frac{-\beta}{\alpha}} + 2b^4e^{\frac{-5\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}}\right),$$

$$a_8 = \left(40e^{\frac{-\beta}{\alpha}} - 58e^{\frac{-3\beta}{\alpha}} - 4b^2e^{\frac{-3\beta}{\alpha}} - 4e^{\frac{\beta}{\alpha}} + 20b^2e^{\frac{-5\beta}{\alpha}}\right),$$

$$a_9 = \left(8e^{\frac{-3\beta}{\alpha}} - 4e^{\frac{-\beta}{\alpha}} - 4b^2e^{\frac{-5\beta}{\alpha}}\right),$$

$$a_{10} = \left(2e^{\frac{-3\beta}{\alpha}} - 12e^{\frac{-5\beta}{\alpha}}\right),$$

$$a_{11} = \left(2e^{\frac{-5\beta}{\alpha}}\right),$$

$$a_{12} = \left\{n\left(-e^{\frac{5\beta}{\alpha}} - e^{\frac{3\beta}{\alpha}} - b^2e^{\frac{5\beta}{\alpha}} + b^2e^{\frac{3\beta}{\alpha}} + 2b^2\right) - e^{\frac{5\beta}{\alpha}} - e^{\frac{3\beta}{\alpha}} - 2b^2e^{\frac{5\beta}{\alpha}} - 6b^2e^{\frac{3\beta}{\alpha}} + 2b^2 + 12b^4e^{\frac{3\beta}{\alpha}}\right\},$$

$$a_{13} = \left\{n\left(5e^{\frac{5\beta}{\alpha}} + 3e^{\frac{3\beta}{\alpha}} - 5b^2e^{\frac{\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} - 7b^2e^{\frac{3\beta}{\alpha}}\right) + 5e^{\frac{5\beta}{\alpha}} + 3e^{\frac{3\beta}{\alpha}} + 2b^2e^{\frac{\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} + 8b^2e^{\frac{3\beta}{\alpha}} + 32b^4e^{\frac{3\beta}{\alpha}} + 48b^4e^{\frac{-\beta}{\alpha}} - 45b^4e^{\frac{\beta}{\alpha}} - 51b^4e^{\frac{\beta}{\alpha}}\right\},$$

$$a_{14} = \left\{n\left(-8e^{\frac{\beta}{\alpha}} - 7e^{\frac{3\beta}{\alpha}} - 8e^{\frac{5\beta}{\alpha}} + 10b^2e^{\frac{-\beta}{\alpha}} + 11b^2e^{\frac{3\beta}{\alpha}} + 3b^2e^{\frac{\beta}{\alpha}} - 5b^2e^{\frac{5\beta}{\alpha}}\right) - 6e^{\frac{\beta}{\alpha}} - 7e^{\frac{5\beta}{\alpha}} - 2b^2e^{\frac{-\beta}{\alpha}} - 13b^2e^{\frac{3\beta}{\alpha}} + 66b^2e^{\frac{\beta}{\alpha}} + 2b^2e^{\frac{5\beta}{\alpha}} - 35b^2e^{\frac{\beta}{\alpha}} + 47b^4e^{\frac{\beta}{\alpha}} + 36b^4e^{\frac{-\beta}{\alpha}} - 12b^4e^{\frac{-3\beta}{\alpha}}\right\},$$

$$a_{15} = \left\{n\left(20e^{\frac{3\beta}{\alpha}} + 22e^{\frac{\beta}{\alpha}} + 3e^{\frac{5\beta}{\alpha}} - 25b^2e^{\frac{-\beta}{\alpha}} - 3b^2e^{\frac{3\beta}{\alpha}} - 5b^2e^{\frac{\beta}{\alpha}} + 2b^2e^{\frac{5\beta}{\alpha}}\right) - e^{\frac{3\beta}{\alpha}} + 35e^{\frac{\beta}{\alpha}} + 3e^{\frac{5\beta}{\alpha}} - 80b^2e^{\frac{-\beta}{\alpha}} - 33b^2e^{\frac{3\beta}{\alpha}} + 119b^2e^{\frac{\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} - 24b^4e^{\frac{\beta}{\alpha}} + 11b^4e^{\frac{-\beta}{\alpha}} + 6b^4e^{\frac{-3\beta}{\alpha}} + 8b^4e^{\frac{3\beta}{\alpha}} + b^4e^{\frac{\beta}{\alpha}} + 43b^4e^{\frac{-\beta}{\alpha}}\right\},$$

$$a_{16}=\Big\{n\Big(-26e^{\frac{3\beta}{\alpha}}+14b^2e^{\frac{-3\beta}{\alpha}}-24e^{\frac{\beta}{\alpha}}+19b^2e^{\frac{-\beta}{\alpha}}-22e^{\frac{-\beta}{\alpha}}+3e^{\frac{5\beta}{\alpha}}+6b^2e^{\frac{\beta}{\alpha}}-2b^2e^{\frac{3\beta}{\alpha}}\Big)+19e^{\frac{3\beta}{\alpha}}+2b^2e^{\frac{-3\beta}{\alpha}}-67e^{\frac{\beta}{\alpha}}+29b^2e^{\frac{-\beta}{\alpha}}-8e^{\frac{-\beta}{\alpha}}+11b^2e^{\frac{\beta}{\alpha}}-8b^2e^{\frac{3\beta}{\alpha}}+4b^4e^{\frac{-\beta}{\alpha}}-b^4e^{\frac{\beta}{\alpha}}+4b^4e^{\frac{-3\beta}{\alpha}}-12b^4e^{\frac{-5\beta}{\alpha}}\Big\},$$

$$a_{17}=\Big\{n\Big(52e^{\frac{-\beta}{\alpha}}+23e^{\frac{\beta}{\alpha}}-27b^2e^{\frac{-3\beta}{\alpha}}+11e^{\frac{3\beta}{\alpha}}-6b^2e^{\frac{-\beta}{\alpha}}-2e^{\frac{5\beta}{\alpha}}-2e^{\frac{3\beta}{\alpha}}\Big)+49e^{\frac{-\beta}{\alpha}}+56e^{\frac{\beta}{\alpha}}-18b^2e^{\frac{-3\beta}{\alpha}}-24e^{\frac{3\beta}{\alpha}}-8b^2e^{\frac{-\beta}{\alpha}}+33b^2e^{\frac{\beta}{\alpha}}+b^4e^{\frac{-\beta}{\alpha}}-5b^4e^{\frac{-3\beta}{\alpha}}+18b^4e^{\frac{-5\beta}{\alpha}}\Big\},$$

$$a_{18}=\Big\{n\Big(-41e^{\frac{-\beta}{\alpha}}+15b^2e^{\frac{-3\beta}{\alpha}}-14e^{\frac{\beta}{\alpha}}-24e^{\frac{-3\beta}{\alpha}}+6b^2e^{\frac{-5\beta}{\alpha}}+2e^{\frac{3\beta}{\alpha}}\Big)-80e^{\frac{-\beta}{\alpha}}+20b^2e^{\frac{-3\beta}{\alpha}}-8e^{\frac{\beta}{\alpha}}+6b^2e^{\frac{-5\beta}{\alpha}}+8e^{\frac{3\beta}{\alpha}}-2b^2e^{\frac{-\beta}{\alpha}}+b^2e^{\frac{\beta}{\alpha}}+b^4e^{\frac{-3\beta}{\alpha}}-8b^4e^{\frac{-5\beta}{\alpha}}\Big\},$$

$$a_{19}=\Big\{n\Big(42e^{\frac{-3\beta}{\alpha}}+14e^{\frac{-\beta}{\alpha}}-2b^2e^{\frac{-3\beta}{\alpha}}-7b^2e^{\frac{-5\beta}{\alpha}}\Big)+15e^{\frac{-3\beta}{\alpha}}+48e^{\frac{-\beta}{\alpha}}-3b^2e^{\frac{-3\beta}{\alpha}}-16b^2e^{\frac{-5\beta}{\alpha}}-10e^{\frac{\beta}{\alpha}}+b^2e^{\frac{-\beta}{\alpha}}+b^4e^{\frac{-5\beta}{\alpha}}\Big\},$$

$$a_{20}=\Big\{n\Big(-21e^{\frac{-3\beta}{\alpha}}-9e^{\frac{-5\beta}{\alpha}}+2b^2e^{\frac{-5\beta}{\alpha}}\Big)-24e^{\frac{-3\beta}{\alpha}}+3e^{\frac{-5\beta}{\alpha}}+8b^2e^{\frac{-5\beta}{\alpha}}-8e^{\frac{-\beta}{\alpha}}+2e^{\frac{\beta}{\alpha}}+b^2e^{\frac{-3\beta}{\alpha}}\Big\},$$

$$a_{21}=\Big(n\big(5e^{\frac{-5\beta}{\alpha}}+6e^{\frac{-3\beta}{\alpha}}\big)+4e^{\frac{-5\beta}{\alpha}}+12e^{\frac{-3\beta}{\alpha}}-2e^{\frac{-\beta}{\alpha}}-b^2e^{\frac{-5\beta}{\alpha}}\Big),$$

$$a_{22}=\Big(2ne^{\frac{-5\beta}{\alpha}}\Big),$$

$$a_{23}=\Big(-2b^2e^{\frac{3\beta}{\alpha}}-4b^2e^{\frac{\beta}{\alpha}}+2b^2e^{\frac{5\beta}{\alpha}}+2b^4e^{\frac{5\beta}{\alpha}}-6b^4e^{\frac{3\beta}{\alpha}}+8b^4e^{\frac{-\beta}{\alpha}}\Big),$$

$$a_{24}=\Big\{n\Big(2e^{\frac{5\beta}{\alpha}}+4e^{\frac{3\beta}{\alpha}}+2b^2e^{\frac{5\beta}{\alpha}}-6b^2e^{\frac{\beta}{\alpha}}+2e^{\frac{\beta}{\alpha}}\Big)+2e^{\frac{5\beta}{\alpha}}+4e^{\frac{3\beta}{\alpha}}+16b^2e^{\frac{\beta}{\alpha}}+2e^{\frac{\beta}{\alpha}}+16b^2e^{\frac{3\beta}{\alpha}}+20b^4e^{\frac{3\beta}{\alpha}}+8b^4e^{\frac{-3\beta}{\alpha}}+12b^4e^{\frac{\beta}{\alpha}}-4b^2e^{\frac{-\beta}{\alpha}}-2b^2e^{\frac{5\beta}{\alpha}}-2b^4e^{\frac{5\beta}{\alpha}}-8b^4e^{\frac{-\beta}{\alpha}}\Big\},$$

$$a_{25} = \Big\{n\Big(-8e^{\frac{5\beta}{\alpha}} - 14e^{\frac{3\beta}{\alpha}} - 6b^2e^{\frac{5\beta}{\alpha}} + 19b^2e^{\frac{\beta}{\alpha}} + 4b^2e^{\frac{3\beta}{\alpha}} - 10e^{\frac{\beta}{\alpha}} + 10b^2e^{\frac{-\beta}{\alpha}}\Big) - 12e^{\frac{5\beta}{\alpha}} - 8e^{\frac{3\beta}{\alpha}} - 8b^2e^{\frac{5\beta}{\alpha}} - 63b^2e^{\frac{\beta}{\alpha}} - 8b^2e^{\frac{3\beta}{\alpha}} - 10e^{\frac{\beta}{\alpha}} + 18b^2e^{\frac{-\beta}{\alpha}} + 2b^4e^{\frac{\beta}{\alpha}} + 114b^4e^{\frac{-\beta}{\alpha}} - 52b^4e^{\frac{-3\beta}{\alpha}} - 22b^4e^{\frac{3\beta}{\alpha}}\Big\},$$

$$a_{26} = \Big\{n\Big(26e^{\frac{\beta}{\alpha}} + 8e^{\frac{5\beta}{\alpha}} + 24e^{\frac{3\beta}{\alpha}} - 20b^2e^{\frac{-\beta}{\alpha}} - 22b^2e^{\frac{\beta}{\alpha}} - 2b^2e^{\frac{3\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} + 14e^{\frac{-\beta}{\alpha}} + 116b^2e^{\frac{-3\beta}{\alpha}}\Big) + 4e^{\frac{\beta}{\alpha}} + 18e^{\frac{5\beta}{\alpha}} + 20e^{\frac{3\beta}{\alpha}} + 46b^2e^{\frac{-\beta}{\alpha}} + 86b^2e^{\frac{\beta}{\alpha}} - 48b^2e^{\frac{3\beta}{\alpha}} + 14b^2e^{\frac{5\beta}{\alpha}} + 14e^{\frac{-\beta}{\alpha}} - 32b^2e^{\frac{-3\beta}{\alpha}} + 42b^4e^{\frac{\beta}{\alpha}} - 216b^4e^{\frac{-\beta}{\alpha}} + 156b^4e^{\frac{-3\beta}{\alpha}} + 16b^4e^{\frac{3\beta}{\alpha}} - 8b^4e^{\frac{-5\beta}{\alpha}}\Big\},$$

$$a_{27} = \Big\{n\Big(-34e^{\frac{3\beta}{\alpha}} - 58e^{\frac{\beta}{\alpha}} + 2e^{\frac{5\beta}{\alpha}} - 40e^{\frac{-\beta}{\alpha}} + 46b^2e^{\frac{-3\beta}{\alpha}} + 12b^2e^{\frac{\beta}{\alpha}} - 4b^2e^{\frac{3\beta}{\alpha}} + 26b^2e^{\frac{-\beta}{\alpha}}\Big) + 2e^{\frac{3\beta}{\alpha}} - 18e^{\frac{\beta}{\alpha}} + 4e^{\frac{5\beta}{\alpha}} - 52e^{\frac{-\beta}{\alpha}} + 186b^2e^{\frac{-3\beta}{\alpha}} - 112b^2e^{\frac{\beta}{\alpha}} + 80b^2e^{\frac{3\beta}{\alpha}} - 168b^2e^{\frac{-\beta}{\alpha}} - 16b^4e^{\frac{\beta}{\alpha}} + 74b^4e^{\frac{-\beta}{\alpha}} - 192b^4e^{\frac{-3\beta}{\alpha}} + 20b^4e^{\frac{-5\beta}{\alpha}} - 6b^2e^{\frac{5\beta}{\alpha}} + 56b^4e^{\frac{-\beta}{\alpha}} - 8b^4e^{\frac{3\beta}{\alpha}}\Big\},$$

$$a_{28} = \Big\{n\Big(14e^{\frac{3\beta}{\alpha}} + 58e^{\frac{\beta}{\alpha}} - 12b^2e^{\frac{-\beta}{\alpha}} - 4e^{\frac{5\beta}{\alpha}} - 52b^2e^{\frac{-3\beta}{\alpha}} + 70e^{\frac{-\beta}{\alpha}} + 30e^{\frac{-3\beta}{\alpha}} - 12b^2e^{\frac{-5\beta}{\alpha}}\Big) + 38e^{\frac{3\beta}{\alpha}} - 58e^{\frac{\beta}{\alpha}} + 232b^2e^{\frac{-\beta}{\alpha}} - 10e^{\frac{5\beta}{\alpha}} - 384b^2e^{\frac{-3\beta}{\alpha}} + 49e^{\frac{-\beta}{\alpha}} + 38e^{\frac{-3\beta}{\alpha}} + 8b^2e^{\frac{-5\beta}{\alpha}} + 128b^2e^{\frac{\beta}{\alpha}} - 40b^4e^{\frac{-\beta}{\alpha}} + 96b^4e^{\frac{-3\beta}{\alpha}} + 16b^4e^{\frac{-5\beta}{\alpha}} - 54b^2e^{\frac{3\beta}{\alpha}} - 8b^4e^{\frac{\beta}{\alpha}}\Big\},$$

$$a_{29} = \Big\{n\Big(-70e^{\frac{-\beta}{\alpha}} - 28e^{\frac{\beta}{\alpha}} + 26b^2e^{\frac{-3\beta}{\alpha}} + 26b^2e^{\frac{-5\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} - 78e^{\frac{-3\beta}{\alpha}}\Big) - 10e^{\frac{-\beta}{\alpha}} + 144e^{\frac{\beta}{\alpha}} + 426b^2e^{\frac{-3\beta}{\alpha}} + 30b^2e^{\frac{-5\beta}{\alpha}} - 74e^{\frac{3\beta}{\alpha}} - 142e^{\frac{-3\beta}{\alpha}} - 154b^2e^{\frac{-\beta}{\alpha}} + 124e^{\frac{-\beta}{\alpha}} - 28b^2e^{\frac{-5\beta}{\alpha}} - 68b^2e^{\frac{\beta}{\alpha}} - 2e^{\frac{5\beta}{\alpha}} - 24b^4e^{\frac{-3\beta}{\alpha}} + 12b^4e^{\frac{-5\beta}{\alpha}} + 8b^4e^{\frac{-\beta}{\alpha}} - 40b^4e^{\frac{-5\beta}{\alpha}}\Big\},$$

$$a_{30} = \Big\{n\Big(28e^{\frac{-\beta}{\alpha}} + 86e^{\frac{-3\beta}{\alpha}} - 4b^2e^{\frac{-3\beta}{\alpha}} + 18e^{\frac{-5\beta}{\alpha}} - 18b^2e^{\frac{-5\beta}{\alpha}}\Big) - 30e^{\frac{-\beta}{\alpha}} + 267e^{\frac{-3\beta}{\alpha}} - 248b^2e^{\frac{-3\beta}{\alpha}} + 6e^{\frac{-5\beta}{\alpha}} - 36b^2e^{\frac{-5\beta}{\alpha}} + 92b^2e^{\frac{-\beta}{\alpha}} - 126e^{\frac{\beta}{\alpha}} + 38e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{-5\beta}{\alpha}} + 8b^4e^{\frac{-3\beta}{\alpha}} + 16b^2e^{\frac{\beta}{\alpha}}\Big\},$$

$$a_{31}=\left\{n\left(-38e^{\frac{-3\beta}{\alpha}}-36e^{\frac{-5\beta}{\alpha}}+4b^2e^{\frac{-5\beta}{\alpha}}\right)-250e^{\frac{-3\beta}{\alpha}}-40e^{\frac{-5\beta}{\alpha}}+46b^2e^{\frac{-5\beta}{\alpha}}+70b^2e^{\frac{-3\beta}{\alpha}}+84e^{\frac{-\beta}{\alpha}}+30e^{\frac{\beta}{\alpha}}-8e^{\frac{3\beta}{\alpha}}-16b^2e^{\frac{-\beta}{\alpha}}\right\},$$

$$a_{32}=\left\{n\left(4e^{\frac{-3\beta}{\alpha}}+22e^{\frac{-5\beta}{\alpha}}\right)+168e^{\frac{-3\beta}{\alpha}}+12e^{\frac{-5\beta}{\alpha}}-56e^{\frac{-\beta}{\alpha}}-6b^2e^{\frac{-5\beta}{\alpha}}-8e^{\frac{\beta}{\alpha}}-16b^2e^{\frac{-3\beta}{\alpha}}\right\},$$

$$a_{33}=\left\{n\left(-4e^{\frac{-5\beta}{\alpha}}\right)+8e^{\frac{-5\beta}{\alpha}}-58e^{\frac{-3\beta}{\alpha}}+8e^{\frac{-\beta}{\alpha}}\right\},$$

$$a_{34}=\left(-2e^{\frac{-5\beta}{\alpha}}+8e^{\frac{-3\beta}{\alpha}}\right),$$

$$a_{35}=\left(-2e^{\frac{5\beta}{\alpha}}-2b^2e^{\frac{5\beta}{\alpha}}+8b^2e^{\frac{3\beta}{\alpha}}+4e^{\frac{3\beta}{\alpha}}-8b^2e^{\frac{\beta}{\alpha}}\right),$$

$$a_{36}=\left(6e^{\frac{5\beta}{\alpha}}-22b^2e^{\frac{3\beta}{\alpha}}+4b^2e^{\frac{5\beta}{\alpha}}-16e^{\frac{3\beta}{\alpha}}+32b^2e^{\frac{\beta}{\alpha}}+4e^{\frac{\beta}{\alpha}}-8b^2e^{\frac{-\beta}{\alpha}}\right),$$

$$a_{37}=\left(10e^{\frac{3\beta}{\alpha}}+6e^{\frac{\beta}{\alpha}}+8b^2e^{\frac{-\beta}{\alpha}}-30b^2e^{\frac{\beta}{\alpha}}+16b^2e^{\frac{3\beta}{\alpha}}-4e^{\frac{5\beta}{\alpha}}-2b^2e^{\frac{5\beta}{\alpha}}\right),$$

$$a_{38}=\left(-2e^{\frac{5\beta}{\alpha}}-50e^{\frac{\beta}{\alpha}}+22b^2e^{\frac{-\beta}{\alpha}}+18e^{\frac{3\beta}{\alpha}}+20e^{\frac{-\beta}{\alpha}}-16b^2e^{\frac{-3\beta}{\alpha}}-2b^2e^{\frac{3\beta}{\alpha}}\right),$$

$$a_{39}=\left(50e^{\frac{\beta}{\alpha}}-26e^{\frac{-\beta}{\alpha}}-18e^{\frac{3\beta}{\alpha}}-24b^2e^{\frac{-\beta}{\alpha}}+2e^{\frac{5\beta}{\alpha}}+4b^2e^{\frac{\beta}{\alpha}}+24b^2e^{\frac{-3\beta}{\alpha}}\right),$$

$$a_{40}=\left(-4e^{\frac{\beta}{\alpha}}-22e^{\frac{-\beta}{\alpha}}-4b^2e^{\frac{-3\beta}{\alpha}}+28e^{\frac{-3\beta}{\alpha}}+2e^{\frac{3\beta}{\alpha}}+4b^2e^{\frac{-\beta}{\alpha}}-8b^2e^{\frac{-5\beta}{\alpha}}\right),$$

$$a_{41}=\left(28e^{\frac{-\beta}{\alpha}}-4e^{\frac{\beta}{\alpha}}-18e^{\frac{-3\beta}{\alpha}}-2b^2e^{\frac{-3\beta}{\alpha}}-20e^{\frac{-3\beta}{\alpha}}+8b^2e^{\frac{-5\beta}{\alpha}}\right),$$

$$a_{42}=\left(6e^{\frac{-3\beta}{\alpha}}-4e^{\frac{-\beta}{\alpha}}+12e^{\frac{-5\beta}{\alpha}}-2b^2e^{\frac{-5\beta}{\alpha}}\right),$$

$$a_{43}=\left(2e^{\frac{-3\beta}{\alpha}}-10e^{\frac{-5\beta}{\alpha}}\right),$$

$$a_{44}=\left(2e^{\frac{-5\beta}{\alpha}}\right).$$

Now we assume that F^n is a Weakly-Berwald space, then B^m_m is hp(1). Since α is irrational in (y^i), the equation (22) is divided into two equations follows,

$$\beta^2F_1B^m_m+\beta^3G_1r_{00}+\alpha^2H_1s_0+\alpha^2\beta^2I_1r_0=0. \tag{23}$$

$$\beta^2F_2B^m_m+\beta G_2r_{00}+\alpha^2H_2s_0+\alpha^2\beta^2I_2r_0=0. \tag{24}$$

Where,

$F_1 = a_0\alpha^{10} + a_2\alpha^8\beta^2 + a_4\alpha^6\beta^4 + a_6\alpha^4\beta^6 + a_8\alpha^2\beta^8 + a_{10}\beta^{10}$,

$G_1 = a_{13}\alpha^8 + a_{15}\alpha^6\beta^2 + a_{17}\alpha^4\beta^4 + a_{19}\alpha^2\beta^6 + a_{21}\beta^8$,

$H_1 = a_{23}\alpha^{10} + a_{25}\alpha^8\beta^2 + a_{27}\alpha^6\beta^4 + a_{29}\alpha^4\beta^6 + a_{31}\alpha^2\beta^8 + a_{33}\beta^{10}$,

$I_1 = a_{35}\alpha^8 + a_{37}\alpha^6\beta^2 + a_{39}\alpha^4\beta^4 + a_{41}\alpha^2\beta^6 + a_{43}\beta^8$,

$F_2 = a_1\alpha^{10} + a_3\alpha^8\beta^2 + a_5\alpha^6\beta^4 + a_7\alpha^4\beta^6 + a_9\alpha^2\beta^8 + a_{11}\beta^{10}$,

$G_2 = a_{12}\alpha^{10} + a_{14}\alpha^8\beta^2 + a_{16}\alpha^6\beta^4 + a_{18}\alpha^4\beta^6 + a_{20}\alpha^2\beta^8 + a_{22}\beta^{10}$,

$H_2 = a_{24}\alpha^{10} + a_{26}\alpha^8\beta^2 + a_{28}\alpha^6\beta^4 + a_{30}\alpha^4\beta^6 + a_{32}\alpha^2\beta^8 + a_{34}\beta^{10}$,

$I_2 = a_{36}\alpha^8 + a_{38}\alpha^6\beta^2 + a_{40}\alpha^4\beta^4 + a_{42}\alpha^2\beta^6 + a_{44}\beta^8$,

Eliminating $\mathrm{B}_{\mathrm{m}}^{\mathrm{m}}$ from these equations, we obtain

$$\mathrm{B}Rr_{00} + \alpha^2 Ss_0 + \alpha^2\beta^2 Tr_0 = 0 \tag{25}$$

Where,

$\mathrm{R} = \beta^2\mathrm{F}_2\mathrm{G}_1 - \mathrm{F}_1\mathrm{G}_2$;

$\mathrm{S} = \mathrm{F}_2\mathrm{H}_1 - \mathrm{F}_1\mathrm{H}_2$;

$\mathrm{T} = \mathrm{F}_2\mathrm{I}_1 - \mathrm{F}_1\mathrm{I}_2$.

Since only the terms $\alpha^2\Big\{\{\alpha^{10}\Big(-2b^2e^{\frac{3\beta}{\alpha}} - 4b^2e^{\frac{\beta}{\alpha}} + 2b^2e^{\frac{5\beta}{\alpha}} + 2b^4e^{\frac{5\beta}{\alpha}} - 6b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{-\beta}{\alpha}}\Big) \times \alpha^{10}\Big(-8e^{\frac{5\beta}{\alpha}} + 32b^2e^{\frac{3\beta}{\alpha}} - 12b^2e^{\frac{5\beta}{\alpha}} + 22b^4e^{\frac{3\beta}{\alpha}} - 32b^4e^{\frac{\beta}{\alpha}} - 4b^4e^{\frac{5\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}} - 8b^2e^{\frac{\beta}{\alpha}} + 8b^4e^{\frac{-\beta}{\alpha}}\Big) - \alpha^{10}\Big(n\Big(2e^{\frac{5\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} + 2b^2e^{\frac{5\beta}{\alpha}} - 6b^2e^{\frac{\beta}{\alpha}} + 2e^{\frac{\beta}{\alpha}}\Big) + 2e^{\frac{5\beta}{\alpha}} + 4e^{\frac{3\beta}{\alpha}} + 16b^2e^{\frac{\beta}{\alpha}} + 2e^{\frac{\beta}{\alpha}} + 16b^2e^{\frac{3\beta}{\alpha}} + 20b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{-3\beta}{\alpha}} + 12b^4e^{\frac{\beta}{\alpha}} - 4b^2e^{\frac{-\beta}{\alpha}} - 2b^2e^{\frac{5\beta}{\alpha}} - 2b^4e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{-\beta}{\alpha}}\Big) \times \alpha^{10}\Big(2e^{\frac{5\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} - 8b^2e^{\frac{3\beta}{\alpha}} + 2b^4e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{\beta}{\alpha}}\Big) s_0\Big\}$ of $\mathrm{S}s_0$ in (25) do not contain β, we must have hp (22) V_{22} such that

$\alpha^{22}s_0 = \beta V_{22}$.

Case 1:

First we are concerned with $\alpha^2 \not\equiv 0 \pmod{\beta}$ and $b^2 \neq 0$. Shows the existence of a function k(x) satisfying $V_{22} = k\alpha^{22}$, and hence $s_0 = k\beta$, then (25) is reduced to

$\mathrm{R}r_{00} + \alpha^2 kS + \alpha^2\beta Tr_0 = 0$.

So the terms

$$\Big\{-\alpha^{10}\Big(2e^{\frac{5\beta}{\alpha}}+4b^2e^{\frac{5\beta}{\alpha}}-8b^2e^{\frac{3\beta}{\alpha}}+2b^4e^{\frac{5\beta}{\alpha}}-8b^4e^{\frac{3\beta}{\alpha}}+8b^4e^{\frac{\beta}{\alpha}}\Big)\times\alpha^{10}\Big(n\Big(-e^{\frac{5\beta}{\alpha}}-e^{\frac{3\beta}{\alpha}}-b^2e^{\frac{5\beta}{\alpha}}+b^2e^{\frac{3\beta}{\alpha}}+2b^2\Big)-e^{\frac{5\beta}{\alpha}}-e^{\frac{3\beta}{\alpha}}-2b^2e^{\frac{5\beta}{\alpha}}-6b^2e^{\frac{3\beta}{\alpha}}+2b^2+12b^4e^{\frac{3\beta}{\alpha}}\Big)\Big\}r_{00}$$
$$+\Big\{\Big(\alpha^{10}(-2b^2e^{\frac{3\beta}{\alpha}}-4b^2e^{\frac{\beta}{\alpha}}+2b^2e^{\frac{5\beta}{\alpha}}+2b^4e^{\frac{5\beta}{\alpha}}-6b^4e^{\frac{3\beta}{\alpha}}+8b^4e^{\frac{-\beta}{\alpha}}\Big)\times\alpha^{10}\Big(-8e^{\frac{5\beta}{\alpha}}+32b^2e^{\frac{3\beta}{\alpha}}-12b^2e^{\frac{5\beta}{\alpha}}+22b^4e^{\frac{3\beta}{\alpha}}-32b^4e^{\frac{\beta}{\alpha}}-4b^4e^{\frac{5\beta}{\alpha}}+2e^{\frac{3\beta}{\alpha}}-8b^2e^{\frac{\beta}{\alpha}}+8b^4e^{\frac{-\beta}{\alpha}}\Big)-\alpha^{10}\Big(n\Big(2e^{\frac{5\beta}{\alpha}}+4e^{\frac{3\beta}{\alpha}}+2b^2e^{\frac{5\beta}{\alpha}}-6b^2e^{\frac{\beta}{\alpha}}+2e^{\frac{\beta}{\alpha}}\Big)+2e^{\frac{5\beta}{\alpha}}+4e^{\frac{3\beta}{\alpha}}+16b^2e^{\frac{\beta}{\alpha}}+2e^{\frac{\beta}{\alpha}}+16b^2e^{\frac{3\beta}{\alpha}}+20b^4e^{\frac{3\beta}{\alpha}}+8b^4e^{\frac{-3\beta}{\alpha}}+12b^4e^{\frac{\beta}{\alpha}}-4b^2e^{\frac{-\beta}{\alpha}}-2b^2e^{\frac{5\beta}{\alpha}}-2b^4e^{\frac{5\beta}{\alpha}}-8b^4e^{\frac{-\beta}{\alpha}}\Big)\times\alpha^{10}\Big(2e^{\frac{5\beta}{\alpha}}+4b^2e^{\frac{5\beta}{\alpha}}-8b^2e^{\frac{3\beta}{\alpha}}+2b^4e^{\frac{5\beta}{\alpha}}-8b^4e^{\frac{3\beta}{\alpha}}+8b^4e^{\frac{\beta}{\alpha}}\Big)\alpha^2ks\Big\}$$

Of the above do not contain β. So there must exist hp(1) U_1 satisfying $\Big\{\Big\{-\Big(2e^{\frac{5\beta}{\alpha}}+4b^2e^{\frac{5\beta}{\alpha}}-8b^2e^{\frac{3\beta}{\alpha}}+2b^4e^{\frac{5\beta}{\alpha}}-8b^4e^{\frac{3\beta}{\alpha}}+8b^4e^{\frac{\beta}{\alpha}}\Big)\times\Big(n\Big(-e^{\frac{5\beta}{\alpha}}-e^{\frac{3\beta}{\alpha}}-b^2e^{\frac{5\beta}{\alpha}}+b^2e^{\frac{3\beta}{\alpha}}+2b^2\Big)-e^{\frac{5\beta}{\alpha}}--e^{\frac{3\beta}{\alpha}}-2b^2e^{\frac{5\beta}{\alpha}}-6b^2e^{\frac{3\beta}{\alpha}}+2b^2+12b^4e^{\frac{3\beta}{\alpha}}\Big)\Big\}r_{00}+\Big(\Big(-2b^2e^{\frac{3\beta}{\alpha}}-4b^2e^{\frac{\beta}{\alpha}}+2b^2e^{\frac{5\beta}{\alpha}}+2b^4e^{\frac{5\beta}{\alpha}}-6b^4e^{\frac{3\beta}{\alpha}}+8b^4e^{\frac{-\beta}{\alpha}}\Big)\times\Big(-8e^{\frac{5\beta}{\alpha}}+32b^2e^{\frac{3\beta}{\alpha}}-12b^2e^{\frac{5\beta}{\alpha}}+22b^4e^{\frac{3\beta}{\alpha}}-32b^4e^{\frac{\beta}{\alpha}}-4b^4e^{\frac{5\beta}{\alpha}}+2e^{\frac{3\beta}{\alpha}}-8b^2e^{\frac{\beta}{\alpha}}+8b^4e^{\frac{-\beta}{\alpha}}\Big)-\Big(n\Big(2e^{\frac{5\beta}{\alpha}}+4e^{\frac{3\beta}{\alpha}}+2b^2e^{\frac{5\beta}{\alpha}}-6b^2e^{\frac{\beta}{\alpha}}+2e^{\frac{\beta}{\alpha}}\Big)+2e^{\frac{5\beta}{\alpha}}+4e^{\frac{3\beta}{\alpha}}+16b^2e^{\frac{\beta}{\alpha}}+2e^{\frac{\beta}{\alpha}}+16b^2e^{\frac{3\beta}{\alpha}}+20b^4e^{\frac{3\beta}{\alpha}}+8b^4e^{\frac{-3\beta}{\alpha}}+12b^4e^{\frac{\beta}{\alpha}}-4b^2e^{\frac{-\beta}{\alpha}}-2b^2e^{\frac{5\beta}{\alpha}}-2b^4e^{\frac{5\beta}{\alpha}}-8b^4e^{\frac{-\beta}{\alpha}}\Big)\times\Big(2e^{\frac{5\beta}{\alpha}}+4b^2e^{\frac{5\beta}{\alpha}}-8b^2e^{\frac{3\beta}{\alpha}}+2b^4e^{\frac{5\beta}{\alpha}}-8b^4e^{\frac{3\beta}{\alpha}}+8b^4e^{\frac{\beta}{\alpha}}\Big)ks\Big)\Big\}\alpha^2=\beta U_1.$

It is a contradiction, which leads to k=0.Hence we obtain $s_0=0$; $s_j=0$.Substituting $s_0=0$ into (25), we have

$$Rr_{00}+\alpha^2\beta Tr_0=0. \tag{26}$$

Then only terms $\beta^3\Big\{\beta^{10}\Big(2e^{\frac{-5\beta}{\alpha}}\Big)\times\beta^8\Big(n\Big(5e^{\frac{-5\beta}{\alpha}}+6e^{\frac{-3\beta}{\alpha}}\Big)+4e^{\frac{-5\beta}{\alpha}}+12e^{\frac{-3\beta}{\alpha}}-2e^{\frac{-\beta}{\alpha}}-b^2e^{\frac{-5\beta}{\alpha}}\Big)\Big\}r_{00}$ of (26) do not contain α^2, and hence we must have hp(21)V_{21} such that $\beta^{21}r_{00}=\alpha^2V_{21}$. From$\alpha^2\not\equiv 0(mod\ \beta)$ there exists a function f(x) such that

$$r_{00}=\alpha^2f(x);r_{ij}=a_{ij}f(x). \tag{27}$$

Transvecting (27) by b^iy^j, we have

$r_0 = \beta f(x); r_j = b_j f(x).$ (28)

Substituting (27) and (28) into (26), we have

$f(x)(R + \beta^2 T) = 0.$ (29)

Let us assumef(x) $\neq$ 0.Then (29) implies

$$\left\{-\left(2e^{\frac{5\beta}{\alpha}} + 4b^2e^{\frac{5\beta}{\alpha}} - 8b^2e^{\frac{3\beta}{\alpha}} + 2b^4e^{\frac{5\beta}{\alpha}} - 8b^4e^{\frac{3\beta}{\alpha}} + 8b^4e^{\frac{\beta}{\alpha}}\right) \times n\left((-e^{\frac{5\beta}{\alpha}} - e^{\frac{3\beta}{\alpha}} - b^2e^{\frac{5\beta}{\alpha}} + b^2e^{\frac{3\beta}{\alpha}} + 2b^2) - e^{\frac{5\beta}{\alpha}} - e^{\frac{3\beta}{\alpha}} - 2b^2e^{\frac{5\beta}{\alpha}} - 6b^2e^{\frac{3\beta}{\alpha}} + 2b2 + 12b^4e^{\frac{3\beta}{\alpha}}\right)\right\}\alpha^{20} = \beta V_{19},$$

Where V_{19} is hp(19). This implies V_{19}=0, provided that $b^2 \neq 0$. Hence f(x)=0 must hold and we obtain

$r_{00} = 0; r_{ij} = 0$ and $r_0 = 0; r_j = 0.$

Conversely, substituting$r_{00} = 0, s_0 = 0$and r_0 =0 into (22), we have B^m_m=0. That is, the the Finsler space with (1) is a weakly-Berwald space.

On the other hand, we suppose that the Finsler space with (1) be a Berwald space. Then we have $r_{00} = 0, s_0 = 0$ and r_0 =0, because the space is a weakly-Berwald space from the above discussion. Substituting the above into (14), we have $B^m = 0$, that is, the Finsler space with (1) is a Berwald space. Hence s_{ij}=0 hold well.

Case 2:

Now consider $\alpha^2 \cong 0(\mathrm{mod}\,\beta)$, Lemma (2.2) shows that, n=2, b^2=0 and $\alpha^2 = \beta\delta, \delta = d_i(x)y^i$. From these conditions (25) is rewritten in the form

$R' r_{00} + \delta S' s_0 = 0,$ (30)

Where,

$$R' = \beta^2\left[\left\{\beta^5\delta^5\left(-8e^{\frac{5\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}}\right) + \beta^6\delta^4\left(-44e^{\frac{3\beta}{\alpha}} + 4e^{\frac{5\beta}{\alpha}} + 16e^{\frac{\beta}{\alpha}}\right) + \beta^7\delta^3\left(-68e^{\frac{\beta}{\alpha}} + 8e^{\frac{3\beta}{\alpha}} + 44e^{\frac{-\beta}{\alpha}}\right) + \beta^8\delta^2\left(-18e^{\frac{-\beta}{\alpha}} - 8e^{\frac{\beta}{\alpha}} + 48e^{\frac{-3\beta}{\alpha}} + 2e^{\frac{3\beta}{\alpha}}\right) + \beta^9\delta\left(8e^{\frac{-3\beta}{\alpha}} - 4e^{\frac{-\beta}{\alpha}}\right) + \beta^{10}\left(2e^{\frac{-5\beta}{\alpha}}\right)\right\} \times \left\{\beta^4\delta^4\left(n(5e^{\frac{5\beta}{\alpha}} + 3e^{\frac{3\beta}{\alpha}}) + 5e^{\frac{3\beta}{\alpha}} + 3e^{\frac{3\beta}{\alpha}}\right) + \beta^5\delta^3\left(n\left(20e^{\frac{3\beta}{\alpha}} + 22e^{\frac{\beta}{\alpha}} + 3e^{\frac{5\beta}{\alpha}}\right) - e^{\frac{3\beta}{\alpha}} + 35e^{\frac{\beta}{\alpha}} + 3e^{\frac{5\beta}{\alpha}}\right) + \beta^6\delta^2\left(n\left(52e^{\frac{-\beta}{\alpha}} + 23e^{\frac{\beta}{\alpha}} + 11e^{\frac{3\beta}{\alpha}} - 2e^{\frac{5\beta}{\alpha}} - 2e^{\frac{3\beta}{\alpha}}\right) + 49e^{\frac{-\beta}{\alpha}} + 56e^{\frac{\beta}{\alpha}} - 24e^{\frac{3\beta}{\alpha}}\right) + \beta^7\delta\left(n\left(42e^{\frac{-3\beta}{\alpha}} + 14e^{\frac{-\beta}{\alpha}}\right) + 15e^{\frac{-3\beta}{\alpha}} + 48e^{\frac{-\beta}{\alpha}} - 10e^{\frac{\beta}{\alpha}}\right) + \beta^8\left(n\left(5e^{\frac{-5\beta}{\alpha}} + 6e^{\frac{-3\beta}{\alpha}}\right) + 4e^{\frac{-5\beta}{\alpha}} + 12e^{\frac{-3\beta}{\alpha}} - 2e^{\frac{-\beta}{\alpha}}\right)\right\}\right] - \left[\left\{\beta^5\delta^5\left(2e^{\frac{5\beta}{\alpha}}\right) + \beta^6\delta^4\left(8e^{\frac{5\beta}{\alpha}} + 8e^{\frac{3\beta}{\alpha}}\right) + \beta^7\delta^3\left(44e^{\frac{3\beta}{\alpha}} - 8e^{\frac{\beta}{\alpha}} - 8e^{\frac{5\beta}{\alpha}}\right) + \beta^8\delta^2\left(74e^{\frac{\beta}{\alpha}} - 60e^{\frac{-\beta}{\alpha}} - 20e^{\frac{3\beta}{\alpha}} + 2e^{\frac{5\beta}{\alpha}}\right) +\right.\right.$$

$$\beta^9\delta\left(40e^{\frac{-\beta}{\alpha}}-58e^{\frac{-3\beta}{\alpha}}-4e^{\frac{\beta}{\alpha}}\right)+\beta^{10}\left(2e^{\frac{-3\beta}{\alpha}}-12e^{\frac{-5\beta}{\alpha}}\right)\times\Bigg(\beta^5\delta^5\left(n\left(-e^{\frac{5\beta}{\alpha}}-e^{\frac{3\beta}{\alpha}}\right)-e^{\frac{5\beta}{\alpha}}-e^{\frac{3\beta}{\alpha}}\right)-+\beta^6\delta^4\left(n\left(-8e^{\frac{\beta}{\alpha}}-7e^{\frac{3\beta}{\alpha}}-8e^{\frac{5\beta}{\alpha}}\right)-6e^{\frac{\beta}{\alpha}}-7e^{\frac{5\beta}{\alpha}}\right)+\beta^7\delta^3\left(n\left(-26e^{\frac{3\beta}{\alpha}}-24e^{\frac{\beta}{\alpha}}-22e^{\frac{-\beta}{\alpha}}+3e^{\frac{5\beta}{\alpha}}\right)+19e^{\frac{3\beta}{\alpha}}-67e^{\frac{\beta}{\alpha}}-8e^{\frac{-\beta}{\alpha}}\right)+\beta^8\delta^2\left(n\left(-41e^{\frac{-\beta}{\alpha}}-14e^{\frac{\beta}{\alpha}}-24e^{\frac{-3\beta}{\alpha}}+2e^{\frac{3\beta}{\alpha}}\right)-80e^{\frac{-\beta}{\alpha}}-8e^{\frac{\beta}{\alpha}}+8e^{\frac{3\beta}{\alpha}}\right)+\beta^9\delta\left(\left(n\left(-21e^{\frac{-3\beta}{\alpha}}-9e^{\frac{-5\beta}{\alpha}}\right)-24e^{\frac{-3\beta}{\alpha}}+3e^{\frac{-5\beta}{\alpha}}-8e^{\frac{-\beta}{\alpha}}+2e^{\frac{\beta}{\alpha}}\right)\right)+\beta^{10}\left(2ne^{\frac{-5\beta}{\alpha}}\right)\Bigg)\Big\}\Big],$$

$$S'=$$

$$\Bigg[\Bigg\{\beta^6\delta^4\left(n\left(-8e^{\frac{5\beta}{\alpha}}-14e^{\frac{3\beta}{\alpha}}-10e^{\frac{\beta}{\alpha}}\right)-12e^{\frac{5\beta}{\alpha}}-8e^{\frac{3\beta}{\alpha}}-10e^{\frac{\beta}{\alpha}}\right)+\beta^7\delta^3\left(n\left(-34e^{\frac{3\beta}{\alpha}}-58e^{\frac{\beta}{\alpha}}+2e^{\frac{5\beta}{\alpha}}-40e^{\frac{-\beta}{\alpha}}\right)+2e^{\frac{3\beta}{\alpha}}-18e^{\frac{\beta}{\alpha}}+4e^{\frac{5\beta}{\alpha}}-52e^{\frac{-\beta}{\alpha}}\right)+\beta^8\delta^2\left(n\left(-70e^{\frac{-\beta}{\alpha}}-28e^{\frac{\beta}{\alpha}}+4e^{\frac{3\beta}{\alpha}}-78e^{\frac{-3\beta}{\alpha}}\right)-10e^{\frac{-\beta}{\alpha}}+144e^{\frac{\beta}{\alpha}}-74e^{\frac{3\beta}{\alpha}}-142e^{\frac{-3\beta}{\alpha}}+124e^{\frac{-\beta}{\alpha}}-2e^{\frac{5\beta}{\alpha}}\right)+\beta^9\delta\left(n\left(-38e^{\frac{-3\beta}{\alpha}}-36e^{\frac{-5\beta}{\alpha}}\right)-250e^{\frac{-3\beta}{\alpha}}-40e^{\frac{-5\beta}{\alpha}}+84e^{\frac{-\beta}{\alpha}}+30e^{\frac{\beta}{\alpha}}-8e^{\frac{3\beta}{\alpha}}\right)+\beta^{10}\left(n\left(-4e^{\frac{-5\beta}{\alpha}}\right)+8e^{\frac{-5\beta}{\alpha}}-58e^{\frac{-3\beta}{\alpha}}+8e^{\frac{-\beta}{\alpha}})\right)\times(\beta^5\delta^5(-8e^{\frac{5\beta}{\alpha}}+2e^{\frac{3\beta}{\alpha}})+\beta^6\delta^4\left(-44e^{\frac{3\beta}{\alpha}}+4e^{\frac{5\beta}{\alpha}}+16e^{\frac{\beta}{\alpha}}\right)+\beta^7\delta^3\left(-68e^{\frac{\beta}{\alpha}}+8e^{\frac{3\beta}{\alpha}}+44e^{\frac{-\beta}{\alpha}}\right)+\beta^8\delta^2\left(-18e^{\frac{-\beta}{\alpha}}-8e^{\frac{\beta}{\alpha}}+48e^{\frac{-3\beta}{\alpha}}+2e^{\frac{3\beta}{\alpha}}\right)+\beta^9\delta\left(8e^{\frac{-3\beta}{\alpha}}-4e^{\frac{-\beta}{\alpha}}\right)+\beta^{10}\left(2e^{\frac{-5\beta}{\alpha}}\right)\Big\}\Big]-\Bigg[\Bigg\{\Bigg(\beta^5\delta^5\left(n\left(2e^{\frac{5\beta}{\alpha}}+4e^{\frac{3\beta}{\alpha}}+2e^{\frac{\beta}{\alpha}}\right)+2e^{\frac{5\beta}{\alpha}}+4e^{\frac{3\beta}{\alpha}}+2e^{\frac{\beta}{\alpha}}\right)\Bigg)+\beta^6\delta^4\left(n\left(26e^{\frac{\beta}{\alpha}}+8e^{\frac{5\beta}{\alpha}}+24e^{\frac{3\beta}{\alpha}}+14e^{\frac{-\beta}{\alpha}}\right)+4e^{\frac{\beta}{\alpha}}+18e^{\frac{5\beta}{\alpha}}+20e^{\frac{3\beta}{\alpha}}\right)+\beta^7\delta^3\left(n\left(14e^{\frac{3\beta}{\alpha}}+58e^{\frac{\beta}{\alpha}}-4e^{\frac{5\beta}{\alpha}}+70e^{\frac{-\beta}{\alpha}}+30e^{\frac{-3\beta}{\alpha}}\right)+38e^{\frac{3\beta}{\alpha}}-58e^{\frac{\beta}{\alpha}}-10e^{\frac{5\beta}{\alpha}}+49e^{\frac{-\beta}{\alpha}}+38e^{\frac{-3\beta}{\alpha}}\right)+\beta^8\delta^2\left(n\left(28e^{\frac{-\beta}{\alpha}}+86e^{\frac{-3\beta}{\alpha}}+18e^{\frac{-5\beta}{\alpha}}\right)-30e^{\frac{-\beta}{\alpha}}+267e^{\frac{-3\beta}{\alpha}}+6e^{\frac{-5\beta}{\alpha}}-126e^{\frac{\beta}{\alpha}}+38e^{\frac{3\beta}{\alpha}}\right)+\beta^9\delta\left(n\left(4e^{\frac{-3\beta}{\alpha}}+22e^{\frac{-5\beta}{\alpha}}\right)+168e^{\frac{-3\beta}{\alpha}}+12e^{\frac{-5\beta}{\alpha}}-56e^{\frac{-\beta}{\alpha}}-8e^{\frac{\beta}{\alpha}}\right)+\beta^{10}\left(-2e^{\frac{-5\beta}{\alpha}}+8e^{\frac{-3\beta}{\alpha}}\right)\Bigg\}\times\Big\{\beta^5\delta^5\left(2e^{\frac{5\beta}{\alpha}}\right)+\beta^6\delta^4\left(8e^{\frac{5\beta}{\alpha}}+8e^{\frac{3\beta}{\alpha}}\right)+\beta^7\delta^3\left(44e^{\frac{3\beta}{\alpha}}-8e^{\frac{\beta}{\alpha}}-8e^{\frac{5\beta}{\alpha}}\right)+\beta^8\delta^2\left(74e^{\frac{\beta}{\alpha}}-60e^{\frac{-\beta}{\alpha}}-20e^{\frac{3\beta}{\alpha}}+2e^{\frac{5\beta}{\alpha}}\right)+\beta^9\delta\left(40e^{\frac{-\beta}{\alpha}}-58e^{\frac{-3\beta}{\alpha}}-4e^{\frac{\beta}{\alpha}}\right)+\beta^{10}\left(2e^{\frac{-3\beta}{\alpha}}-12e^{\frac{-5\beta}{\alpha}}\right)\Big\}\Bigg].$$

Since the only term which does not contain δ is $\beta^2\left\{\beta^{10}\left(2e^{\frac{-5\beta}{\alpha}}\right)\times\beta^8\left(n\left(5e^{\frac{-5\beta}{\alpha}}+6e^{\frac{-3\beta}{\alpha}}\right)+4e^{\frac{-5\beta}{\alpha}}+12e^{\frac{-3\beta}{\alpha}}-2e^{\frac{-\beta}{\alpha}}\right)-\beta^{10}\left(2e^{\frac{-3\beta}{\alpha}}-12e^{\frac{-5\beta}{\alpha}}\right)\times\beta^{10}\left(2ne^{\frac{-5\beta}{\alpha}}\right)\right\}r_{00}$ in (30), we must have hp(1)v_1 such that $r_{00}=\delta v_1$. We have $s_0=0; sj=0$, now (30) becomes

$$R' r_{00} = 0, \tag{31}$$

This implies that

$r_{00}=0$; $r_{ij}=0$ and $r_0=0$; $r_j=0$.

Conversely, from $r_{00}=0$, $r_0=0$ and $s_0=0$ we have $B^m_m=0$. Thus the space with (1) is weakly-Berwald space. Thus we state that

Theorem 3.1. A Finsler space with the special exponential (α,β)-metric is weakly Berwald space if and only if the following conditions holds;

1. $\alpha^2 \not\cong 0(mod\beta)$ $implies\; r_{ij}=0\; and\; s_j=0$.
2. $\alpha^2 \cong 0(mod\beta) implies, n=2, b^2=0\; and\; r_{ij}=0, s_j=0$ are satisfied, where $\alpha^2=\beta\delta, \delta=d_i y^i$.

References

[1] B´asc´o, S. and Matsumoto, M. (1997). On the Finsler spaces of Douglas type. Ageneralization of the notion of Berwald space, Publ. Math. Debreren, **51**, 385-406.

[2] Berwald, L. (1929). Über die n-dimensionalen Geometrien konstanter Krümmung, in denen die Geraden die kürzesten sind. Math. Z. **30**, 449-469.

[3] Hashiguchi, M., Hojo S. and Matsumoto, M. (1996). Landsberg spaces of dimension two with (α, β)-metric, Tensor N.S. **57**(2), 145-153.

[4] Lee, I.Y. and Lee, M.H. (2006). On Weakly-Berwald spaces of special (α, β)-metric, Bull. Korean Math. Soc. **43**(2), 425-441.

[5] Lee, I.Y. and Park, H.S. (2004). Finsler spaces with infinite series (α, β)-metric, J.Korean Math. Society, **41**(3), 567-589.

[6] Matsumoto, M. (1992). Theory of Finsler spaces with (α, β)-metric, Rep. on Math, Phys. **31**, 43-83.

[7] Matsumoto, M. (1989). A slope of Mountain is a Finsler surface with respect to time measure, J. Math. Kyoto Univ. **29**(1), 17-25.

[8] Matsumoto, M. (1991). The Berwald connection of a Finsler space with an (α, β)-metric, Tensor, N. S., **50**, 18-21.

[9] Park, H. S. and Choi, E. S. (1999). Finsler spaces with an approximate Matsumoto metric of Douglas type, Comm. of Korean Math. Soc., **14**, 535-544.

[10] Park, H.S. and Choi, E.S. (2002). Finsler spaces with the second approximate Matsumoto metric, Bull. Korean Math. Soc., **39**, 153-163.

[11] Pandey, S.B., Chaubey, V.K. and Tripathi, S.K. (2009). Berwald connection and geodesic of a Finsler space with generalized (α, β)-Metric, J. Rajasthan Acad. Phy. Sci., **8**(1) 39-48.

[12] Shen, Z.M. and Yu, C.T. (2014). On Einstein square metrics, Publ. Math. Debr., **85**(34), 413-424.

[13] Tripathi, B.K. (2020). Hypersurfaces of a Finsler Space with exponential form of (α, β)-Metric, Annals of the University of Craivo Mathematics and Computer Science Series, volume **47**(1), 132-140.

[14] Tripathi, B.K. and Kumar, P. (2022). Douglas Spaces for Some (α, β)-Metric of a Finsler Spaces, Journal of Advanced Mathematical Studies, **15**(4) 444-455.

[15] Tripathi, B.K. and Khan, S. (2023). On weakly Berwald space with a special cubic (α, β)-metric, Surveys in Mathematics and its applications, **18**, 1-11.