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CERTAIN RESULTS ON MULTIPARAMETER S-K-MITTAG-LEFFLER FUNCTION

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Abstract. In this paper, we define a Multiparameter S-K-Mittag-Leffler function and have investigated the properties of the above new function such as integral representations of ${}_p^qK_{s,k}^{(\beta,\eta)_m}[z]$, recurrence relations and the relation between Multiparameter S-K-Mittag-Leffler function and K-series. Some particular cases have been obtained by particularizing the values of the parameters.

Key Words : Multiparameter S-K-Mittag-Leffler function, K-Series, S-K-Pochhammer symbol, S-K- Gamma function.

MSC 2010 : 26A33, 33B15, 33E12.

1. Introduction

Gehlot[3] established the functional relation between Multiparameter K-Mittag-Leffler function and K-series. Various properties of Multiparameter K-Mittag-Leffler function are also proved. Gehlot[4] introduced the generalized S-K-Gamma function ${}_s\Gamma_k(x)$ as

$${}_s\Gamma_k(x) = \frac{1}{k}\lim_{n\to\infty}\frac{n!s^{n+1}(ns)^{\frac{x}{k}-1}}{{}_s(x)_{n,k}}, s,k > 0, x \in C\backslash kZ^-, \tag{1}$$

where

$${}_s(x)_{n,k} = (\frac{xs}{k})(\frac{xs}{k}+s)(\frac{xs}{k}+2s).....(\frac{xs}{k}+(n-1)s), x \in C, s,k \in R, n \in N^+. \tag{2}$$

and the integral representation of S-K-Gamma function

$${}_s\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{s}}dt, \quad x \in C, s,k \in R, Re(x) > 0, \tag{3}$$

and it follows easily that

$${}_s\Gamma_k(x) = \frac{s^{\frac{x}{k}}}{k}\Gamma(\frac{x}{k}). \tag{4}$$

$${}_s\Gamma_k(x+k) = \frac{xs}{k} \quad {}_s\Gamma_k(x). \tag{5}$$

$${}_s(x)_{n,k} = s^n (\frac{x}{k})_n. \quad (6)$$

$${}_s(x)_{n,k} = \frac{{}_s\Gamma_k(x+nk)}{{}_s\Gamma_k(x)}. \tag{7}$$

$$ns \ {}_s(x)_{n-1,k} = {}_s\,(x)_{n,k} - {}_s(x-k)_{n,k}. \tag{8}$$

$${}_s(x)_{n+j,k} = {}_s\,(x)_{j,k} \ {}_s(x+jk)_{n,k} \tag{9}$$

2. Definition

Let $s,k \in R_+ = (0,\infty)$; $a_j, b_r, \beta_i \in C$; $\eta_i \in R$ $(j = 1,2,..,p; r = 1,2,..,q; i = 1,2,..,m)$. Then the Multiparameter S-K-Mittag-Leffler function is defined in the following form :

$${}^q_pK^{(\beta,\eta)_m}_{s,k}[z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \ {}_s(a_j)_{n,k}\, z^n}{\prod_{r=1}^{q} \ {}_s(b_r)_{n,k} \quad \prod_{i=1}^{m} \ {}_s\Gamma_k(\eta_i n+\beta_i)}, \tag{10}$$

Where ${}_s\Gamma_k(x)$ is the S-K-Gamma function and ${}_s(\gamma)_{n,k}$ is the S-K-Pochhammer symbol.

The series (10) is converges when no parameter $b_r(r = 1,2,..,q)$ is zero or a negative integer. Next, if any numerator parameter $a_j(j = 1,2,..,p)$ is zero or negative integer, the series terminates and the question of convergence does not arise.

For cases other than these by D'Alembert's ratio test :

(i) If $p < q + \sum_{i=1}^{m} (\frac{\eta_i}{k})$, RHS of (10) is absolutely convergent for all $z \in C$.

(ii) If $p = q + \sum_{i=1}^{m} (\frac{\eta_i}{k})$, then the RHS of (10) is absolutely convergent for all $|s^{p-q-\sum_{i=1}^{m}(\frac{\eta_i}{k})} z| < \prod_{i=1}^{m} (|\frac{\eta_i}{k}|)^{\frac{\eta_i}{k}}$ and $|s^{p-q-\sum_{i=1}^{m}(\frac{\eta_i}{k})} z| = \prod_{i=1}^{m} (|\frac{\eta_i}{k}|)^{\frac{\eta_i}{k}}$,

$Re\,(\sum_{r=1}^{q} (\frac{b_r}{k}) + \sum_{i=1}^{m} (\frac{\beta_i}{k}) - \sum_{j=1}^{p} (\frac{a_j}{k})) > \frac{2+q+m-p}{2}$.

3. Relation between Multiparameter S-K-Mittag-Leffler function and K-series

In this section we established the functional relation between Multiparameter S-K-Mittag-Leffler Function and K-Series.

3.1 Theorem

$${}^q_pK^{(\beta,\eta)_m}_{s,k}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i,\eta_i)_{i=1}^{m}; z]$$

$$= k^m s^{\sum_{i=1}^{m}(-\frac{\beta_i}{k})} \ {}_pK_q^{(\beta,\eta)_m}[(\frac{a_j}{k})_{j=1}^{p}; (\frac{b_r}{k})_{r=1}^{q}, (\frac{\beta_i}{k}, \frac{\eta_i}{k})_{i=1}^{m}; z s^{p-q-\sum_{i=1}^{m}\frac{\eta_i}{k}}]. \tag{11}$$

And its counter part is given by

$${}_pK_q^{(\beta,\eta)_m}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i,\eta_i)_{i=1}^{m}; z]$$

$$= k^{-m} s^{\sum_{i=1}^{m}(\beta_i)}\, {}_{p}^{q}K_{s,k}^{(\beta,\eta)_m}[(ka_j)_{j=1}^{p}; (kb_r)_{r=1}^{q}, (k\beta_i, k\eta_i)_{i=1}^{m}; zs^{\sum_{i=1}^{m}\eta_i+q-p}]. \quad (12)$$

Proof : Using equations (4) and (6), (10) becomes

$${}_{p}^{q}K_{s,k}^{(\beta,\eta)_m}[z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} s^{pn}(\frac{a_j}{k})_n\, z^n}{\prod_{r=1}^{q} s^{qn}(\frac{b_r}{k})_n \prod_{i=1}^{m} \frac{s^{\frac{\eta_i n+\beta_i}{k}}}{k}\Gamma(\frac{\eta_i}{k}n+\frac{\beta_i}{k})},$$

after some simplification, we obtain

$$= k^m s^{-\sum_{i=1}^{m}(\frac{\beta_i}{k})} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\ (\frac{a_j}{k})_n\, (zs^{p-q-\sum_{i=1}^{m}\frac{\eta_i}{k}})^n}{\prod_{r=1}^{q}(\frac{b_r}{k})_n \prod_{i=1}^{m}\Gamma(\frac{\eta_i}{k}n+\frac{\beta_i}{k})},$$

using the definition of K-series, we get

$$= k^m s^{-\sum_{i=1}^{m}(\frac{\beta_i}{k})}\, {}_{p}K_{q}^{(\beta,\eta)_m}[(\frac{a_j}{k})_{j=1}^{p}; (\frac{b_r}{k})_{r=1}^{q}, (\frac{\beta_i}{k}, \frac{\eta_i}{k})_{i=1}^{m}; zs^{p-q-\sum_{i=1}^{m}\frac{\eta_i}{k}}].$$

This completes the proof of (11).

Similarly we can prove the counter part.

4. Reccurence Relations

In this section, we proved some reccurence relations of Multiparameter S-K-Mittag-Leffler function.

4.1 Theorem

Let $b \in C, \beta \in R$ and the convergent conditions of Multiparameter S-K-Mittag-Leffler Function are satisfies, then

$$\frac{k}{s}\ {}_{p}^{q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i, \eta_i)_{i=1}^{m}, (b, \beta); z]$$

$$= b\, {}_{p}^{q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i, \eta_i)_{i=1}^{m}, (b+k, \beta); z]$$

$$+\beta z \frac{d}{dz}\, {}_{p}^{q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i, \eta_i)_{i=1}^{m}, (b+k, \beta); z]. \quad (13)$$

Proof : Consider the right hand side of equation (13) and using equation (10), we have

$$b\, {}_{p}^{q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i, \eta_i)_{i=1}^{m}, (b+k, \beta); z]$$

$$+\beta z \frac{d}{dz}\, {}_{p}^{q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i, \eta_i)_{i=1}^{m}, (b+k, \beta); z]$$

$$= b \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_{s}(a_j)_{n,k}\, z^n}{\prod_{r=1}^{q}\ {}_{s}(b_r)_{n,k}\ \prod_{i=1}^{m}\ {}_{s}\Gamma_k(\eta_i n+\beta_i)\, {}_{s}\Gamma_k(\beta n+b+k)}$$

$$+\beta z \frac{d}{dz} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_{s}(a_j)_{n,k}\, z^n}{\prod_{r=1}^{q} {}_{s}(b_r)_{n,k}\ \prod_{i=1}^{m}\ {}_{s}\Gamma_k(\eta_i n+\beta_i)\, {}_{s}\Gamma_k(\beta n+b+k)},$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k} (\beta n+b)\, z^n}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \ \prod_{i=1}^{m} {}_s\Gamma_k(\eta_i n+\beta_i)\ {}_s\Gamma_k(\beta n+b+k)},$$

using equation (5), we obtain LHS of (13)

$$= \frac{k}{s}\ {}_p^q K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i,\eta_i)_{i=1}^{m}, (b,\beta); z].$$

Hence.

4.2 Theorem

Let $a \in C$ and the convergent conditions of Multiparameter S-K-Mittag-Leffler Function are satisfies, then

$${}_{p+1}^{\ q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}, a+k; (b_r)_{r=1}^{q}, (\beta_i,\eta_i)_{i=1}^{m}, (k,1); z]$$
$$- {}_{p+1}^{\ q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}, a; (b_r)_{r=1}^{q}, (\beta_i,\eta_i)_{i=1}^{m}, (k,1); z]$$
$$= kz\left(\frac{s}{k}\right)^{p-q} \frac{\prod_{j=1}^{p}(a_j)}{\prod_{r=1}^{q}(b_r)}\ {}_{p+1}^{\ q}K_{s,k}^{(\beta+\eta,\eta)_{m+1}}[(a_j+k)_{j=1}^{p}, a+k; (b_r+k)_{r=1}^{q},$$
$$(\beta_i+\eta_i,\eta_i)_{i=1}^{m}, (1,1); z]. \tag{14}$$

Proof : Consider the left hand side of equation (14) and using equation (10), we have

$${}_{p+1}^{\ q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}, a+k; (b_r)_{r=1}^{q}, (\beta_i,\eta_i)_{i=1}^{m}, (k,1); z]$$
$$- {}_{p+1}^{\ q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}, a; (b_r)_{r=1}^{q}, (\beta_i,\eta_i)_{i=1}^{m}, (k,1); z]$$
$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k}\, z^n}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} {}_s\Gamma_k(\eta_i n+\beta_i) {}_s\Gamma_k(n+k)} [{}_s(a+k)_{n,k} - {}_s(a)_{n,k}],$$

By making use of (8), it yields

$$= \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k}\, z^n}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} {}_s\Gamma_k(\eta_i n+\beta_i) {}_s\Gamma_k(n+k)} [ns\ {}_s(a+k)_{n-1,k}],$$

replacing n by $n+1$, we obtain

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n+1,k}\, z^{n+1}}{\prod_{r=1}^{q} {}_s(b_r)_{n+1,k} \prod_{i=1}^{m} {}_s\Gamma_k(\eta_i (n+1)+\beta_i)\ {}_s\Gamma_k(n+1+k)} [(n+1)k\ {}_s(a+k)_{n,k}],$$

using equations (5) and (9), we obtain

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{1,k}\ {}_s(a_j+k)_{n,k}\, z^{n+1}[(n+1)s\ {}_s(a+k)_{n,k}]}{\prod_{r=1}^{q} {}_s(b_r)_{1,k}\ {}_s(b_r+k)_{n,k} \prod_{i=1}^{m} {}_s\Gamma_k(\eta_i n+\beta_i+\eta_i) \frac{(n+1)s}{k}\ {}_s\Gamma_k(n+1)},$$

after some simplification, the above expression becomes equal to

$$= kz\left(\frac{s}{k}\right)^{p-q} \frac{\prod_{j=1}^{p}(a_j)}{\prod_{r=1}^{q}(b_r)}\ {}_{p+1}^{\ q}K_{s,k}^{(\beta+\eta,\eta)_{m+1}}[(a_j+k)_{j=1}^{p}, a+k; (b_r+k)_{r=1}^{q}, (\beta_i +$$

$\eta_i, \eta_i)_{i=1}^{m}, (1,1); z]$.

The RHS of (14) is proved.

4.3 Theorem

Let $\beta \in C, Re(\beta) > 0, \alpha \in R$ and the convergent conditions of Multiparameter S-K-Mittag-Leffler Function are satisfies, then

$$\frac{k^2}{s^2} \, {}_p^q K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i, \eta_i)_{i=1}^{m}, (\beta + k, \alpha); z]$$

$$-\frac{k^2}{s} \, {}_p^q K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i, \eta_i)_{i=1}^{m}, (\beta + 2k, \alpha); z]$$

$$= z^2\alpha^2 \, {}_p^q \ddot{\mathbf{K}}_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i, \eta_i)_{i=1}^{m}, (\beta + 3k, \alpha); z]$$

$$+z\{\alpha^2 + 2\alpha(\beta + k)\} \, {}_p^q \dot{\mathbf{K}}_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i, \eta_i)_{i=1}^{m}, (\beta + 3k, \alpha); z]$$

$$+\beta(\beta + 2k) \, {}_p^q K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i, \eta_i)_{i=1}^{m}, (\beta + 3k, \alpha); z]. \quad (15)$$

Proof : From equations (10) and (5), we have

$${}_p^q K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i, \eta_i)_{i=1}^{m}, (\beta + k, \alpha); z]$$

$$= \frac{k}{s} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k} \, z^n}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} {}_s\Gamma_k(\eta_i n + \beta_i) \ (\alpha n + \beta) \ {}_s\Gamma_k(\alpha n + \beta)}. \quad (16)$$

Again,

$${}_p^q K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i, \eta_i)_{i=1}^{m}, (\beta + 2k, \alpha); z]$$

$$= \frac{k^2}{s^2} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k} \, z^n}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} {}_s\Gamma_k(\eta_i n + \beta_i)(\alpha n + \beta + k)(\alpha n + \beta) {}_s\Gamma_k(\alpha n + \beta)}. \quad (17)$$

$$= \frac{k^2}{s^2} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k} \, z^n}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} \ {}_s\Gamma_k(\eta_i n + \beta_i) {}_s\Gamma_k(\alpha n + \beta)} \frac{1}{k} \left[\frac{1}{(\alpha n + \beta)} - \frac{1}{(\alpha n + \beta + k)}\right]$$

by making use of (16), we obtain

$$\frac{k^3}{s^3} \mathbf{S} = {}_p^q K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i, \eta_i)_{i=1}^{m}, (\beta + k, \alpha); z]$$

$$-k \, {}_p^q K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p}; (b_r)_{r=1}^{q}, (\beta_i, \eta_i)_{i=1}^{m}, (\beta + 2k, \alpha); z]. \quad (18)$$

Where

$$\mathbf{S} = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k} \, z^n}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} \ {}_s\Gamma_k(\eta_i n + \beta_i) {}_s\Gamma_k(\alpha n + \beta)(\alpha n + \beta + k)}. \quad (19)$$

using the identity $\frac{1}{u}=\frac{k}{u(u+k)}+\frac{1}{(u+k)}$, for $u=\alpha n+\beta+k$ to equation (19), we obtain

$$S=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}{}_s(a_j)_{n,k}\,z^n}{\prod_{r=1}^{q}{}_s(b_r)_{n,k}\,\prod_{i=1}^{m}\ {}_s\Gamma_k(\eta_i n+\beta_i)\,{}_s\Gamma_k(\alpha n+\beta)}$$
$$\times\left[\frac{k}{(\alpha n+\beta+k)(\alpha n+\beta+2k)}+\frac{1}{(\alpha n+\beta+2k)}\right],$$
$$\mathbf{S}=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}{}_s(a_j)_{n,k}\,z^n}{\prod_{r=1}^{q}{}_s(b_r)_{n,k}\prod_{i=1}^{m}\ {}_s\Gamma_k(\eta_i n+\beta_i)\,{}_s\Gamma_k(\alpha n+\beta)}$$
$$\times\left[\frac{k(\alpha n+\beta)}{(\alpha n+\beta)(\alpha n+\beta+k)(\alpha n+\beta+2k)}+\frac{(\alpha n+\beta)(\alpha n+\beta+k)}{(\alpha n+\beta)(\alpha n+\beta+k)(\alpha n+\beta+2k)}\right],$$

using equation (5), we have

$$\mathbf{S}=\frac{s^3}{k^3}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}{}_s(a_j)_{n,k}\,z^n[n^2\alpha^2+2n\alpha(\beta+k)+\beta(\beta+2k)]}{\prod_{r=1}^{q}{}_s(b_r)_{n,k}\,\prod_{i=1}^{m}\ {}_s\Gamma_k(\eta_i n+\beta_i)\,{}_s\Gamma_k(\alpha n+\beta+3k)}. \tag{20}$$

We express each summation in right side of (15) as follows;

$$\frac{d}{dz}\{z\,{}_p^qK_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]\}$$
$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}{}_s(a_j)_{n,k}(n+1)\,z^n}{\prod_{r=1}^{q}{}_s(b_r)_{n,k}\prod_{i=1}^{m}{}_s\Gamma_k(\eta_i n+\beta_i)\,{}_s\Gamma_k(\alpha n+\beta+3k)},$$
$$z\,{}_p^q\dot{\mathbf{K}}_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$
$$+{}_p^qK_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$
$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}{}_s(a_j)_{n,k}(n+1)\,z^n}{\prod_{r=1}^{q}{}_s(b_r)_{n,k}\prod_{i=1}^{m}\ {}_s\Gamma_k(\eta_i n+\beta_i)\,{}_s\Gamma_k(\alpha n+\beta+3k)},$$
$$z\,{}_p^q\dot{\mathbf{K}}_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$
$$=\sum_{n=0}^{\infty}\frac{n\prod_{j=1}^{p}{}_s(a_j)_{n,k}\,z^n}{\prod_{r=1}^{q}{}_s(b_r)_{n,k}\prod_{i=1}^{m}\ {}_s\Gamma_k(\eta_i n+\beta_i)\,{}_s\Gamma_k(\alpha n+\beta+3k)}. \tag{21}$$

Again

$$\frac{d^2}{dz^2}\{z^2\,{}_p^qK_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]\}$$
$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}{}_s(a_j)_{n,k}(n+2)(n+1)\,z^n}{\prod_{r=1}^{q}{}_s(b_r)_{n,k}\,\prod_{i=1}^{m}\ {}_s\Gamma_k(\eta_i n+\beta_i)\ {}_s\Gamma_k(\alpha n+\beta+3k)},$$
$$z^2\,{}_p^q\ddot{\mathbf{K}}_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$
$$+4z\,{}_p^q\dot{\mathbf{K}}_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p;(b_r)_{r=1}^q,(\beta_i,\eta_i)_{i=1}^m,(\beta+3k,\alpha);z]$$

$$+2\,{}_{p}^{q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$
$$=\sum_{n=0}^{\infty}\frac{(n^2+3n+2)\prod_{j=1}^{p}{}_{s}(a_j)_{n,k}\,z^n}{\prod_{r=1}^{q}{}_{s}(b_r)_{n,k}\prod_{i=1}^{m}\,{}_{s}\Gamma_k(\eta_i n+\beta_i)\,{}_{s}\Gamma_k(\alpha n+\beta+3k)},$$

using equation (21)

$$z^2\,{}_{p}^{q}\ddot{\mathbf{K}}_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$
$$+z\,{}_{p}^{q}\dot{\mathbf{K}}_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$
$$=\sum_{n=0}^{\infty}\frac{n^2\prod_{j=1}^{p}{}_{s}(a_j)_{n,k}\,z^n}{\prod_{r=1}^{q}{}_{s}(b_r)_{n,k}\prod_{i=1}^{m}\,{}_{s}\Gamma_k(\eta_i n+\beta_i)\,\,{}_{s}\Gamma_k(\alpha n+\beta+3k)}. \tag{22}$$

using equations (21), (22) in equation (20), we obtain

$$\frac{k^3}{s^3}\mathbf{S}=z^2\alpha^2\,{}_{p}^{q}\ddot{\mathbf{K}}_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$
$$+z\{\alpha^2+2\alpha(\beta+k)\}\,{}_{p}^{q}\dot{\mathbf{K}}_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z]$$
$$+\beta(\beta+2k)\,{}_{p}^{q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+3k,\alpha);z].$$

Hence.

5. Integral Representation of ${}_{p}^{q}K_{s,k}^{(\beta,\eta)_m}[z]$.

In this section, we investigated the integral representation of Multiparameter S-K-Mittag-Leffler function.

5.1 Theorem

Let $\beta\in C, Re(\beta)>0, \alpha\in R$ and the convergent conditions of Multiparameter S-K-Mittag-Leffler Function are satisfies, then

$$\int_0^1 t^{\beta+k-1}\,{}_{p}^{q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta,\alpha);t^{\alpha}]dt$$
$$=\frac{s}{k}\,{}_{p}^{q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+k,\alpha);1]$$
$$-\frac{s^2}{k}\,{}_{p}^{q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+2k,\alpha);1]. \tag{23}$$

Proof : Put $z=1$ in equations (18) and (19), we have

$$\mathbf{S}=\frac{s}{k}\,{}_{p}^{q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+k,\alpha);1]$$
$$-\frac{s^2}{k}\,{}_{p}^{q}K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^{p};(b_r)_{r=1}^{q},(\beta_i,\eta_i)_{i=1}^{m},(\beta+2k,\alpha);1]$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k}}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \quad \prod_{i=1}^{m} \quad {}_s\Gamma_k(\eta_i n+\beta_i) {}_s\Gamma_k(\alpha n+\beta)(\alpha n+\beta+k)}. \tag{24}$$

Consider the left hand side of (23) and by making use of (10), we obtain,

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k}}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} \quad {}_s\Gamma_k(\eta_i n + \beta_i) {}_s\Gamma_k(\alpha n + \beta)} \int_0^1 t^{\alpha n+\beta+k-1} dt$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k}}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} \quad {}_s\Gamma_k(\eta_i n+\beta_i) \ {}_s\Gamma_k(\alpha n+\beta)(\alpha n+\beta+k)},$$

from equation (24), we obtain RHS of (23),

$$\frac{s}{k} \ {}_p^q K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p; (b_r)_{r=1}^q, (\beta_i, \eta_i)_{i=1}^m, (\beta + k, \alpha); 1]$$

$$- \frac{s^2}{k} \ {}_p^q K_{s,k}^{(\beta,\eta)_{m+1}}[(a_j)_{j=1}^p; (b_r)_{r=1}^q, (\beta_i, \eta_i)_{i=1}^m, (\beta + 2k, \alpha); 1].$$

This completes the proof of (23).

6. Particular Cases

The particular cases of this paper are given by particularzing the values of parameters, we obtain the result for different known Mittag-Leffler Functions, given as:

(a) If we set $s = k$, then we obtain the results for Multiparameter K Mittag-Leffler function defined by Gehlot[2].

(b) If we set $s = k = 1$, then we get the results for K-Series defined by Gehlot and Ram[5].

(c) If we set $s = k = 1, p = q = m$ and $b_1 = b_2 = \ldots = b_m = 1$, we get the results for the 3M-Parameter Multi-Index Mittag-Leffler function defined by Konovska[6].

(d) If we set $k = 1, p = q = 1, a_1 = \rho, b_1 = 1$, then we obtain the results for the Generalized Mittag-Leffler function studied by Luchko[7].

(e) If we set $s = k = 1, p = q = 1, a_1 = b_1 = 1$ and $\eta_i = \frac{1}{\alpha_i}$, then we arrive at the results for the Multi-Index Mittag-Leffler function studied by Kiryakova[11].

(f) If we set $s = k = 1, m = 1$, then we get the results for Generalized M-Series defined by Sharma and Jain[10]

(g) If we set $s = k, p = q = m = 1, a_1 = \delta, b_1 = k$, then we obtain the results for the K- Mittag-Leffler function studied by Dorrego and Cerutti[1].

(h) If we set $s = k = 1, p = q = m = 1, a_1 = \delta, b_1 = 1$, then we get the results for the Generalized Mittag-Leffler function studied by Prabhakar[9].

(i) If we set $s = k = 1, p = q = m = 1, a_1 = b_1 = 1$, then we obtain the results for the Mittag-Leffler function studied by Wiman[12].

(j) If we set $s = k = 1, p = q = m = 1, a_1 = b_1 = 1$ and $\beta = 1$, then we obtain the results for the Mittag-Leffler function studied by Mittag-Leffler[8].

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References

[1] Dorrego, G.A. and Cerutti, R.A (2012). *The K-Mittag-Leffler Function*, Int. J. Contemp. Math. Sciences, **7**(15), 705-716.

[2] Gehlot, K.S. (2013). *Multiparameter K-Mittag-Leffler function*, International Mathematical Forum, **8**(34), 1691-1702.

[3] Gehlot, K.S. (2014). *Reccurence relations of Multiparameter K-Mittag-Leffler function*, Malaya J. Mat. **2**(2), 165-171.

[4] Gehlot, K.S. (2017). *Two Parameter Gamma Function and its Properties*, arXiv:1701.0105v1[math.CA], 1-7.

[5] Gehlot, K.S. and Ram, C. (2012). *Integral Representation of K-Series*, International Transactions in Mathematical Sciences and computers, **4**(2), 387-396.

[6] Konovska, P.J. (2012). *Three-Multi-Index Mittag-Leffler Functions, Series and Convergence Theorem*, Proceeding of FDA'2012 symposium, Nanjing, China, Paper No. **284**, FDA 2012.

[7] Luchko, Yu. F. and Yakubovich, S.B., *An operational method for solving some class of introdifferential equations*, (Russian). Differ. Uravn., 30, 269-280; translation in Differential Equation, **30**, 247-256.

[8] Mittag-Leffler, G.M. (1903). *Sur la nouvellefonction* $E_\alpha(z)$, C.R. Acad. Sci. Paris, **137**, 554-558.

[9] Prabhakar, T.R. (1971). *A singular integral equation with a generalized Mittag-Leffler function in the kernal*, Yokohama Math. J. **19**, 7-15.

[10] Sharma, M., Jain, Renu (2009). *A note on generalized M-Series as a special function of fractional calculus*, Fractional Calculus and applied Analysis, **12**(4), 449-452.

[11] Virginia, S.K. (2000). *Multiple (multiondex) Mittag-Leffler functions and relations to generalized fractional calculus*, Journal of Computational and applied Mathematics, **118**, 241-259.

[12] Wiman, A. (1905). *Uber de fundamental satz in der theorie der funktionen* $E_\alpha(x)$, Acta Math. **29**, 191-201.