Journal of Rajasthan Academy of Physical Sciences
ISSN: 0972-6306; URL: http://raops.org.in
Vol.22, No.1&2, January-June, 2023, 52-56

SOME COMMON FIXED POINT RESULTS IN 2-BANACH SPACES FOR NEW RATIONAL EXPRESSION

[1]**A.K. Goyal and Gaurav Kumar Garg**
Department of Mathematics, M.S.J. Govt. P.G. College, Bharatpur-321001, Rajasthan, India
Email: akgbpr67@gmail.com (Corresponding Author)

Abstract: The study of non-contraction mapping concerning the existence of fixed points draws attention of various authors in non-linear analysis. It is well known that the differential and integral equations that arise in physical problems are generally non-linear. Therefore the fixed point methods specially Banach's contraction principle provides a powerful tool for obtaining the solutions of these equations which were very difficult to solve by any other methods. Recently Yadava, Rajput, Bhardwaj [5] proved a results in 2-Banach spaces for non-contraction mappings. In this paper we prove a common fixed point theorem for non-contraction mapping in 2-Banach spaces, which contains new rational expressions.

2010 AMS Mathematics Subject Classification: 47H10, 54H25.

Keywords: 2-Banach Spaces, Common Fixed Point, non-contraction.

1. Introduction

Browder [2] was the first mathematician to study non-expansive mappings.

Several mathematicians have done the generalization of non-expansive mappings as well as non-contraction mappings. Kirk [4] gave the comprehensive survey concerning fixed point theorems for non-expansive mappings.

Gahlar [3] introduced the concept of 2-Banach spaces. Badshah and Gupta [1], Yadava et al. [6] worked for Banach and 2-Banach spaces for non contraction mappings. Also Yadava et al. [5] proved a results in 2-Banach spaces for non-contraction mappings as follows:

Theorem 1.1. Let F be a mapping of Banach spaces X into itself. If F satisfies the following conditions:

(i) $F^2 = I$

(ii) $$\|F(x)-F(y),a\| \le \alpha \frac{\|x-F(x),a\|\|y-F(x),a\|+\|y-F(y),a\|\|x-F(y),a\|}{\|x-F(x),a\|+\|y-F(x),a\|+\|y-F(y),a\|+\|x-F(y),a\|}$$
$$+\beta \frac{\|x-F(x),a\|\|y-F(y),a\|+\|y-F(x),a\|\|x-F(y),a\|}{\|x-F(x),a\|+\|y-F(x),a\|+\|y-F(y),a\|+\|x-F(y),a\|}$$

$$+\gamma \frac{\|x-F(x),a\|\|y-F(y),a\|+\|y-F(x),a\|\|x-F(y),a\|}{\|x-F(x),a\|+\|y-F(x),a\|+\|y-F(y),a\|+\|x-F(y),a\|}$$

$\forall x,\ y\varepsilon X,\ x \neq y,\ a \succ 0,\ 0 \leq \alpha, \beta, \gamma,\ \delta \prec 1$ and $\alpha + 7\beta + 8\gamma + 4\delta \prec 8$

Then F has fixed point, further if $\beta + 2\delta \prec 2$, then F has unique fixed point.

In this paper, we prove another common fixed point theorem in 2-Banach spaces in which multiplication of two mappings is an identity mapping.

That is *TG* = *I* = *GT*

2. Preliminaries

Definition 2.1. In a paper Gahler [6] defined a linear 2-normed space to be pair (L, ||., ||), where L is a Linear space and ||., . || is non negative, real valued function defined on L such that $a, b, c \in L$

(i) $\|a, b\| = 0$ if and only if *a* and *b* are linearly dependent

(ii) $\|a, b\| = \|b, a\|$

(iii) $\|a, \beta b\| = |\beta|\|a, \beta b\|$, β is real

(iv) $\|a, b + c\| \leq \|a, b\| + \|a, c\|$

Hence ||.,. || is called a 2-norm.

Definition 2.2. A sequence $\{x_n\}$ in a linear 2-Normed space L, is called a convergent sequence if there is, $x \in L$, such that

$\lim_{n\to\infty} \|x_n - x, y\| = 0$ for all $y \in L$.

Definition 2.3. A sequence $\{x_n\}$ in a linear 2-Normed space L, is called a Cauchy sequence if there exist $y, z \in L$, such that y and z are linearly independent and $\lim_{n\to\infty} \|x_m - x_n, y\| = 0$.

Definition 2.4. A Linear 2-normed space in which every Cauchy sequence is convergent is called 2-Banach Spaces.

3. Main Result

Theorem 3.1. Let T and G be two non-expansive mappings of a 2-Banach space X into itself. T and G satisfy the following conditions:

$TG = I = GT$ where I is identity mapping. (1)

$$\|T(x) - G(y), a\| \leq \alpha \frac{\|x-T(x),a\|\|y-G(y),a\|}{\|x-y\|}$$

$$+\beta \frac{\|T-T(x),a\|\|x-G(y),a\|+\|y-T(x),a\|\|y-G(y),a\|+\|x-y,a\|^2}{\|x-T(x),a\|+\|y-G(y),a\|+\|x-G(y),a\|+\|y-T(y),a\|+\|x-y,a\|}$$

$$+\gamma\|x - y, a\| + \delta[\|x - T(x), a\| + \|y - G(y), a\|]$$

$$+\eta[\|x-G(y),a\|+\|y-T(x),a\|] \tag{2}$$

For all $x \neq y$, $\alpha,\beta,\gamma,\delta,\eta \varepsilon [0,1[$ with $[\|x-T(x),a\|+\|y-G(y),a\|]+\|x-G(y),a\|+\|y-T(x),a\|+\|x-y,a\| \neq 0$

Then *T* and *G* have common fixed point.

Proof.

Taking $y=\frac{1}{2}\|(T+I)(x)\|$, $z=G(y)$, $u=2y-z\|x-y,a\|$,

then $\|z-x,a\|=\|G(y)-TG(x)x,a\|$

So by using (1) and (2), we get,

$$\|z-x,a\| \leq \alpha \frac{\|y-G(y),a\|\|T(x)-G(T(x)),a\|}{\|y-T(x)\|}$$

$$+\beta \frac{\|y-G(y),a\|\|y-G(T(x)),a\|+\|T(x)-G(y),a\|\|T(x)-G(T(x)),a\|+\|y-T(x),,a\|^2}{\|y-G(y),a\|+\|T(x)-G(T(x)),a\|+\|y-G(T(x)),a\|+\|T(x)-G(y),a\|+\|y-T(x),a\|}$$

$$+\gamma\|y-T(x),a\|+\delta\left[\|y-G(y),a\|+\|T(x)-G(T(x)),a\|\right]$$

$$+\eta\left[\|y-G(T(x)),a\|+\|T(x)-G(y),a\|\right]$$

$$\leq \alpha \frac{\|y-G(y),a\|\|T(x)-x,a\|}{\frac{1}{2}\|x-T(x),a\|}$$

$$+\beta \frac{\|y-G(y),a\|\frac{1}{2}\|x-Tx,a\|+\|T(x)-y+y-G(y),a\|\|T(x)-x,a\|+\frac{1}{4}\|x-T(x),,a\|^2}{\|y-T(x),a\|+\|T(x)-x,a\|+\|y-x,a\|+\|y-T(x),a\|}$$

$$+\gamma\|y-T(x),a\|+\delta[\|y-G(y),a\|+\|T(x)-x,a\|]$$

$$+\eta[\|y-x,a\|+\|T(x)-y+y-G(y),a\|]$$

$$=2\alpha\|y-G(y),a\|+\frac{2}{5}\beta\left[\frac{3}{2}\|y-G(y),a\|+\frac{3}{4}\|T(x)-x,a\|\right]$$

$$+\frac{1}{2}\gamma\|x-T(x),a\|+\delta\|y-G(y),a\|+\|T(x)-x,a\|$$

$$+\eta[\|x-T(x),a\|+\|y-G(y),a\|]$$

$$\|z-x,a\| \leq \|x-T(x),a\|\left[\frac{3\beta}{10}+\frac{\gamma}{2}+\delta+\eta\right]$$

$$+\|y-G(y),a\|\left[2\alpha+\frac{6\beta}{10}+\delta+\eta\right] \tag{3}$$

Now we calculate $\|u-x,a\|$,

$$\|u-x,a\|=\|2y-z,a\|=\|T(x)-G(y),a\|$$

$$\leq \alpha \frac{\|x-T(x),a\|\|y-G(y),a\|}{\|x-y-a\|}$$

$$+\beta \frac{\|T-T(x),a\|\|x-G(y),a\|+\|y-T(x),a\|\|y-G(y),a\|+\|x-y,a\|^2}{\|x-T(x)\|+\|y-G(y)\|+\|x-G(y)\|+\|y-T(x),a\|+\|x-y\|}$$

$$+\gamma\|x-y,a\| + \delta[\|x-T(x),a\| + \|y-G(y),a\|]$$

$$+\eta[\|x-G(y),a\| + \|y-T(x),a\|]$$

$$\leq \alpha \frac{\|x-T(x),a\|\|y-G(y),a\|}{\frac{1}{2}\|x-Tx-a\|}$$

$$+\beta \frac{\|T-T(x),a\|[\|x-y,a\|+\|y-Gy,a\|]+\frac{1}{2}\|x-T(x),a\|+\|y-G(y),a\|+\frac{1}{4}\|x-T(x),a\|^2}{\frac{5}{2}\|x-T(x),a\|}$$

$$+\frac{1}{2}\gamma\|x-T(x),a\| + \delta[\|x-T(x),a\| + \|y-G(y),a\|]$$

$$+\eta\left[\frac{1}{2}\|x-T(x),a\| + \|G(y)-y,a\| + \frac{1}{2}\|x-T(x),a\|\right]$$

$$\|u-x,a\| \leq \|x-T(x),a\|\left[\frac{3\beta}{10}+\frac{\gamma}{2}+\delta+\eta\right]$$

$$+\|y-G(y),a\|\left[2\alpha+\frac{6\beta}{10}+\delta+\eta\right] \qquad (4)$$

Now,

$$\|z-u,a\| \leq \|z-x,a\| + \|u-x,a\|$$

$$\|z-u,a\| \leq \|x-T(x),a\|\left[\frac{3\beta}{5}+\frac{\gamma}{1}+2\delta+2\eta\right]$$

$$+\|y-G(y),a\|\left[4\alpha+\frac{6\beta}{5}+2\delta+2\eta\right] \qquad (5)$$

But $\|z-u,a\| = \|G(y)-2y+z\| = 2\|G(y)-y\|$ (6)

By (5) and (6)

$$2\|y-G(y),a\| \leq \|x-T(x),a\|\left[\frac{3\beta}{5}+\gamma+2\delta+2\eta\right]$$

$$+\|y-G(y),a\|\left[4\alpha+\frac{6\beta}{5}+2\delta+2\eta\right]$$

$\|y-G(y),a\| \leq S\|x-T(x),a\|$, where $S = \frac{\left[\frac{3\beta}{5}+\frac{\gamma}{1}+2\delta+2\eta\right]}{2-\left[4\alpha+\frac{6\beta}{5}+2\delta+2\eta\right]} \prec 1$ (7)

Because $20\alpha+9\beta+5\gamma+20\delta+20\eta \prec 10$

Let $F = \frac{1}{2}[T+I]$, then for any $x\varepsilon X$

$$\|F^2(x)-F(x),a\| = \|F(Fx)-F(x),a\| = \|F(y)-y,a\| = \frac{1}{2}\|y-T(y),a\|$$

$$= \frac{1}{2}\|TG(y)-T(y),a\| \leq \frac{1}{2}\|G(y)-(y),a\|$$

Because *T* is non-expansive. So $\|F^2(x) - F(x), a\| \leq \frac{S}{2}\|x - T(x), a\|$

By the definition of *S*, we claim that $F_n(x)$ is a Cauchy sequence in *X*. Also by the completeness, $F_n(x)$ converges to some element (x_0) in *X*.

i.e $\lim_{n\to\infty} F^n(x) = x_0 \Rightarrow F(x_0) = x_0$.

Hence $T(x_0) = x_0$. That is x_0 is fixed point of *T*.

Again

$$\|F^2(x) - F(x), a\| \leq \frac{S}{2}\|x - T(x), a\|$$

$$= \frac{S}{2}\|TG(x) - T(x), a\|$$

$$\leq \frac{S}{2}\|x - G(x), a\|$$

We can conclude that $G(x_0) = x_0$. That is x_0 is fixed point of *G*.

So, $T(x_0) = G(x_0) = x_0$. So x_0 is common fixed point of *T* and *G*. The uniqueness part is trival.

Acknowledgement: We express our grateful thanks to the Editor and Reviewer for their valuable suggestions to bring the manuscript in its present form.

References

1. Badshah, V.H. and Gupta, O.P. (2005). Fixed point theorems in Banach and 2-Banach spaces, Jnanabha **35**, 73-78.

2. Brouwder, F.E. (1965). Non-expansive non-linear operators in Banach spaces, Proc. Nat. Acad. Sci. U.S.A. **54**, 1041-1044.

3. Gahlar, S. (1963-64). 2-Metrche raume and ihre topologiscche structure, Math. Nadh. **26**, 115-148.

4. Kirk, W.A. (1983). Fixed point theorem for non-expansive mappings, Contem Math. **18**, 121-140.

5. Yadava, R.N., Rajput, S.S. and Bhardwaj, R.K. (2007). Some fixed point and common fixed point theorems in Banach spaces, Acta Ciencia Indica **33**(2), 453-460.

6. Yadava, R.N., Rajput, S.S., Choudhary, S. and Bhardwaj, R.K. (2007). Some fixed point and common fixed point theorems for non-contraction mapping on 2-Banach spaces, Acta Ciencia Indica, **33(**3), 737-744.