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COEFFICIENT INEQUALITY FOR A COMBINED SUBCLASS OF VARIOUS CLASSES OF REGULAR FUNCTIONS

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Abstract. Here, we take functions of the type $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and solve the Fekete – Szegö inequality for a new class of analytic functions.

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1. Introduction

In this paper we define an inequality called Fekete – Szegö Inequality for a new class of analytic functions. This is an inequality which relates to those coefficients which are related to univalent analytic functions [8],[16]. M. Fekete and G. Szego proved this inequality in 1933[5]. It originates from bieberbach conjecture([6], [13], [14], [15]), which was given by Bieberbach [2] in 1916 but finally proved by him [3] in 1985.

Firstly, let us discuss some classes and some basic results :

Let A be the family of functions f of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, having conditions of normalisation $f(0) = 0, f'(0) = 1$; analytic in open unit disc $E = \{ z \in C : |z| < 1\}$.

Let S be the family of functions f univalent in the open disk $\{ z \in C : |z| < 1\}$ with conditions

$f(0) = 0, f'(0) = 1; f(z) = z + \sum_{k=2}^{\infty} a_k z^k$.

Any function f belonging to the class A is said to be a Starlike function if $f(E)$ is starlike domain with respect to the origin and this class is denoted by S^* [1]. The essential condition for this class as given by Duren [4], is $Re\left(\frac{zf'(z)}{f(z)}\right) > 0; z \in E$, and $S^*(\phi)$ be the class of functions in $f \in S$, for which $\frac{zf'(z)}{f(z)} \prec \phi(z)$, given by Ma and Minda [10].

A convex function $f \in A$ is a conformal mapping which maps the unit disk E onto a convex domain. The subclass of S consisting of all univalent convex functions is class K. The essential condition for any function to be in class K is $Re\left(1+\frac{zf''(z)}{f'(z)}\right) > 0; z \in E$ as given by Nehari [12].

Let $w(z)$ be any analytic function in E of the form $w(z)= \sum_{n=1}^{\infty} c_n z^n$, then it is said to be Schwarzian function and the class is denoted by U, if the conditions $w(0) = 0$ and $|w(z)| < 1$ hold. These conditions, which were given by Miller et. al. [11], are $|c_1| \leq 1$, $|c_2| \leq 1-|c_1|^2$

Let $u(z)$ and $v(z)$ are two analytic functions in E. If there exists a Schwarzian function $F(z)$ (analytic in E) in such a way that $|F(z)| < 1$, $F(0) = 0$ and $u(z) = v(F(z))$; $z \in E$ then the function $u(z)$ is subordinate to $v(z)$ written as $u(z) \prec v(z)$ and this concept (called subordination) was given by Lindelof [9].

We introduce a new class $TS^*[p, \alpha]$ of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$; defined as

$$(1- p - q)\, f'(z)+p\frac{zf'(z)}{f(z)} + q\frac{\{zf'(z)\}'}{\{f(z)\}'} = \frac{1+(t+2)w(z)}{1+tw(z)};\ z \in E. \tag{1}$$

2. Main Results

THEOREM 1: Let $f(z) \in TK[p,q,t]$ and $\phi(z) = \frac{1+(t+2)w(z)}{1+tw(z)}$; $w(z)$ is a Schwarzian function, then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{[4(p+4q)-(2t)(2-p)^2]}{(2-p)^2(3-p+3q)} - \frac{4\mu}{(2-p)^2}\ ;\ \mu \leq \frac{[4(p+4q)-(2t+2)(2-p)^2]}{4(3-p+3q)}; \\ \frac{2}{(3-p+3q)}\ ; \frac{[4(p+4q)-(2t+2)(2-p)^2]}{4(3-p+3q)} \leq \mu \leq \frac{[4(p+4q)+(2-2t)(2-p)^2]}{4(3-p+3q)}; \\ \frac{4\mu}{(2-p)^2} - \frac{[4(p+4q)-2A(2-p)^2]}{(2-p)^2(3-p+3q)}\ ;\ \mu \geq \frac{[4(p+4q)+(2-2t)(2-p)^2]}{4(3-p+3q)}. \end{cases}$$

PROOF : By definition of $TK[p,q,t]$, given by (1)

and using $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \ldots$

$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$

$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \ldots$

$f''(z) = 2a_2 + 6a_3 z + 12a_4 z^2 + \ldots$

we get

$$1+ [2- p]\, a_2 z + [\{3-p+3q\}a_3 - (4q+p)a_2^2]z^2 + \ldots$$
$$= 1 + 2c_1 z + 2(c_2 - tc_1^2)z^2 + \ldots$$

Comparing like coefficients, one can easily obtain

$$a_2 = \frac{2c_1}{2-p} \text{ and } a_3 = \frac{2(2-p)^2 c_2 - [2t(2-p)^2 - 4(p+4q)]\, c_1^2}{(2-p)^2(3-p+3q)}$$

Using these values of a_2 and a_3, one can construct

$$a_3-\mu a_2^2=\frac{2c_2}{(3-p+3q)}+\left(\frac{[4(p+4q)-2t(2-p)^2]}{(2-p)^2(3-p+3q)}-\frac{4\mu}{(2-p)^2}\right)c_1^2$$

After applying mode on both sides, we get

$$\left|a_3-\mu a_2^2\right|\le\left(\frac{2}{(3-p+3q)}\right)|c_2|+\left|\frac{[4(p+4q)-2t(2-p)^2]}{(2-p)^2(3-p+3q)}-\frac{4\mu}{(2-p)^2}\right||c_1|^2$$

Using $|c_2|\le 1-|c_1|^2$, we get

$$\left|a_3-\mu a_2^2\right|\le$$

$$\frac{2}{(3-p+3q)}+\left\{\left|\frac{[4(p+4q)-2t(2-p)^2]}{(2-p)^2(3-p+3q)}-\frac{4\mu}{(2-p)^2}\right|-\frac{2}{(3-p+3q)}\right\}|c_1|^2$$

Case 1 : If $\mu\le\frac{[2(p+4q)-t(2-p)^2]}{2(3-p+3q)}$

Then, $\left|a_3-\mu a_2^2\right|\le\frac{2}{(3-p+3q)}+\left\{\frac{[4(p+4q)-(2t+2)(2-p)^2]}{(2-p)^2(3-p+3q)}-\frac{4\mu}{(2-p)^2}\right\}|c_1|^2$

Subcase – 1 (a) : When $\mu\le\frac{[4(p+4q)-(2t+2)(2-p)^2]}{4(3-p+3q)}$

By using $|c_1|\le 1$, we get

$$\left|a_3-\mu a_2^2\right|\le\frac{[4(p+4q)-(2t)(2-p)^2]}{(2-p)^2(3-p+3q)}-\frac{4\mu}{(2-p)^2}\qquad(2)$$

Subcase – 1 (b) : When $\mu\ge\frac{[4(p+4q)-(2t+2)(2-p)^2]}{4(3-p+3q)}$

Then, $\left|a_3-\mu a_2^2\right|\le\frac{2}{(3-p+3q)}$ (3)

Case – 2 : If $\mu\ge\frac{[2(p+4q)-t(2-p)^2]}{2(3-p+3q)}$

Then, $\left|a_3-\mu a_2^2\right|\le\frac{2}{(3-p+3q)}+\left\{\frac{4\mu}{(2-p)^2}-\frac{[4(p+4q)+(2-2t)(2-p)^2]}{(2-p)^2(3-p+3q)}\right\}|c_1|^2$

Subcase – 2 (a) : When $\mu\ge\frac{[4(p+4q)+(2-2t)(2-p)^2]}{4(3-p+3q)}$

Then, $\left|a_3-\mu a_2^2\right|\le\frac{4\mu}{(2-p)^2}-\frac{[4(p+4q)-2A(2-p)^2]}{(2-p)^2(3-p+3q)}$ (4)

Subcase – 2 (b) : When $\mu\le\frac{[4(p+4q)+(2-2t)(2-p)^2]}{4(3-p+3q)}$

Then, $\left|a_3-\mu a_2^2\right|\le\frac{2}{(3-p+3q)}$ (5)

Combining (2), (3), (4) and (5), we get the required result.

Corollary 2 : $TS^*[p,q,t] = TS^*[0,1,-1]$, as by substituting p = 0, q = 1 and t = -1, the result becomes

$$|a_3 - \mu\, a_2^2| \leq \begin{cases} 1 - 4\mu;\ \mu \leq \frac{2}{3}; \\ \frac{1}{3};\ \frac{2}{3} \leq \mu \leq \frac{4}{3}; \\ 4\mu - 1;\ \mu \geq \frac{4}{3}\ . \end{cases}$$

which is the required result for any function to be convex function, given by Keogh and Merkes [7].

Corollary 3 : $TS^*[p,q,t] = TS^*[0,0,-1]$, as by substituting p = 0, q = 0 and t = -1, the result becomes

$$|a_3 - \mu\, a_2^2| \leq \begin{cases} \frac{2}{3} - \mu;\ \mu \leq 0; \\ \frac{2}{3};\ 0 \leq \mu \leq \frac{2}{3}; \\ \mu - \frac{2}{3};\ \mu \geq \frac{2}{3}\ . \end{cases}$$

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