

PARA-SASAKIAN MANIFOLDS ADMITTING A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract: In this paper, para-Sasakian manifolds that admit a semi-symmetric non-metric connection has been studied. Some curvature properties of para-Sasakian manifolds in the defined connection are investigated. The association between semi-symmetric and locally φ -symmetric para-Sasakian manifolds with respect to a semi-symmetric non-metric connection are investigated. Finally, an example is present for demonstrated outcomes.

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1. Introduction

In 20th century, Carton established connection as a certain kind of differential form. In 1950, Koszul gave an algebraic framework for a connection as differential operator. The main invariants of an affine connection are its torsion and curvature ([11], [13]). Hayden [13] established the concept of metric connection having torsion. If a Riemannian metric \bar{g} having $\nabla_{\bar{g}} = 0$, is called metric and if $\nabla_{\bar{g}} \neq 0$, then it is non-metric. In a connection, torsion tensor T is given as

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad (1)$$

for all $X, Y \in \chi(M)$. If the torsion tensor T vanishes, it is symmetric and if it is not vanish, then it is non-symmetric.

Friedmann and Schouten [11] proposed the concept of semi-symmetric linear connection on a differentiable manifold in 1924. On M^n , a linear connection is defined as semi-symmetric if

$$\tilde{T}(X, Y) = \tilde{\eta}(Y)X - \tilde{\eta}(X)Y \quad (2)$$

for all X, Y on M^n , where $\bar{\eta}$ is a 1-form connected to the vector field $\bar{\xi}$ and satisfies

$$\bar{\eta}(X) = \bar{g}(X, \bar{\xi}) \quad (3)$$

In a Riemannian manifold, Yano [28] introduced the systematic study of semi-symmetric connection. The notion of semi-symmetric non-metric connection is proposed by Agashe and Chafle [1]. In a differentiable manifold, Golab [12] introduced the quarter-symmetric connection with. Rastogi [21], Mishra and Panday [17], Yano and Imai [29], Kumar et al. [14] and many other geometers examined various features of quarter-symmetric metric connection. Tripathi [23] investigated the semi-symmetric non-metric connection in Kenmotsu manifold. In line with this, Chaubey et al. [8] create a new type of semi-symmetric non-metric connection in Riemannian manifold. Afterwards, some other authors ([5], [6], [7], [18], [19], [20], [26]) studied several manifolds with several connections. Also Tripathi [24] also studied a connection in Riemannian manifold.

On the other hand, a para-Sasakian manifold was defined by Adati and Matsumoto [15]. Para-Sasakian manifold, which are special illustration of an almost para contact manifold defined by Sato [22]. Matsumoto, Ianus and Mihai [16], Yildiz, Turan and Acet [27], Barman ([2], [3]) and many others have researched on para-Sasakian manifolds. After the introduction, para-Sasakian manifolds have been studied in section-2. We have studied a semi-symmetric non-metric connection in section-3. In Section-4, we demonstrated the basic properties of curvature tensor with a semi-symmetric non-metric connection. We defined semi-symmetric in section-5 and locally φ -symmetric para-Sasakian manifolds studied in section-6. In the last section-7 we concern the example of para-Sasakian manifolds that admits semi-symmetric non-metric connection and validate some our results.

1. Preliminaries

A $(2n + 1)$ -dimensional differentiable manifold M^n [4] with almost para contact metric structure $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$, if it admits a tensor field $\bar{\varphi}$ of a type (1,1), a vector field $\bar{\xi}$ and 1-form $\bar{\eta}$ satisfying:

$$\bar{\varphi}^2 X = X - \bar{\eta}(X)\bar{\xi}, \quad \bar{\eta}(\bar{\xi}) = 1, \quad \bar{\varphi}\bar{\xi} = 0, \quad \bar{\eta}(\bar{\varphi}X) = 0, \quad (4)$$

$$\bar{g}(\bar{\varphi}X, \bar{\varphi}Y) = \bar{g}(X, Y) - \bar{\eta}(X)\bar{\eta}(Y), \quad (5)$$

$$\bar{g}(\bar{\varphi}X, Y) = \bar{g}(X, \bar{\varphi}Y), \quad \bar{g}(X, \bar{\xi}) = \bar{\eta}(X), \quad (6)$$

for all $X, Y \in TM$.

An almost para-contact metric manifold M^n is para-Sasakian manifolds if $(\nabla_X \bar{\varphi})Y = -\bar{g}(X, Y)\bar{\xi} - \bar{\eta}(Y)X + 2\bar{\eta}(X)\bar{\eta}(Y)\bar{\xi}$, (7)

from equations (4), (5), (6) and (2.4), we have

$$d\bar{\eta} = 0, \quad \nabla_X \bar{\xi} = \bar{\varphi}X, \quad (8)$$

$$(\nabla_X \bar{\eta})Y = \bar{g}(X, \bar{\varphi}Y) = (\nabla_Y \bar{\eta})X. \quad (9)$$

In para-Sasakian manifolds, the following relation holds:

$$\bar{\eta}(R(X, Y)Z) = \bar{g}(X, Z)\bar{\eta}(Y) - \bar{g}(Y, Z)\bar{\eta}(X), \quad (10)$$

$$R(X, Y)\bar{\xi} = \bar{\eta}(X)Y - \bar{\eta}(Y)X, \quad (11)$$

$$R(\bar{\xi}, X) = \bar{\eta}(Y)X - \bar{g}(X, Y)\bar{\xi}, \quad (12)$$

$$R(\bar{\xi}, X)\bar{\xi} = X - \bar{\eta}(X)\bar{\xi}, \quad (13)$$

$$S(X, \bar{\xi}) = -2n\bar{\eta}(X), \quad (14)$$

$$Q\bar{\xi} = -2n\bar{\xi}, \quad (15)$$

$$S(\bar{\varphi}X, \bar{\varphi}Y) = S(X, Y) + 2n\bar{\eta}(X)\bar{\eta}(Y), \quad (16)$$

where, R, S, Q are the curvature tensor, Ricci tensor and Ricci operator respectively. S and Q related by

$$S(X, Y) = \bar{g}(QX, Y). \quad (17)$$

3. A semi-symmetric non-metric connection

Suppose M^n be a para-Sasakian manifolds with the Levi-Civita connection ∇ and a linear connection $\tilde{\nabla}$ on M^n ([8], [10]) is given as

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}[\bar{\eta}(Y)X - \bar{\eta}(X)Y] \quad (18)$$

satisfying

$$\tilde{T}(X, Y) = \bar{\eta}(Y)X - \bar{\eta}(X)Y \quad (19)$$

$$\text{and } (\tilde{\nabla}_X \bar{g})(Y, Z) = \frac{1}{2}[2\bar{\eta}(X)\bar{g}(Y, Z) - \bar{\eta}(Y)\bar{g}(X, Z) - \bar{\eta}(Z)\bar{g}(X, Y)] \quad (20)$$

for arbitrary vector field X, Y and Z , known as a semi-symmetric non-metric connection.

Now,

$$(\tilde{\nabla}_X \bar{\varphi})(Y) = \frac{1}{2}[2(\nabla_X \bar{\varphi})Y - \bar{\eta}(Y)(\bar{\varphi}Y)], \quad (21)$$

$$(\tilde{\nabla}_X \bar{\eta})(Y) = (\nabla_X \bar{\eta})(Y), \quad (22)$$

$$(\tilde{\nabla}_X \bar{g})(\bar{\varphi}Y, Z) = \frac{1}{2}[2\bar{\eta}(X)\bar{g}(\bar{\varphi}Y, Z) - \bar{\eta}(Z)\bar{g}(X, \bar{\varphi}Y)]. \quad (23)$$

Replacing Y by $\bar{\xi}$ in (18), we find

$$\tilde{\nabla}_X \bar{\xi} = \nabla_X \bar{\xi} + \frac{1}{2}\bar{\varphi}^2 X. \quad (24)$$

Now, replacing X by $\bar{\xi}$ in (20), we have

$$(\tilde{\nabla}_{\bar{\xi}} \bar{g})(Y, Z) = \bar{g}(\bar{\varphi}X, \bar{\varphi}Y) = \bar{g}(Y, Z) - \bar{\eta}(Y)\bar{\eta}(Z). \quad (25)$$

We arrive the following proposition:

Proposition 3.1 *In a para-contact metric manifolds admitting a connection $\tilde{\nabla}$, co-variant differentiation of a metric \bar{g} with respect to contra-variant vector field $\bar{\xi}$ dose not vanish.*

4. Para-Sasakian manifolds with semi-symmetric non-metric connection

The curvature tensor \tilde{R} endowed with $\tilde{\nabla}$ given as

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]} Z, \quad (26)$$

where $X, Y, Z \in TM$. Using (18) in (26), we have

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + \frac{1}{2}[(\nabla_X \bar{\eta})(Z)Y - (\nabla_X \bar{\eta})(Y)Z - (\nabla_Y \bar{\eta})(Z)X \\ &\quad + (\nabla_Y \bar{\eta})(X)Z] + \frac{1}{4}[\bar{\eta}(Y)\bar{\eta}(Z)X - \bar{\eta}(X)\bar{\eta}(Z)Y], \end{aligned} \quad (27)$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad (28)$$

Using (6) and (9) in (27), we find

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + \frac{1}{2}[\bar{g}(X, \bar{\varphi}Z)Y - \bar{g}(Y, \bar{\varphi}Z)X] \\ &\quad + \frac{1}{4}[\bar{\eta}(Y)\bar{\eta}(Z)X - \bar{\eta}(X)\bar{\eta}(Z)Y]. \end{aligned} \quad (29)$$

Contracting equation (29) in terms of X , we have

$$\tilde{S}(Y, Z) = S(Y, Z) - n\bar{g}(Y, \bar{\varphi}Z) + \frac{n}{2}\bar{\eta}(Y)\bar{\eta}(Z). \quad (30)$$

Equations (17) and (30) together yield

$$\tilde{Q}(Y) = Q(Y) - n(\bar{\varphi}Y) + \frac{n}{2}\bar{\eta}(Y)\bar{\xi}. \quad (31)$$

Again, contracting (30), we have

$$\tilde{r} = r + \frac{n}{2}. \quad (32)$$

where $\tilde{S}; S, \tilde{Q}; Q$ and $\tilde{r}; r$ are the Ricci tensor, Ricci operators and scalar curvature endowed with $\tilde{\nabla}$ and ∇ .

Putting $X = \bar{\xi}$ in (29) and using equations (4) & (6), we have

$$\tilde{R}(\bar{\xi}, Y)Z = R(\bar{\xi}, Y)Z - \frac{1}{2}\bar{g}(\bar{\varphi}Y, Z)\bar{\xi} + \frac{1}{4}[\bar{\eta}(Y)\bar{\eta}(Z)\bar{\xi} - \bar{\eta}(Z)Y]. \quad (33)$$

In view of (12) and (33), we have

$$\tilde{R}(\bar{\xi}, Y)Z = -\bar{g}(Y, Z)\bar{\xi} - \frac{1}{2}\bar{g}(\bar{\varphi}Y, Z)\bar{\xi} + \frac{1}{4}\bar{\eta}(Y)\bar{\eta}(Z)\bar{\xi} + \frac{3}{4}\eta(Z)Y. \quad (34)$$

Again putting Z by $\bar{\xi}$ in (29) and using (4), (6) & (11), we have

$$\begin{aligned} \tilde{R}(X, Y)\bar{\xi} &= \frac{3}{4}R(X, Y)\bar{\xi} \\ &= \frac{3}{4}(\bar{\eta}(X)Y - \bar{\eta}(Y)X) \end{aligned} \tag{35}$$

In view of (29), (6) and $\bar{g}(R(X, Y, Z), W) = -\bar{g}(R(X, Y, W), Z)$, we have

$$\begin{aligned} \bar{\eta}(\tilde{R}(X, Y)Z) &= \frac{1}{2}[2\bar{\eta}(Y)\bar{g}(X, Z) - 2\bar{\eta}(X)\bar{g}(Y, Z) \\ &\quad + \bar{g}(\bar{\varphi}X, Z)\bar{\eta}(Y) - \bar{g}(\bar{\varphi}Y, Z)\bar{\eta}(X)]. \end{aligned} \tag{36}$$

Contracting (35) in terms of X , we obtain

$$\tilde{S}(Y, \bar{\xi}) = -\frac{3n}{2}\bar{\eta}(Y). \tag{37}$$

Suppose $\tilde{R}(X, Y)Z = 0$ in (29), we have

$$R(X, Y)Z = \frac{1}{2}[\bar{g}(\bar{\varphi}Y, Z)X - \bar{g}(\bar{\varphi}X, Z)Y] + \frac{1}{4}[\bar{\eta}(X)\bar{\eta}(Z)Y - \bar{\eta}(Y)\bar{\eta}(Z)X]. \tag{38}$$

In view of $R(X, Y, Z, W) = \bar{g}(R(X, Y)Z, W)$ and (38), we have

$$\begin{aligned} R(X, Y, Z, W) &= \frac{1}{2}[\bar{g}(\bar{\varphi}Y, Z)\bar{g}(X, W) - \bar{g}(\bar{\varphi}X, Z)\bar{g}(Y, W)] \\ &\quad + \frac{1}{4}[\bar{\eta}(X)\bar{\eta}(Y)\bar{g}(Y, W) - \bar{\eta}(Y)\bar{\eta}(Z)\bar{g}(X, W)]. \end{aligned} \tag{39}$$

Contracting the above equation in terms of X , we have

$$S(Y, Z) = n\bar{g}(\bar{\varphi}Y, Z) - \frac{n}{2}\bar{\eta}(Y)\bar{\eta}(Z). \tag{40}$$

Thus, we have the following theorem:

Theorem 4.1. *For a para-Sasakian manifolds with respect to the semi-symmetric non-metric connection $\tilde{\nabla}$*

- 1) *The curvature tensor \tilde{R} is given by (29),*
- 2) *The Ricci tensor \tilde{S} is given by (30),*
- 3) *The scalar curvature \tilde{r} is given by (32),*
- 4) $\tilde{S}(Y, \bar{\xi}) = -\frac{3n}{2}\bar{\eta}(Y),$
- 5) *If \tilde{R} is vanish then Ricci tensor is given by (40).*

5. Semi-symmetric para-Sasakian manifolds admitting a semi-symmetric non-metric connection

A $(2n + 1)$ -dimensional para contact metric manifold M^n with a connection $\tilde{\nabla}$ is semi-symmetric if

$$(\tilde{R}(X, Y) \cdot \tilde{R})(Z, U)V = 0$$

that is

$$\tilde{R}(X, Y)\tilde{R}(Z, U)V - \tilde{R}(\tilde{R}(X, Y)Z, U)V - \tilde{R}(Z, \tilde{R}(X, Y)U)V - \tilde{R}(Z, U)\tilde{R}(X, Y)V = 0. \quad (41)$$

Replacing X by $\bar{\xi}$, we have

$$\begin{aligned} &\tilde{R}(\bar{\xi}, Y)\tilde{R}(Z, U)V - \tilde{R}(\tilde{R}(\bar{\xi}, Y)Z, U)V \\ &- \tilde{R}(Z, \tilde{R}(\bar{\xi}, Y)U)V - \tilde{R}(Z, U)\tilde{R}(\bar{\xi}, Y)V = 0. \end{aligned} \quad (42)$$

In view of equation (34), we have

$$\begin{aligned} &-4\bar{g}(Y, \tilde{R}(Z, U)V)\bar{\xi} - 2\bar{g}(\bar{\varphi}Y, \tilde{R}(Z, U)V)\bar{\xi} + \eta(Y)\eta(\tilde{R}(Z, U)V)\bar{\xi} + 3\eta(\tilde{R}(Z, U)V)Y \\ &+ 4\bar{g}(Y, Z)\tilde{R}(\bar{\xi}, U)V + 2\bar{g}(\bar{\varphi}Y, Z)\tilde{R}(\bar{\xi}, U)V - \bar{\eta}(Y)\bar{\eta}(Z)\tilde{R}(\bar{\xi}, U)V - 3\bar{\eta}(Z)\tilde{R}(Y, U)V \\ &+ 4\bar{g}(Y, U)\tilde{R}(Z, \bar{\xi})V + 2\bar{g}(\bar{\varphi}Y, U)\tilde{R}(Z, \bar{\xi})V - \bar{\eta}(Y)\bar{\eta}(U)\tilde{R}(Z, \bar{\xi})V - 3\bar{\eta}(U)\tilde{R}(Z, Y)V \\ &+ 4\bar{g}(Y, V)\tilde{R}(Z, U)\bar{\xi} + 2\bar{g}(\bar{\varphi}Y, V)\tilde{R}(Z, U)\bar{\xi} - \bar{\eta}(Y)\bar{\eta}(V)\tilde{R}(Z, U) \\ &- 3\bar{\eta}(V)\tilde{R}(Z, U)Y = 0, \end{aligned} \quad (43)$$

which implies

$$\begin{aligned} 4\tilde{R}(Z, U, V, Y) &= -2\tilde{R}(Z, U, V, \bar{\varphi}Y) + \bar{\eta}(Y)\bar{\eta}(\tilde{R}(Z, U)V) + 3\bar{\eta}(Y)\bar{\eta}(\tilde{R}(Z, U)V) \\ &+ 4\bar{g}(Y, Z)\bar{\eta}(\tilde{R}(\bar{\xi}, U)V) + 2\bar{g}(\bar{\varphi}Y, Z)\bar{\eta}(\tilde{R}(\bar{\xi}, U)V) - \bar{\eta}(Y)\bar{\eta}(Z)\bar{\eta}(\tilde{R}(\bar{\xi}, U)V) \\ &- 3\bar{\eta}(Z)\bar{\eta}(\tilde{R}(Y, U)V) + 4\bar{g}(Y, U)\bar{\eta}(\tilde{R}(Z, \bar{\xi})V) + 2\bar{g}(\bar{\varphi}Y, U)\bar{\eta}(\tilde{R}(Z, \bar{\xi})V) \\ &- \bar{\eta}(Y)\bar{\eta}(U)\bar{\eta}(\tilde{R}(Z, \bar{\xi})V) - 3\bar{\eta}(U)\bar{\eta}(\tilde{R}(Z, Y)V) + 4\bar{g}(Y, V)\bar{\eta}(\tilde{R}(Z, U)\bar{\xi}) \\ &+ 2\bar{g}(\bar{\varphi}Y, V)\bar{\eta}(\tilde{R}(Z, U)\bar{\xi}) - \bar{\eta}(Y)\bar{\eta}(V)\bar{\eta}(\tilde{R}(Z, U)\bar{\xi}) - 3\bar{\eta}(V)\bar{\eta}(\tilde{R}(Z, U)Y). \end{aligned} \quad (44)$$

By using (4), (6), (34), (35) and (36) in (44), we find

$$\begin{aligned} 4\tilde{R}(Z, U, V, Y) &= -2\tilde{R}(Z, U, V, \bar{\varphi}Y) + 4[\bar{g}(Y, U)\bar{g}(Z, V) - \bar{g}(Y, Z)\bar{g}(U, V)] \\ &- 2[\bar{g}(Y, Z)\bar{g}(\bar{\varphi}U, V) + \bar{g}(U, V)\bar{g}(\bar{\varphi}Y, Z) - \bar{g}(Y, U)\bar{g}(\bar{\varphi}Z, V) \\ &- \bar{g}(Z, V)\bar{g}(\bar{\varphi}Y, U)] + \frac{1}{2}[\bar{\eta}(U)\bar{\eta}(V)\bar{g}(\bar{\varphi}Y, Z) - \bar{\eta}(Z)\bar{\eta}(V)\bar{g}(\bar{\varphi}Y, U)] \\ &+ [\bar{\eta}(U)\bar{\eta}(V)\bar{g}(Y, Z) - \bar{\eta}(Z)\bar{\eta}(V)\bar{g}(Y, U)] \\ &+ [\bar{g}(\bar{\varphi}Y, U)\bar{g}(\bar{\varphi}Z, V) - \bar{g}(\bar{\varphi}Y, Z)\bar{g}(\bar{\varphi}U, V)]. \end{aligned} \quad (45)$$

Hence, we have

$$\begin{aligned} 4\tilde{R}(Z, U)V &= -2\tilde{R}(Z, U, \bar{\varphi}V) + 4[\bar{g}(Z, V)U - \bar{g}(U, V)Z] \\ &- 2[\bar{g}(\bar{\varphi}U, V)Z + \bar{g}(U, V)\bar{\varphi}Z - \bar{g}(\bar{\varphi}Z, V)U - \bar{g}(Z, V)\bar{\varphi}U] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} [\bar{\eta}(U)\bar{\eta}(V)\bar{\varphi}Z - \bar{\eta}(Z)\bar{\eta}(V)\bar{\varphi}U] \\
& + [\bar{\eta}(U)\bar{\eta}(V)Z - \bar{\eta}(Z)\bar{\eta}(V)U] + [\bar{g}(\bar{\varphi}Z, V)\bar{\varphi}U - \bar{g}(\bar{\varphi}U, V)\bar{\varphi}Z]. \quad (46)
\end{aligned}$$

Contracting (46) in terms of Z , we have

$$\tilde{S}(U, V) = -\frac{1}{2}\tilde{S}(U, \bar{\varphi}V) - \frac{(8n-1)}{4}\bar{g}(U, V) - \frac{(2n-1)}{4}[2\bar{g}(\bar{\varphi}U, V) - \bar{\eta}(U)\bar{\eta}(V)] \quad (47)$$

and

$$\tilde{Q}U = -\frac{1}{2}[\bar{\varphi}(\tilde{Q}U) + \frac{(8n-1)}{4}U + (2n-1)\bar{\varphi}U - \frac{(2n-1)}{2}\bar{\eta}(U)\bar{\xi}]. \quad (48)$$

Theorem 5.1. *In semi-symmetric para-Sasakian manifolds admitting $\tilde{\nabla}$, Ricci tensor and Ricci operator defined by the equations (47) and (48) respectively.*

6. Locally φ -symmetric para-Sasakian manifolds admitting a semi-symmetric non-metric connection

Definition 6.1 A para-Sasakian manifolds with respect to $\tilde{\nabla}$ is locally φ -symmetric if ([3], [9], [25])

$$\bar{\varphi}^2 \left((\tilde{\nabla}_W \tilde{R})(X, Y)Z \right) = 0$$

for all vector fields X, Y, Z, W orthogonal to $\bar{\xi}$.

Now,

$$(\tilde{\nabla}_W R)(X, Y)Z = \tilde{\nabla}_W R(X, Y)Z - R(\tilde{\nabla}_W X, Y)Z - R(X, \tilde{\nabla}_W Y)Z - R(X, Y)(\tilde{\nabla}_W Z). \quad (49)$$

Using (18) and (10) in (29), we have

$$\begin{aligned}
(\tilde{\nabla}_W R)(X, Y)Z &= (\nabla_W R)(X, Y)Z + \frac{1}{2}[2\bar{\eta}(W)R(X, Y)Z - \bar{\eta}(X)R(W, Y)Z \\
&\quad - \bar{\eta}(Y)R(X, W)Z - \bar{\eta}(Z)R(X, Y)W + \bar{g}(X, Z)\bar{\eta}(Y)W - \bar{g}(Y, Z)\bar{\eta}(X)W]. \quad (50)
\end{aligned}$$

Taking covariant differentiation of (29) with respect to W and using (20), (21), (22) and (50), we have

$$\begin{aligned}
(\tilde{\nabla}_W \tilde{R})(X, Y)Z &= (\nabla_W R)(X, Y)Z + \frac{1}{2}[2\bar{\eta}(W)R(X, Y)Z - \bar{\eta}(X)R(W, Y)Z \\
&\quad - \bar{\eta}(Y)R(X, W)Z - \bar{\eta}(Z)R(X, Y)W + \bar{g}(X, Z)\bar{\eta}(Y)W \\
&\quad - \bar{g}(Y, Z)\bar{\eta}(X)W + 2\bar{\eta}(W)\bar{\eta}(Z)\bar{\eta}(X) - 2\bar{\eta}(W)\bar{\eta}(Z)\bar{\eta}(Y)X \\
&\quad + \bar{\eta}(W)\bar{g}(X, \bar{\varphi}Z)Y - \bar{\eta}(W)\bar{g}(Y, \bar{\varphi}Z)X - \bar{\eta}(X)\bar{g}(Z, \bar{\varphi}W)Y \\
&\quad + \bar{\eta}(Y)\bar{g}(Z, \bar{\varphi}W)X - \bar{\eta}(X)\bar{g}(Z, W)Y + \bar{\eta}(Y)\bar{g}(Z, W)X \\
&\quad - \bar{\eta}(Z)\bar{g}(X, W)Y + \bar{\eta}(Z)\bar{g}(Y, W)X - \bar{\eta}(Z)\bar{g}(X, \bar{\varphi}W)Y \\
&\quad + \bar{\eta}(Z)\bar{g}(Y, \bar{\varphi}W)X]. \quad (51)
\end{aligned}$$

Now, applying $\bar{\varphi}^2$ on both side of equation (51) and using equations (4), (5), (6), we obtain

$$\begin{aligned}
\bar{\varphi}^2 \left((\tilde{\nabla}_W \tilde{R})(X, Y)Z \right) &= \bar{\varphi}^2 \left((\nabla_W R)(X, Y)Z \right) + \frac{1}{2} [2\bar{\eta}(W)R(X, Y)Z \\
&\quad - 2\bar{\eta}(W)\bar{\eta}(R(X, Y)Z)\bar{\xi} - \bar{\eta}(X)R(W, Y)Z - \bar{\eta}(X)\bar{\eta}(R(W, Y)Z)\bar{\xi} \\
&\quad - \bar{\eta}(Y)R(X, W)Z + \bar{\eta}(Y)\bar{\eta}(R(X, W)Z)\bar{\xi} - \bar{\eta}(Z)R(X, Y)W \\
&\quad + \bar{\eta}(Z)\bar{\eta}(R(X, Y)W)\bar{\xi} + \bar{\eta}(Y)\bar{g}(X, Z)W - \bar{\eta}(Y)\bar{\eta}(W)\bar{g}(X, Z)\bar{\xi} \\
&\quad - \bar{\eta}(X)\bar{g}(Y, Z)W + \bar{\eta}(X)\bar{\eta}(W)\bar{g}(Y, Z)\bar{\xi} + 2\bar{\eta}(X)\bar{\eta}(Z)\bar{\eta}(W)Y \\
&\quad - 2\bar{\eta}(Y)\bar{\eta}(Z)\bar{\eta}(W)X + \bar{\eta}(W)\bar{g}(X, \bar{\varphi}Z)Y - \bar{\eta}(Y)\bar{\eta}(W)\bar{g}(X, \bar{\varphi}Z)\bar{\xi} \\
&\quad - \bar{\eta}(W)\bar{g}(Y, \bar{\varphi}Z)X + \bar{\eta}(X)\bar{\eta}(W)\bar{g}(Y, \bar{\varphi}Z)\bar{\xi} - \bar{\eta}(X)\bar{g}(Z, \bar{\varphi}W)Y \\
&\quad + \bar{\eta}(Y)\bar{g}(Z, \bar{\varphi}W)X - \bar{\eta}(X)\bar{g}(Z, W)Y + \bar{\eta}(Y)\bar{g}(Z, W)X \\
&\quad - \bar{\eta}(Z)\bar{g}(X, W)Y + \bar{\eta}(Y)\bar{\eta}(Z)\bar{g}(X, W)\bar{\xi} - \bar{\eta}(Z)\bar{g}(X, \bar{\varphi}W)Y \\
&\quad + \bar{\eta}(Y)\bar{\eta}(Z)\bar{g}(X, \bar{\varphi}W)\bar{\xi} + \bar{\eta}(Z)\bar{g}(Y, W)X - \bar{\eta}(X)\bar{\eta}(Z)\bar{g}(Y, W)\bar{\xi} \\
&\quad + \bar{\eta}(Z)\bar{g}(Y, \bar{\varphi}W)X - \bar{\eta}(X)\bar{\eta}(Z)\bar{g}(Y, \bar{\varphi}W)\bar{\xi}]. \tag{52}
\end{aligned}$$

If we take X, Y, Z and W orthogonal to $\bar{\xi}$, then (52) reduces to

$$\bar{\varphi}^2 \left((\tilde{\nabla}_W \tilde{R})(X, Y)Z \right) = \bar{\varphi}^2 \left((\nabla_W R)(X, Y)Z \right).$$

Theorem 6.1. *Necessary and sufficient condition for a para-Sasakian manifolds to be locally φ -symmetric equipped with $\tilde{\nabla}$ is that manifold is also locally φ -symmetric equipped with ∇ .*

7. Example of para-Sasakian Manifolds

Let us consider 3-dimensional manifold $M^3 = (x, y, z) \in \mathbb{R}^3: z \neq 0$ with the standard coordinates (x, y, z) of \mathbb{R}^3 . Let $v_1 = e^x \frac{\partial}{\partial y}$, $v_2 = e^x \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right)$, $v_3 = -\frac{\partial}{\partial x} = \bar{\xi}$ be linear independent.

Let g be the Riemannian metric defined by

$$\begin{aligned}
\bar{g}(v_1, v_2) &= \bar{g}(v_2, v_3) = \bar{g}(v_3, v_1) = 0, \\
\bar{g}(v_1, v_1) &= \bar{g}(v_2, v_2) = \bar{g}(v_3, v_3) = 1 \tag{53}
\end{aligned}$$

and $\bar{\varphi}$ is a $(1,1)$ -tensor field, defined by

$$\bar{\varphi}(v_1) = v_1, \bar{\varphi}(v_2) = v_2, \bar{\varphi}(v_3) = 0. \tag{54}$$

By using linearity of $\bar{\varphi}$ and g , we have

$$\bar{\eta}(v_3) = 1, \quad \bar{\varphi}^2 X = X - \bar{\eta}(X)v_3 \tag{55}$$

for each $X \in TM$. Here $\bar{\eta}(X) = \bar{g}(X, v_3)$ defines a 1-form on M^3 . As a result for $\bar{\xi} = v_3$, the structure $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ defined on M^3 . By definition of Lie bracket $[X, Y] = X(Yf) - Y(Xf)$, we have

$$\begin{aligned} [v_1, v_1] &= 0, & [v_1, v_2] &= 0, & [v_1, v_3] &= v_1, \\ [v_2, v_1] &= 0, & [v_2, v_2] &= 0, & [v_2, v_3] &= v_2, \\ [v_3, v_1] &= -v_1, & [v_3, v_2] &= -v_2, & [v_3, v_3] &= 0. \end{aligned} \quad (56)$$

Let ∇ be a Levi-Civita connection with regard to metric \bar{g} . By Koszul equation

$$\begin{aligned} 2\bar{g}(\nabla_X Y, Z) &= X\bar{g}(Y, Z) + Y\bar{g}(Z, X) - Z\bar{g}(X, Y) \\ + \bar{g}([X, Y], Z) &- \bar{g}([Y, Z], X) + \bar{g}([Z, X], Y) \end{aligned} \quad (57)$$

we can easily calculate

$$\begin{aligned} \nabla_{v_1} v_1 &= -v_3, & \nabla_{v_1} v_2 &= 0, & \nabla_{v_1} v_3 &= v_1, \\ \nabla_{v_2} v_1 &= 0, & \nabla_{v_2} v_2 &= -v_3, & \nabla_{v_2} v_3 &= v_2, \\ \nabla_{v_3} v_1 &= 0, & \nabla_{v_3} v_2 &= 0, & \nabla_{v_3} v_3 &= 0. \end{aligned} \quad (58)$$

Now $X = X^1 v_1 + X^2 v_2 + X^3 v_3$ and $\bar{\xi} = v_3$, we have

$$\begin{aligned} \nabla_X \bar{\xi} &= \nabla_{X^1 v_1 + X^2 v_2 + X^3 v_3} v_3 \\ &= X^1 \nabla_{v_1} v_3 + X^2 \nabla_{v_2} v_3 + X^3 \nabla_{v_3} v_3 \\ &= X^1 v_1 + X^2 v_2, \end{aligned} \quad (59)$$

$$\begin{aligned} \bar{\varphi} X &= \bar{\varphi}(X^1 v_1 + X^2 v_2 + X^3 v_3) \\ &= X^1 \bar{\varphi} v_1 + X^2 \bar{\varphi} v_2 + X^3 \bar{\varphi} v_3 \\ &= X^1 v_1 + X^2 v_2, \end{aligned} \quad (60)$$

where X^1, X^2, X^3 are scalars. According to (59) and (60), the structure $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is para-Sasakian structure. Therefore $M^3(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is para-Sasakian manifolds. With regard to the equations (4), (6), (18) and (58), we have the following results:

$$\begin{aligned} \tilde{\nabla}_{v_1} v_1 &= -v_3, & \tilde{\nabla}_{v_1} v_2 &= 0, & \tilde{\nabla}_{v_1} v_3 &= \frac{3}{2} v_1, \\ \tilde{\nabla}_{v_2} v_1 &= 0, & \tilde{\nabla}_{v_2} v_2 &= -v_3, & \tilde{\nabla}_{v_2} v_3 &= \frac{3}{2} v_2, \\ \tilde{\nabla}_{v_3} v_1 &= -\frac{1}{2} v_1, & \tilde{\nabla}_{v_3} v_2 &= -\frac{1}{2} v_2, & \tilde{\nabla}_{v_3} v_3 &= 0. \end{aligned} \quad (61)$$

From (19) and (20), we have

$$\tilde{T}(v_1, v_3) = \bar{\eta}(v_3)v_1 - \bar{\eta}(v_1)v_3 = v_1 \neq 0$$

and

$$\begin{aligned} (\tilde{\nabla}_{v_1} \bar{g})(v_1, v_3) &= \frac{1}{2} [2 \bar{\eta}(v_1) \bar{g}(v_1, v_3) - \bar{\eta}(v_1) \bar{g}(v_1, v_3) - \bar{\eta}(v_3) \bar{g}(v_1, v_1)] \\ &= -\frac{1}{2} \neq 0. \end{aligned}$$

Consequently, a semi-symmetric non-metric connection defined in (18).

Also, we have

$$\begin{aligned} \tilde{\nabla}_X \xi &= \tilde{\nabla}_{X^1 v_1 + X^2 v_2 + X^3 v_3} v_3 \\ &= X^1 \tilde{\nabla}_{v_1} v_3 + X^2 \tilde{\nabla}_{v_2} v_3 + X^3 \tilde{\nabla}_{v_3} v_3 \\ &= \frac{3}{2} X^1 v_1 + \frac{3}{2} X^2 v_2 \end{aligned} \quad (62)$$

The equation (24) can be verified using equations (59) and (62).

Using (56) and (58) to compute R of connection ∇ .

$$\begin{aligned} R(v_1, v_2)v_1 &= v_2, \quad R(v_1, v_3)v_1 = v_3, \quad R(v_2, v_3)v_1 = 0, \\ R(v_1, v_2)v_2 &= -v_1, \quad R(v_1, v_3)v_2 = 0, \quad R(v_2, v_3)v_2 = v_3, \\ R(v_1, v_2)v_3 &= 0, \quad R(v_1, v_3)v_3 = -v_1, \quad R(v_2, v_3)v_3 = -v_2. \end{aligned} \quad (63)$$

also $R(v_i, v_i)v_i = 0; i = 1, 2, 3$. From simple calculations, The equations (11), (12) and (13) are simply demonstrated to be true.

Similarly, using (56) and (61) to calculated \tilde{R} with connection $\tilde{\nabla}$.

$$\begin{aligned} \tilde{R}(v_1, v_2)v_1 &= \frac{3}{2} v_2, \quad \tilde{R}(v_1, v_3)v_1 = \frac{3}{2} v_3, \quad \tilde{R}(v_2, v_3)v_1 = 0, \\ \tilde{R}(v_1, v_2)v_2 &= -\frac{3}{2} v_1, \quad \tilde{R}(v_1, v_3)v_2 = 0, \quad \tilde{R}(v_2, v_3)v_2 = \frac{3}{2} v_3, \\ \tilde{R}(v_1, v_2)v_3 &= 0, \quad \tilde{R}(v_1, v_3)v_3 = -\frac{3}{4} v_1, \quad \tilde{R}(v_2, v_3)v_3 = -\frac{3}{4} v_2. \end{aligned} \quad (64)$$

Along with $R(v_i, v_i)v_i = 0; i = 1, 2, 3$.

In consequence of equations (63) and (64), we can verify the equations (29), (33), (34), (35) and (36).

The Ricci tensor S of connection ∇ can be derived by using (63) in $S(X, Y) = \sum_{i=1}^3 \bar{g}(R(v_i, X)Y, v_i)$. It is as under:

$$S(v_1, v_1) = S(v_2, v_2) = S(v_3, v_3) = -2. \quad (65)$$

Ricci tensor \tilde{S} of a connection $\tilde{\nabla}$ can be derived by using equation (64) in $\tilde{S}(X, Y) = \sum_{i=1}^3 \bar{g}(R(v_i, X)Y, v_i)$. It is as follows:

$$\tilde{S}(v_1, v_1) = -3, \tilde{S}(v_2, v_2) = -3, \tilde{S}(v_3, v_3) = -\frac{3}{2}. \quad (66)$$

Hence, we can say that given example is perfect match of our investigations.

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