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ALMOST HERMITE MANIFOLD ADMITTING RICCI QUARTER SYMMETRIC CONNECTION

Nayan Goel[1], Sudhir Kumar Srivastava[2], Sunil Kumar Srivastava[3]
[1&2]Deen Dayal Upadhyaya Gorakhpur University, Gorakhpur, India
[3]Jaipur Engineering College and Research Centre, Jaipur, Rajasthan, India
Email: [1]nayangoel1233@gmail.com, [2]sudhirpr66@rediffmail.com,
[3]sunilk537@gmail.com

Abstract: In the present paper we obtained certain results on an almost Hermite manifold admitting a Ricci quarter symmetric connection. Necessary and sufficient conditions for existence of covariant almost analytic vector field has been discussed. The properties of contravariant almost analytic vector field and almost Hermitian manifold are also the part of the study.

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1. Introduction

Ghosh [3] introduced the notion of quarter-symmetric linear connection on a differentiable Riemannian manifold. Hayden [5] introduced the idea of semi-symmetric linear connection with non-zero torsion tensor on a Riemannian manifold which was further studied by Yano and Imai [14].

As a generalization of semi-symmetric connection, quarter-symmetric connection was introduced and studied in detail by Rastogi [12], Baishya [1] and many more.

The Riemannian Manifold equipped with a quarter symmetric metric connection has been studied by Golab [4]. From there after many geometers studied this connection in different manifold. Mukhopadhyay [10] studied quarter symmetric metric connection on a Riemannian manifold with almost complex structure and Biswas & De [2] studied quarter symmetric metric connection on a SP Sasakian manifold. Para-Sasakian manifold with respect to quarter symmetric metric connection was studied by Mondal & De [9]. Kankarej [7] studied the properties of quarter symmetric connection in a Hermitian manifold.

Ghosh [3] introduced the idea of Ricci quarter symmetric metric connection in a Riemannian manifold which was further studied by Mishra and Pandey [8], Kamilya et al. [6], Baishya [1], Tang et al. [13] established the properties of Ricci quarter symmetric connection and projective Ricci quarter symmetric connection in Riemannian manifold and many more geometers have been contributing to this study further.

Motivated by above studies in this article we study, properties of almost Hermite manifold admitting Ricci quarter symmetric connection which is extended work of Kankarej [7]. The organization of the paper is as follows. In section 2, we studied about existence of covariant almost analytic vector field. In section 3, we discussed properties of an almost Hermite manifold when $\widetilde{\nabla}_X F = 0$. Section 4, explains properties of curvature tensor when $\widetilde{\nabla}_X F = 0$. Section 5, deals with contravariant almost analytic vector field whereas section 6, discusses geodesics on an Almost Hermite Manifold.

Let M^{2n} be an *2n* dimensional almost Hermite Manifold with structure *{F,g}* such that

$\bar{\bar{X}} + X = 0$ and

$$g(\bar{X},\bar{Y}) = g(X,Y)\text{s} \tag{1}$$

where $\bar{X} = FX$, F is tensor of type (1,1), g is a metric tensor and *(X,Y)* are arbitrary vector fields by Golab [4].

Further we have

$$'F(X,Y) + {}'F(Y,X) = g(\bar{X},Y) + g(X,\bar{Y}) = 0 \tag{2}$$

where $'F(X,Y) = g(\bar{X},Y)$

By Yano[14] if ω is a covariant almost analytic vector field then we have

$$\omega((\nabla_X F)(Y) - (\nabla_Y F)(X)) = (\nabla_{\bar{X}}\omega)(Y) - (\nabla_X \omega)(\bar{Y}) \tag{3}$$

The ***Nijenhuis tensor*** *N(X,Y)* on an almost Hermite manifold is given by,

$$N(X,Y) = (\nabla_{\bar{X}}F)(Y) - (\nabla_{\bar{Y}}F)(X) - \overline{(\nabla_X F)(Y)} + \overline{(\nabla_Y F)(X)} \tag{4}$$

Taking,

$$M(X,Y) = \nabla_{\bar{X}}\bar{Y} - \nabla_X Y - \overline{\nabla_{\bar{X}}Y} - \overline{\nabla_X \bar{Y}} \tag{5}$$

We have

$$N(X,Y) = M(X,Y) - \text{M}(\text{Y},\text{X}) \tag{6}$$

where ∇ is ***Levi-Civita connection*** on almost Hermite manifold.

Let K be the ***curvature tensor*** of symmetric connection ∇ on almost Hermite manifold, then we have

$$(K_{X\,Y}F)(Z) = K(X,Y,\bar{Z}) - \overline{K(X,Y,Z)} \tag{7}$$

A linear connection $\widetilde{\nabla}$ on (M^{2n},g) is said to be ***Ricci quarter symmetric connection*** if the torsion tensor T of connection $\widetilde{\nabla}$ and the metric tensor g of the manifold satisfy the following conditions:

$\left(\widetilde{\nabla}g\right)(Y,Z) = 0$ and (8)

$T(X,Y) = \omega(Y)LX - \omega(X)LY$ (9)

where L is the *(1, 1)* ***Ricci tensor***.

For the vector field X,Y , ω is the 1-form associated with torsion tensor of connection $\widetilde{\nabla}$. Golab [4], Ghosh [3]

$\widetilde{\nabla}_X Y = \nabla_X Y + \omega(Y)LX - S(X,Y)\rho$ (10)

Where,

$S(X,Y) = g(LX,Y)$ (11)

S is the ***Ricci tensor*** on M^n $Ghosh$[3].

By virtue of Mondal and De [9] the 1-form $\boldsymbol{\omega}$ and the vector field $\boldsymbol{\rho}$ are usually called ***1-form vector field*** associated with metric tensor g by the relation,

$\omega(X) = g(X,\rho)$ and (12)

$(\widetilde{\nabla}_X \omega)(Y) = (\nabla_X\omega)(Y) + \omega(LX)\,\omega(Y) - \omega(\rho)\,S(X,Y)$ (13)

Biswas & De [2] studied quarter symmetric metric connection on a SP Sasakian manifold inspired by which we extended the work to Ricci quarter symmetric metric connection.

2. Existence of covariant almost analytic vector field

Theorem 2.1: On almost Hermite manifold if 1-form ω is covariant almost analytic vector field with respect to connection ∇, then ω is also covariant almost analytic vector field with respect to Ricci quarter symmetric metric connection $\widetilde{\nabla}$ if and only if

$\omega(LX)\,\omega(\bar{Y}) = \omega(\overline{LX})\,\omega(Y)$

and $S(X,\bar{Y})\omega(\rho) = -S(\bar{X},Y)\,\omega(\rho)$ (14)

Proof: Barring Y in (1.10) we have

$\widetilde{\nabla}_X\bar{Y} = \nabla_X\bar{Y} + \omega(\bar{Y})LX - S(X,\bar{Y})\rho$ (15)

Using (2.2) we get

$(\widetilde{\nabla}_X F)(Y) = \widetilde{\nabla}_X\bar{Y} - \overline{\widetilde{\nabla}_X Y}$

$= \nabla_X\bar{Y} + \omega(\bar{Y})LX - S(X,\bar{Y})\rho - \overline{(\nabla_X Y)} - \overline{\omega(Y)LX} + \overline{S(X,Y)\rho}$

$= \nabla_X\bar{Y} + \omega(\bar{Y})LX - S(X,\bar{Y})\rho - \overline{(\nabla_X Y)} - \omega(Y)\overline{LX} + S(X,Y)\bar{\rho}$

$(\widetilde{\nabla}_X F)(Y) = (\nabla_X F)(Y) + \omega(\bar{Y})LX - S(X,\bar{Y})\rho - \overline{(\nabla_X Y)} - \omega(Y)\overline{LX} + S(X,Y)\bar{\rho}$ (16)

Interchanging X and Y we have,

$(\widetilde{\nabla}_Y F)(X) = (\nabla_Y F)(X) + \omega(\bar{X})LY - S(Y,\bar{X})\rho - \overline{(\nabla_Y X)} - \omega(X)\overline{LY} + S(Y,X)\bar{\rho}$ (17)

From (16) and (17) we have

$$\big(\tilde{\nabla}_X F\big)(Y) - \big(\tilde{\nabla}_Y F\big)(X) = (\nabla_X F)Y + \omega(\bar{Y})LX - S(X,\bar{Y})\rho - \overline{(\nabla_X Y)} - \omega(Y)\overline{LX} + S(X,Y)\bar{\rho}$$

$$- \quad (\nabla_Y F)X - \omega(\bar{X})LY + S(Y,\bar{X})\rho + \overline{(\nabla_Y X)} + \omega(X)\overline{LY} - S(Y,X)\bar{\rho}$$

$$(\tilde{\nabla}_X F)(Y) - (\tilde{\nabla}_Y F)(X) = (\nabla_X F)(Y) - (\nabla_Y F)(X) + \omega(\bar{Y})LX - \omega(\bar{X})LY - \omega(Y)\overline{LX} + \omega(X)\overline{LY} + 2\,S(X,Y)\bar{\rho}$$

If $(S(X,Y) = -S(Y,X) \; and \; S(X,\bar{Y}) = S(Y,\bar{X}))$ (18)

Operating ω on (18) we have,

$$\omega[(\tilde{\nabla}_X F)(Y) - (\tilde{\nabla}_Y F)(X)] = \omega(\nabla_X F)(Y) - \omega(\nabla_Y F)(X) + 2\,S(X,Y)\omega(\overline{\rho}) \tag{19}$$

Replacing X with $\bar{X}$ and Y with $\bar{Y}$ in (13) we get

$$(\tilde{\nabla}_{\bar{X}}\,\omega)(Y) = (\nabla_{\bar{X}}\omega)(Y) + \omega(L\bar{X})\,\omega(Y) - \omega(\rho)\,S(\bar{X},Y) \text{ and} \tag{20}$$

$$(\tilde{\nabla}_X\,\omega)(\bar{Y}) = (\nabla_X\omega)(\bar{Y}) + \omega(LX)\,\omega(\bar{Y}) - \omega(\rho)\,S(X,\bar{Y}) \tag{21}$$

From (20) and (21) we get

$$(\tilde{\nabla}_{\bar{X}}\,\omega)(Y) - (\tilde{\nabla}_X\,\omega)(\bar{Y}) = (\nabla_{\bar{X}}\omega)(Y) - (\nabla_X\omega)(\bar{Y}) + \omega(L\bar{X})\,\omega(Y) - \omega(LX)\,\omega(\bar{Y}) - \omega(\rho)\,S(\bar{X},Y) + \omega(\rho)\,S(X,\bar{Y})$$

$$(\tilde{\nabla}_{\bar{X}}\,\omega)(Y) - (\tilde{\nabla}_X\,\omega)(\bar{Y}) = (\nabla_{\bar{X}}\omega)(Y) - (\nabla_X\omega)(\bar{Y}) + \omega(L\bar{X})\,\omega(Y) - \omega(LX)\,\omega(\bar{Y}) - 2\omega(\rho)\,S(\bar{X},Y) \tag{22}$$

Since ω is covariant almost analytic with respect to Levi-Civita connection ∇, so from (19) and (22) we get ω is also covariant almost analytic vector field with respect to Ricci quarter symmetric connection if and only if (14) holds.

Theorem 2.2: On an almost Hermite manifold if 1-form ω is covariant analytic vector field with respect to connection ∇, then we have

$\tilde{d}\omega(\bar{X},\bar{Y}) = d\omega(\bar{X},\bar{Y})$if and only if

$$\omega(L\bar{X})\omega(\bar{Y}) - \omega(L\bar{Y})\omega(\bar{X}) = 2\,S(\bar{X},\bar{Y})\,\omega(\rho) \tag{23}$$

where

$$\tilde{d}\omega(X,Y) = (\tilde{\nabla}_X\,\omega)(Y) - (\tilde{\nabla}_Y\,\omega)(X) \tag{24}$$

Proof: From (13) we get

$$(\tilde{\nabla}_{\bar{X}}\,\omega)(\bar{Y}) = (\nabla_{\bar{X}}\omega)(\bar{Y}) - \omega(L\bar{X})\,\omega(\bar{Y}) + \omega(\rho)\,S(\bar{X},\bar{Y}) \tag{25}$$

Interchanging $\bar{X} \; and \; \bar{Y}$, we get

$$(\tilde{\nabla}_{\bar{Y}}\,\omega)(\bar{X}) = (\nabla_{\bar{Y}}\omega)(\bar{X}) - \omega(L\bar{Y})\,\omega(\bar{X}) + \omega(\rho)\,S(\bar{Y},\bar{X}) \tag{26}$$

From (25) and (26) we get

$$\left(\widetilde{\nabla}_{\bar{X}}\omega\right)(\bar{Y}) - \left(\widetilde{\nabla}_{\bar{Y}}\omega\right)(\bar{X}) =$$

$$(\nabla_{\bar{X}}\omega)(\bar{Y}) - (\nabla_{\bar{Y}}\omega)(\bar{X}) - \omega(L\bar{X})\,\omega(\bar{Y}) + \omega(L\bar{Y})\,\omega(\bar{X}) + 2\,\omega(\rho)\,S(\bar{X},\bar{Y}) \tag{27}$$

Since 1-form ω is covariant almost analytic vector field with respect to connection ∇, then we have

$$d\omega(X,Y) = (\nabla_X\omega)(Y) - (\nabla_Y\omega)(X) \tag{28}$$

Barring X and Y in (2.15) and using it in (2.14) we get

$$\tilde{d}\omega(\bar{X},\bar{Y}) = d\omega(\bar{X},\bar{Y}) - \omega(L\bar{X})\,\omega(\bar{Y}) + \omega(L\bar{Y})\,\omega(\bar{X}) + 2\,\omega(\rho)\,S(\bar{X},\bar{Y})$$

Thus, we get (23).

Corollary (2.2): If ω is closed with respect to connection ∇, then ω is also closed with respect to Ricci quarter symmetric metric connection $\widetilde{\nabla}$ if and only if

$$\omega(L\bar{X})\,\omega(\bar{Y}) - \omega(L\bar{Y})\,\omega(\bar{X}) = 2\,\omega(\rho)\,S(\bar{X},\bar{Y})$$

Proof: Since we know that if 1-form ω is closed with respect to connection ∇ i.e.

$d\omega = 0$, then (29)

From (15) on using (29) in (23) we get corollary (15).

3. Properties of an almost Hermite manifold when $\widetilde{\nabla}_X F = 0$

Theorem 3.1: By Pandey and Chaturvedi [11], on an almost Hermite manifold if $(\widetilde{\nabla}_X F)(Y) = 0$, we have

$$(\nabla_{\bar{X}}F)(\bar{Y}) = (\nabla_X F)(Y) \tag{30}$$

Proof: Let $(\widetilde{\nabla}_X F)(Y) = 0$, then from (2.3) we have

$$(\nabla_X F)(Y) = S(X,\bar{Y})\rho - S(X,Y)\bar{\rho} + \omega(Y)\overline{LX} - \omega(\bar{Y})LX \tag{31}$$

Barring X and Y in (31)

$$(\nabla_{\bar{X}}F)(\bar{Y}) = S(\bar{X},\bar{\bar{Y}})\rho - S(\bar{X},\bar{Y})\bar{\rho} + \omega(\bar{Y})\overline{\overline{LX}} - \omega(\bar{\bar{Y}})\overline{LX}$$

$$= -S(\bar{X},Y)\rho - S(X,Y)\bar{\rho} - \omega(\bar{Y})LX + \omega(Y)\,\overline{LX}$$

$$(\nabla_{\bar{X}}F)(\bar{Y}) = S(X,\bar{Y})\rho - S(X,Y)\bar{\rho} - \omega(\bar{Y})LX + \omega(Y)\,\overline{LX} \tag{32}$$

It is clear from (31) and (32) that

$$(\nabla_X F)(Y) = (\nabla_{\bar{X}}F)(\bar{Y})$$

Theorem 3.2: On an almost Hermite manifold when $\widetilde{\nabla}_X F = 0$, then Nijenhuis tensor $N(X,Y)$ with respect to Levi-Civita connection ∇ vanishes i.e. an almost Hermite manifold becomes Hermite manifold and then, we have

$$'M(X,Y,Z) = {}'M(Y,X,Z) \tag{33}$$

Proof: From (4) we have

$$N(X,Y) = (\nabla_{\bar{X}}F)(Y) - (\nabla_{\bar{Y}}F)(X) - \overline{(\nabla_X F)(Y)} + \overline{(\nabla_Y F)(X)} \quad (34)$$

Also, from theorem (3.1) we have

$$(\nabla_{\bar{X}}F)(Y) = -(\nabla_X F)(\bar{Y}) \text{ and} \quad (35)$$

$$(\nabla_{\bar{Y}}F)(X) = -(\nabla_Y F)(\bar{X}) \quad (36)$$

Let

$$F(\bar{Y}) = \bar{\bar{Y}} = -Y \quad (37)$$

Operating ∇, we get

$$(\nabla_X F)(\bar{Y}) = -\overline{(\nabla_X F)(Y)} \quad (38)$$

Using (35), (36) and (38) in (34), we get

$$N(X,Y) = 0 \quad (39)$$

Define $'N(X,Y,Z) \stackrel{\text{def}}{=} g(N(X,Y),Z)$

Then from (6)

$$N(X,Y) = M(X,Y) - M(Y,X)$$

We have

$$'N(X,Y,Z) = {}'M(X,Y,Z) - {}'M(Y,X,Z) \quad (40)$$

Using (39) in (40), we get (33).

4. Properties of curvature tensor when $\tilde{\nabla}_X F = 0$

By Baishya [1], Ghosh [3] let us denote the curvature tensor for connection ∇ and with respect to Ricci quarter symmetric metric connection $\tilde{\nabla}$ by $K(X,Y,Z)$ and $\tilde{K}(X,Y,Z)$ where

$$K(X,Y,Z) = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z \text{ and}$$

$$\tilde{K}(X,Y,Z) = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z$$

From (10) we have

$$\tilde{\nabla}_Y Z = \nabla_Y Z + H(Y,Z) \quad (41)$$

where

$$H(Y,Z) = \omega(Z)LY - S(Y,Z)\rho \quad (42)$$

Theorem 4.1: An almost Hermite manifold equipped with Ricci quarter symmetric connection $\tilde{\nabla}$ admits the following:

1. The torsion tenor *T(X, Y)* of the connection $\tilde{\nabla}$ is 2 times the skew symmetric part *A'(X, Y)* of the tensor. *T(X, Y) = A'(X, Y)* where *A(X, Y)* and *A'(X, Y)* are symmetric and skew symmetric part of tensor *H(X, Y)*.
2. The symmetric part of the tensor *T(X, Y)* vanishes if and only if $\omega(Y)LX = -\omega(X)LY$.

Proof: By virtue of Kankarej [8] and using (9) and (10) we can prove above theorem.

Theorem 4.2: On an almost Hermite manifold, if $\tilde{\nabla}_X\omega = 0$, then the curvature tensor with respect to Ricci quarter symmetric metric connection $\tilde{\nabla}$, is given by

$$\tilde{K}(X,Y,Z) = K(X,Y,Z) + \omega(\rho)\{S(X,Z)LY - S(Y,Z)LX\} \tag{43}$$

Proof: On differentiating (41) we get,

$$\tilde{\nabla}_X\tilde{\nabla}_Y Z = \tilde{\nabla}_X(\nabla_Y Z) + \tilde{\nabla}_X(H(Y,Z))$$

$$= \nabla_X(\nabla_Y Z) + H(X,\nabla_Y Z) + \nabla_X(H(Y,Z)) + H(X,H(Y,Z)) \tag{44}$$

Interchanging X and Y in (44) we have

$$\tilde{\nabla}_Y\tilde{\nabla}_X Z == \nabla_Y(\nabla_X Z) + H(Y,\nabla_X Z) + \nabla_Y(H(X,Z)) + H(Y,H(X,Z)) \text{ and} \tag{45}$$

$$\tilde{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z + H([X,Y],Z) \tag{46}$$

Adding (44) subtracting (45) and (46) from the result we have

$$\tilde{K}(X,Y,Z) = K(X,Y,Z) + H(X,\nabla_Y Z) - H(Y,\nabla_X Z) - H([X,Y],Z) + \nabla_X(H(Y,Z)) - \nabla_Y(H(X,Z)) + H(X,H(Y,Z)) - H(Y,H(X,Z))$$

$$\tilde{K}(X,Y,Z) = K(X,Y,Z) + \nabla_X(H(Y,Z)) - \nabla_Y(H(X,Z)) + H(X,H(Y,Z)) - H(Y,H(X,Z)) \tag{47}$$

where $H(X,\nabla_Y Z) - H(Y,\nabla_X Z) - H([X,Y],Z) = 0$

On differentiating (42) we have

$$(\nabla_X H)(Y,Z) = (\nabla_X\omega)(Z)LY - (\nabla_X\rho)S(Y,Z) \tag{48}$$

Let

$$\tilde{\nabla}_X\omega = 0 \tag{49}$$

Then from (12) and (13) we have

$$(\nabla_X\omega)(Z) = -\omega(LX)\,\omega(Z) + \omega(\rho)\,S(X,Z) \text{ and} \tag{50}$$

$$\nabla_X\rho = -\omega(LX)\,\rho + \omega(\rho)LX \tag{51}$$

Using (50) and (51) in (48) we get

$$(\nabla_X H)(Y,Z) = (-\omega(LX)\,\omega(Z) + \omega(\rho)\,S(X,Z))LY - (-\omega(LX)\rho + \omega(\rho)LX)S(Y,Z)$$

$$(\nabla_X H)(Y,Z) = -\omega(LX)\,\omega(Z)\,LY + \omega(LX)\,\rho\,S(Y,Z) + \omega(\rho)\,S(X,Z)\,LY - \omega(\rho)LX\,S(Y,Z) \quad (52)$$

and interchanging X and Y we have,

$$(\nabla_Y H)(X,Z) = -\omega(LY)\,\omega(Z)\,LX + \omega(LY)\,\rho\,S(X,Z) + \omega(\rho)\,S(Y,Z)\,LX - \omega(\rho)LY\,S(X,Z) \quad (53)$$

Also, from (42) we have

$$H(X,H(Y,Z)) - H(Y,H(X,Z)) = \omega(H(Y,Z))LX - S(X,H(Y,Z))\rho - \omega(H(X,Z))LY + S(Y,H(X,Z))\rho$$

$$= \omega(\omega(Z)LY - S(Y,Z)\rho)LX - S(X,\omega(Z)LY - S(Y,Z)\rho)\rho - \omega(\omega(Z)LX - S(X,Z)\rho)LY + S(Y,\omega(Z)LX - S(X,Z)\rho)\rho$$

$$H(X,H(Y,Z)) - H(Y,H(X,Z)) = \omega(Z)\omega(LY)LX - S(Y,Z)\omega(\rho)LX - S(X,\omega(Z)LY)\rho$$

$$+S(X,S(Y,Z)\rho)\rho - \omega(Z)\omega(LX)LY + S(X,Z)\omega(\rho)LY + S(Y,\omega(Z)LX)\rho - S(Y,S(X,Z)\rho)\rho \quad (54)$$

Using (52), (53) and (54) in (47) we get

$$\widetilde{K}(X,Y,Z) = K(X,Y,Z) + \omega(\rho)\{S(X,Z)LY - S(Y,Z)LX\}$$

Which is required result (4.3).

Theorem 4.3: On an almost Hermite manifold if the curvature tensor with respect to Ricci quarter symmetric metric connection $\widetilde{\nabla}$ vanishes i.e. $\widetilde{K}(X,Y,Z) = 0$, then we have

$$(K_{\bar{X}\bar{Y}}F)(Z) + (K_{XY}F)(Z) = 0 \quad (55)$$

which yields

$$K(\bar{X},\bar{Y},\bar{Z}) + K(X,Y,\bar{Z}) = \overline{K(\bar{X},\bar{Y},Z)} + \overline{K(X,Y,Z)} \quad (56)$$

$$\overline{K(X,Y,Z)} = K(\bar{X},\bar{Y},\bar{Z}) = K(\bar{X},Y,Z) + K(X,\bar{Y},Z) + K(X,Y,\bar{Z}) \quad (57)$$

Proof: Let

$$\widetilde{K}(X,Y,Z) = 0 \quad (58)$$

Then from (43) we get

$$K(X,Y,Z) = -\omega(\rho)\{S(X,Z)LY - S(Y,Z)LX\} \quad (59)$$

Barring Z and using (1) we get

$$K(X,Y,\bar{Z}) = -\omega(\rho)\{S(X,\bar{Z})LY - S(Y,\bar{Z})LX\} \text{ and} \quad (60)$$

$$\overline{K(X,Y,Z)} = K(\bar{X},\bar{Y},\bar{Z}) = -\omega(\rho)\{S(\bar{X},\bar{Z})L\bar{Y} - S(\bar{Y},\bar{Z})L\bar{X}\} \quad (61)$$

Adding (60) and (61) we have

$K(X,Y,\bar{Z}) + K(\bar{X},\bar{Y},\bar{Z}) = -\omega(\rho)\{ S(X,\bar{Z})LY - S(Y,\bar{Z})LX + S(\bar{X},\bar{Z})L\bar{Y} - S(\bar{Y},\bar{Z})L\bar{X}\}$

$= -\omega(\rho)\{ \overline{S(\bar{X},Z)}L\bar{\bar{Y}} - \overline{S(\bar{Y},Z)}L\bar{\bar{X}} + S(\bar{X},\bar{Z})L\bar{Y} - S(\bar{Y},\bar{Z})L\bar{X}\}$

$= \overline{K(\bar{X},\bar{Y},Z)} + \overline{K(X,Y,Z)}$ (62)

This proves (56).

Replacing X with $\bar{X}$ and Y with $\bar{Y}$ in (59) we have,

$$K(\bar{X},Y,Z) + K(X,\bar{Y},Z) + K(X,Y,\bar{Z}) = -\omega(\rho)\{ S(\bar{X},Z)LY - S(Y,Z)L\bar{X} + S(X,Z)L\bar{Y} - S(\bar{Y},Z)LX + S(X,\bar{Z})LY - S(Y,\bar{Z})LX\}$$

$= -\omega(\rho)\{- S(Y,Z)L\bar{X} + S(X,Z)L\bar{Y}\}$

$= -\omega(\rho)\{S(\bar{X},\bar{Z})L\bar{Y} - S(\bar{Y},\bar{Z})L\bar{X}\}$

$= -\omega(\rho)\{S(X,Z)LY - S(Y,Z)LX\} = K(X,Y,Z)$

This proves (57).

Using (7) we have $(K_{X\,Y}F)(Z) = K(X,Y,\bar{Z}) - \overline{K(X,Y,Z)}$

Barring X and Y we have

$(K_{\bar{X}\bar{Y}}F)(Z) = K(\bar{X},\bar{Y},\bar{Z}) - \overline{K(\bar{X},\bar{Y},Z)}$ (63)

Adding (7) and (63) we have

$(K_{X\,Y}F)(Z) + (K_{\bar{X}\bar{Y}}F)(Z) = K(X,Y,\bar{Z}) + K(\bar{X},\bar{Y},\bar{Z}) - \overline{K(X,Y,Z)} - \overline{K(\bar{X},\bar{Y},Z)}$

Implies that $(K_{X\,Y}F)(Z) + (K_{\bar{X}\bar{Y}}F)(Z) = 0$ using (56)

5. Contravariant almost analytic vector field

Let V be a contravariant almost analytic vector field in an almost complex manifold, then

$L_V\,F = 0\ implies\ [V,\bar{X}] = \overline{[V,X]}$ (64)

where

$L_V\,X = [V,X]$ (65)

where L denotes Lie Derivative.

If a Riemannian connection ∇ satisfies

$(\nabla_X\,F)(Y) = 0$ (66)

The almost Hermite manifold becomes K$\ddot{a}$hler manifold from (64) and (66), we have

$\nabla_{\bar{X}}V = \overline{\nabla_X\,V}$

Theorem 5.1: If V is contravariant almost analytic vector field associated to connection ∇, then it is also contravariant and almost analytic associated to Ricci quarter symmetric metric connection $\tilde{\nabla}$ if and only if

$$\omega(\bar{X})LV = \omega(X)\overline{LV} \tag{67}$$

and $S(\bar{X},V)\rho = S(X,V)\bar{\rho}$

Proof: Let V be contravariant almost analytic vector field and X be any arbitrary vector field, then from (10) and (11), we have

$$\tilde{\nabla}_V X = \nabla_V X + \omega(X)LV - S(V,X)\rho \text{ and} \tag{68}$$

$$\tilde{\nabla}_X V = \nabla_X V + \omega(V)LX - S(X,V)\rho \tag{69}$$

From (5.5) and (5.6), we get

$$[V,X]_s = [V,X] + \omega(X)LV - \omega(V)LX + 2S(X,V)\rho \tag{70}$$

equivalent to $T(V,X) = \omega(X)LV - \omega(V)LX$

where $[\,]_s$ is Lie Bracket with respect to Ricci quarter symmetric $\tilde{\nabla}$ from (70) we get

$$[V,\bar{X}]_s = [V,\bar{X}] + \omega(\bar{X})LV - \omega(V)\overline{LX} + 2S(\bar{X},V)\rho \quad \text{and} \tag{71}$$

$$\overline{[V,X]_s} = \overline{[V,X]} + \omega(X)\overline{LV} - \omega(V)\overline{LX} + 2S(X,V)\bar{\rho} \tag{72}$$

From (71) and (72), by virtue of (64) we get,

$$[V,\bar{X}]_s = \overline{[V,X]_s} + \omega(\bar{X})LV - \omega(X)\overline{LV} + 2\{S(\bar{X},V)\rho - S(X,V)\bar{\rho}\} \tag{73}$$

From (73) we have

$$\omega(\bar{X})LV = \omega(X)\overline{LV}$$

$$S(\bar{X},V)\rho = S(X,V)\bar{\rho}$$

6. Geodesics on an Almost Hermite Manifold

By Pandey and Chaturvedi [11], let σ be a curve in M^{2n} with the tangent field $\check{T}$, vector field Y is parallel if and only if $\nabla_{\check{T}}Y=0$ on σ and the curve is geodesic if and only if $\nabla_{\check{T}}\check{T} = 0$ on σ.

Now, we propose:

Theorem 6.1. In an almost Hermite manifold equipped with Ricci quarter symmetric connection $\tilde{\nabla}$ and associated Riemannian connection ∇ have same geodesic if and only if

$$\omega(\check{T})L\check{T} = S(\check{T},\check{T})\rho.$$

Proof: Putting $X = Y = \check{T}$ in equation (1.10), we have

$$\tilde{\nabla}_{\check{T}}\check{T} = \nabla_{\check{T}}\check{T} + \omega(\check{T})L\check{T} - S(\check{T},\check{T})\rho \tag{74}$$

From equation (74),

$\nabla_{\check{T}}\check{T} = 0$ implies $\tilde{\nabla}_{\check{T}}\check{T} = \omega(\check{T})L\check{T} - S(\check{T},\check{T})\rho.$

So, $\tilde{\nabla}_{\check{T}}\check{T} = 0$ if and only if $\omega(\check{T})L\check{T} = S(\check{T},\check{T})\rho.$

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