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BILINEAR AND BILATERAL GENERATING RELATIONS OF GENERALIZED HYPERGEOMETRIC POLYNOMIALS

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Abstract: The present paper mainly concerns with three theorems involving generating functions expressed in terms of single and double Laplace and Beta Integrals. These theorems have been applied to obtain Bilinear and Bilateral Generating functions involving polynomials of generalized Rice, Jacobi, Lagrange, Gegenbauer and other polynomials hypergeometric in nature. A number of generating functions have also been obtained as special cases. One variable special cases of hypergeometric polynomials are important in several applied problems.

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Key words: Bilinear and Bilateral Generating Functions; Eulerian integrals of first and second kind; Hankel's contour integral; Kampe' de Fe'riet's double hypergeometric function $F^{(2)}$; Srivastava's triple hypergeometric function $F^{(3)}$ and Orthogonal Polynomials.

1. Introduction

Hypergeometric polynomials occupy the pride place in the literature on special functions. One variable special functions namely generalized Rice polynomials[15], Jacobi polynomials[14], Generalized Laguerre polynomials[1], Legendre polynomials, Gegenbauer polynomials and other polynomials hypergeometric in nature, are closely associated with problems of applied in nature. For example, Ultraspherical polynomials are deeply connected with axially symmetric potential in n dimensions and contain Legendre and Chebyshev polynomials as special cases. The generalized Laguerre polynomials play an important role in finding the wave function associated with the electron in a hydrogen atom. Further Laguerre polynomials are encountered in the solution of the problems on propagation of electromagnetic waves and in the analysis of the motion of electrons on Coulomb field, as well as in certain other problems of theoretical physics.

Further, Bessel functions are closely associated with the problem possessing circular or cylindrical symmetry. For example, they arise in the study of free vibration of a circular membrane and in finding the temperature distribution in a circular cylinder. They also occur electromagnetic field theory and numerous other areas of physics and engineering.

This paper aims at making applications of three theorems given by Chaudhary [5] in obtaining Bilinear and Bilateral Generating Functions for a variety of hypergeometric polynomials [12]. The theorems used in our work, are as follows:

Theorem 1. Let $F(x,t)$ be a function having formal power series expansion in t, given by

$$F(x,t) = \sum_{n=0}^{\infty} c_n \, f_n\,(x) t^n \tag{1}$$

where $\{c_n\}$ is a specified sequence of parameters, independent of x and t, and $f_n(x); n = 0,1,2,3, \ldots$ are polynomials of degree n in x; with restrictions on x_1, x_2, x_3 and t such that triple hypergeometric series of Srivastava and $F\left(x, \frac{tz}{z-1}\right)$ remain uniformly convergent for $z \in (0,1)$, then

$$\sum_{n=0}^{\infty} \frac{c_n (p)_n}{(1+p-q)_n} \, F^{(3)} \left[\begin{matrix} p+n, (a_A) :: (b_B); (b'_{B'}); (b''_{B''}) : (c_C); (c'_{C'}); (c''_{C''}); \\ q, (d_D) :: (e_E); (e'_{E'}); (e''_{E''}) : (f_F); (f'_{F'}); (f''_{F''}); \end{matrix} x_1, x_2, x_3 \right] f_n(x) t^n$$

$$=$$

$$= \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \int_0^1 z^{p-1} (1 -$$

$$z)^{q-p-1} \, F^{(3)} \left[\begin{matrix} (a_A) :: (b_B); (b'_{B'}); (b''_{B''}) : (c_C); (c'_{C'}); (c''_{C''}); \\ (d_D) :: (e_E); (e'_{E'}); (e''_{E''}) : (f_F); (f'_{F'}); (f''_{F''}); \end{matrix} x_1 z, x_2 z, x_3 z \right] \times$$

$$\times \; F\left(x, \frac{tz}{z-1}\right) \, \mathrm{dz} \tag{2}$$

$$(q-p \neq 0, \pm 1, \pm 2, \pm 3 \ldots, R(q) > R(p) > 0)$$

where $F^{(3)}[x_1, x_2, x_3]$ is Srivastava's triple hypergeometric function[29-34].

Theorem 2. : Let

$$G(x,t) = \sum_{n=0}^{\infty} c_n f_n\,(x) t^n \tag{3}$$

where $f_n(x)$ are the polynomials of degree n in x, then

$$\sum_{n=0}^{\infty} c_n (\lambda)_n \;\; F_2[\lambda + n; a, c; d, b; z, y] \, f_n(x) \; t^n =$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^{\infty} e^{-p} p^{\lambda - 1} \, {}_1F_1\,[c; b; yp] \; {}_1F_1[a; d; zp] \; G(x, tp) \; dp \; ; \; (\, R(\,\lambda\,) > 0\,) \tag{4}$$

where F_2 is Appell's function of second kind [34].

Theorem 3. : Let

$$G(x,t) = \sum_{n=0}^{\infty} c_n f_n\,(x)\,t^n$$

where $f_n(x)$ are polynomials of degree n in x, then

$$\sum_{n=0}^{\infty} c_n (a)_n (b)_n\, F_4\,[a+n, b+n; c, d; y, z]\, f_n(x)\, t^n =$$

$$= \frac{1}{\Gamma(a)\Gamma(b)} \int_0^{\infty}\int_0^{\infty} e^{-(p+q)} p^{a-1} q^{b-1}\, {}_0F_1\,[-; c; ypq]\ {}_0F_1[-; d; zpq]\, G(x, tpq)\, dp\ dq\ (R(a) > 0, R(b) > 0) \tag{5}$$

where F_4 is Appell's function of fourth kind[34].

Corollary : On taking $B = B' = B'' = E = E' = E'' = C'' = F'' = 0$, Theorem 1, with $x_3 = 0$ reduces to:

$$\sum_{n=0}^{\infty} \frac{c_n (p)_n}{(1+p-q)_n}\, F^{(2)} \left[\begin{matrix} p+n, (a_A) : (c_C); ({c'}_{C'}); \\ q, (d_D) : (f_F); ({f'}_{F'}); \end{matrix} x_1, x_2 \right] f_n(x) t^n =$$

$$= \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \int_0^1 z^{p-1} (1-z)^{q-p-1}\, F^{(2)} \left[\begin{matrix} (a_A) : (c_C); ({c'}_{C'}); \\ (d_D) : (f_F); ({f'}_{F'}); \end{matrix} x_1 z, x_2 z \right] F\left(x, \frac{tz}{z-1}\right) dz \tag{6}$$

where

$$F(x,t) = \sum_{n=0}^{\infty} c_n f_n(x) t^n\ , (q-p \neq 0, \pm 1, \pm 2, \pm 3, \dots, (R(q) > R(p)) > 0)$$

and $F^{(2)}$ is Kampe' de Fe'riet's double hypergeometric function [4,8,9, See also 2, p.150, eq. (29)].

Again on adjusting parameters and variables suitably, the results for Lauricella's function

${F_A}^{(3)}, {F_B}^{(3)}, {F_C}^{(3)}, {F_D}^{(3)}$ [16, See also 20, pp.222-223], equations (8)-(11) follow as special cases of Theorem 1.

Further the known results of Brafman [3], Chaudhary [5], Chaudhary et al. [6], Chaundy [7], Manocha [17-18], Manocha and Sharma [19], Sharma and Mittal [27] are obtained as special cases of our findings.

Again on making applications of these theorems, many more known and new generating functions can be obtained by specializing the parameters or variables or sometimes both

It is to be noted that $F^{(3)}$ is a generalization of $F_1\ to\ F_{14}$ [16] series of Lauricella, Kampe' de Fe'riet's double series $F^{(2)}$ [2 eq.(29), See also 4,p.112], $H_A, H_B\ and\ H_C$ of Srivastava [28, See also 30] and F_K of Sharma [26,p.613,eq.(2)].

2. Definitions

Some of the hypergeometric functions and hypergeometric polynomials used in our work are as follows:

The generalized Rice polynomial [13] is defined by:

$$\boldsymbol{H_n}^{(\alpha,\beta)}\,[\nu,\sigma,x] = \frac{(1+\alpha)_n}{n!}\;{}_3F_2[\,-\mathrm{n}, \mathrm{n}+\alpha+\beta+1, \nu; 1+\alpha, \sigma; \mathrm{x}] \tag{7}$$

has Jacobi polynomials $\boldsymbol{P_n}^{(\alpha,\beta)}(x)$ [22], Ultraspherical (or Gegenbauer) polynomials $\boldsymbol{C_n}^{\alpha}(x)$ [22, p.277], Legendre polynomial $\boldsymbol{C_n}^{\frac{1}{2}}(x)$ and the generalized Lagueree polynomials $\boldsymbol{L_n}^{(\alpha)}(x)$ [22,p.200] as special cases defined below.

$$\boldsymbol{P}_n{}^{(\alpha,\beta)}(x) = H_n{}^{(\alpha,\beta)}\left[\sigma,\sigma,\frac{1-x}{2}\right] = (-1)^n \boldsymbol{P}_n{}^{(\beta,\alpha)}(-x) \tag{8}$$

$$\boldsymbol{P_n}^{(\boldsymbol{\alpha},\boldsymbol{\alpha})}(x) = \frac{(1+\alpha)_n}{(1+2\alpha)_n}\,\boldsymbol{C_n}^{\alpha+\frac{1}{2}}(x) \;=\; \frac{(1+\alpha)_n}{n!}\;{}_2F_1\left[\begin{matrix}-n,1+2\alpha+n; \\ 1+\alpha;\end{matrix}\,\frac{1-x}{2}\right] \tag{9}$$

$$\boldsymbol{L_n}^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_1F_1\left[\begin{matrix}-n; \\ 1+\alpha;\end{matrix}\,x\right] = \lim_{|\beta|\to\infty}\left\{\boldsymbol{P_n}^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right)\right\} \tag{10}$$

$$\boldsymbol{P_n}(x) = \boldsymbol{P_n}^{(0,0)}(x) = \boldsymbol{C_n}^{\frac{1}{2}}(\mathrm{x}) = (-1)^n\,\boldsymbol{P_n}(-x) \tag{11}$$

$$\boldsymbol{P_n}^{(\alpha-n,\beta-n)}(x) \;=\; \frac{1}{n!}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n)}\left(\frac{1+x}{2}\right)^n {}_2F_1\left[\begin{matrix}-n,-\beta; \\ 1+\alpha-n;\end{matrix}\,\frac{x-1}{x+1}\right]$$

$$= \binom{\alpha+\beta}{n}\left(\frac{x-1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n,-\alpha; \\ -\alpha-\beta;\end{matrix}\,\frac{2}{1-x}\right] \tag{12}$$

Further szegö [35, p.64] defined Jacobi polynomials $\boldsymbol{P_n}^{(\alpha,\beta)}(x)$ as follows:

$$\boldsymbol{P_n}^{(\alpha,\beta)}(x) \;=\left(\frac{1-x}{2}\right)^n \boldsymbol{P_n}^{(-\alpha-\beta-2n-1,\beta)}\left(\frac{x+3}{x-1}\right) \tag{13}$$

and

$$\boldsymbol{P_n}^{(\alpha-n,\beta-n)}(x) \;=\left(\frac{1-x}{2}\right)^n \boldsymbol{P_n}^{(-\alpha-\beta-1,\beta-n)}\left(\frac{x+3}{x-1}\right) \tag{14}$$

where in each of the equations $(7)-(14), Re(\alpha) > -1, Re(\beta) > -1$ and n is a non-negative integer.

The Lagrange's polynomial $g_n{}^{(\alpha,\beta)}(x,y)$ [10, p. 267] is defined as :

$$g_n{}^{(\alpha,\beta)}(x,y) = \sum_{r=0}^{n}\frac{(\alpha)_r(\beta)_{n-r}}{r!(n-r)!}\,x^r y^{n-r} \;=\; \frac{(\beta)_n y^n}{n!}\;\;{}_2F_1\left[\begin{matrix}-n,\alpha; \\ 1-\beta-n;\end{matrix}\,\frac{x}{y}\right] \tag{15}$$

$$=\; g_n{}^{(\beta,\alpha)}(y,x).$$

The Chebyshev polynomials [34, p. 125, eqns. (4) and (5)] of first and second kind denoted by $\boldsymbol{T_n}(x)$ and $\boldsymbol{U_n}(x)$, are defined as:

$$T_n(x) = \binom{n-\frac{1}{2}}{n}^{-1} \boldsymbol{P_n}^{\left(\frac{-1}{2},\frac{-1}{2}\right)}(x) \tag{16}$$

and

$$U_n(x) = \frac{1}{2}\binom{n+\frac{1}{2}}{n+1}^{-1} \boldsymbol{P_n}^{\left(\frac{1}{2},\frac{1}{2}\right)}(x). \tag{17}$$

The triple hypergeometric functions F_E *and* F_G [34, eqns. (26) and (28)] in the notation of Saran[23-25], indicating also the numbering of Lauricella [16] on the left, are given as follows:

$$F_4\colon F_E[a,a,a,b,c,c;d,e,f;x,y,z] = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}(b)_m(c)_{n+p}}{(d)_m(e)_n(f)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}; \tag{18}$$

where $(|x| < r, |y| < s, |z| < t)$ such that $r + \left(s^{\frac{1}{2}} + t^{\frac{1}{2}}\right)^2 = 1$

an

$$F_8\colon F_G[a,a,a;b,c,d;e,f,f;x,y,z] = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}(b)_m(c)_n(d)_p}{(e)_m(f)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}; \tag{19}$$

where $(|x| < r, |y| < s, |z| < t)$ such that $r + s = 1 = r + t$.

Further for the definitions of following hypergeometric functions and hypergeometric polynomials used in our subsequent work, we refer to the monographs of Erdelyi et al. [10], Exton [11], Rainville [22],Shrivastava and Manocha [34] and Szego [35].

i. Kampe$'$ de Fe$'$riet's double hypergeometric function $F^{(2)}$[x,y]

ii. Srivastava's triple hypergeometric function $F^{(3)}[x_1,x_2,x_3]$

iii. Lauricella's triple hypergeometric functions $F_A{}^{(3)}, F_B{}^{(3)}, F_C{}^{(3)}$ *and* $F_D{}^{(3)}$

iv. Appell's functions F_1, F_2, F_3 *and* F_4

v. Generalized hypergeometric function of one variable

3. Applications

Case (I) Consider the generating relation [34, eq.(34), See also 15,p.1206,eq.(4.2)]

$$\sum_{n=0}^{\infty} \frac{[(g_G)]_n}{(-\alpha-\beta)_n[(h_H)]_n} H_n{}^{(\alpha-n,\beta-n)}[\nu,\sigma,x]\, t^n = F^{(2)}\left[\begin{smallmatrix}(g_G)\,:\,\nu;-\alpha;\\(h_H)\,:\,\sigma;-\alpha-\beta;\end{smallmatrix}\, xt\,,-t\right], \tag{20}$$

where $H_n{}^{(\alpha,\beta)}\,[\nu,\sigma,x]$ are generalized Rice's polynomials, defined by (7).

In (1.1), we take $F(x,t) = F^{(2)}\left[\begin{smallmatrix}(g_G):\,\nu;-\alpha;\\(h_H):\,\sigma;-\alpha-\beta;\end{smallmatrix}\, xt,-t\right]$, combining (20) with (1) then application of the Corollary of Theorem 1, would give us

$$\sum_{n=0}^{\infty}\frac{(p)_n}{(1+p-q)_n(-\alpha-\beta)_n}\frac{[(g_G)]_n}{[(h_H)]_n}\,F^{(2)}\left[\begin{matrix}p+n,(a_A):(c_C);(c'_{C'});\\ q,(d_D):(f_F);(f'_{F'});\end{matrix}x_1;x_2\right]H_n{}^{(\alpha-n,\beta-n)}[\nu,\sigma,x]\,t^n$$

$$=$$

$$=\sum_{u,v,r,s=0}^{\infty}\frac{(p)_{r+s+u+v}}{(1+p-q)_{r+s}(q)_{u+v}}\,\frac{[(a_A)]_{u+v}[(c_C)]_u[(c'_{C'})]_v[(g_G)]_{r+s}}{[(d_D)]_{u+v}[(f_F)]_u[(f'_{F'})]_v[(h_H)]_{r+s}}\times$$

$$\times\frac{(\nu)_r(-\alpha)_s}{(\sigma)_r(-\alpha-\beta)_s}\,\frac{x_1{}^u}{u!}\frac{x_2{}^v}{v!}\frac{(xt)^r}{r!}\frac{(-t)^s}{s!} \tag{20a}$$

which on taking $H=0\ and\ G=1$ such that replacing $g_1\ by\ 1+p-q$, reduces to :

$$\sum_{n=0}^{\infty}\frac{(p)_n}{(-\alpha-\beta)_n}\,F^{(2)}\left[\begin{matrix}p+n,(a_A):(c_C);(c'_{C'});\\ q,(d_D):(f_F);(f'_{F'});\end{matrix}x_1,x_2\right]H_n{}^{(\alpha-n,\beta-n)}[\nu,\sigma,x]\,t^n ==$$

$$\sum_{r=0}^{\infty}\frac{(p)_r(\nu)_r}{(\sigma)_r}\frac{(xt)^r}{r!}\,F^{(3)}\left[\begin{matrix}p+r::-;(a_A);-:-\alpha;(c_C);(c'_{C'});\\ -::-;q,(d_D);-:-\alpha-\beta;(f_F);(f'_{F'});\end{matrix}-t,x_1,x_2\right]. \tag{21}$$

Again on putting $A=C=C'=F=F'=1, D=0$ and then replacing

$a_1\ by\ q, \nu\ by\ \sigma\ , x\ by\ \frac{1-x}{2}, c_1\ by\ \lambda_1\ , c_1'\ by\ \lambda_2\ , f_1\ by\ \mu_1\ , f_1'\ by\ \mu_2, x_1\ by\ y, and\ x_2\ by$, equation (21) in the light of result (18) and the binomial theorem reduces to a generating relation for special Jacobi polynomials in the form:

$$\sum_{n=0}^{\infty}\frac{(p)_n}{(-\alpha-\beta)_n}F_2\,[p+n,\lambda_1,\lambda_2{:}\,\mu_1,\mu_2;y,z]\,\boldsymbol{P_n}{}^{(\alpha-n,\beta-n)}(x)t^n=\left(1-\left(\frac{1-x}{2}\right)t\right)^{-p}\times$$

$$\times F_A^{(3)}\left[p;\lambda_1,\lambda_2,-\alpha;\mu_1,\mu_2,-\alpha-\beta;\frac{2y}{2-(1-x)t},\frac{2z}{2-(1-x)t},\frac{-2t}{2-(1-x)t}\right], \tag{22}$$

which is a known result of Manocha [17].

Similarly on taking $C=C'=D=0, F=F'=1, A=2$ and letting one of the A's parameter be q and other be λ, equation (3.2) reduces to a known result of Sharma and Mittal [27] in the form:

$$\sum_{n=0}^{\infty}\frac{(p)_n}{(-\alpha-\beta)_n}F_4\,[p+n,,\lambda;\mu_1,\mu_2;y,z]\,\boldsymbol{P_n}{}^{(\alpha-n,\beta-n)}(x)\,t^n$$

$$=(v)^{-p}F_E\left[p,p,p,-\alpha,\lambda,\lambda;-\alpha-\beta,\mu_1,\mu_2\,;\frac{-t}{v},\frac{y}{v},\frac{z}{v}\right], \tag{23}$$

where, for convenience, we put

$$v=\left(1-\tfrac{1}{2}(1-x)t\right).$$

Since on replacing $\frac{1}{2}(x+1)$ by – X, α by – α, β by – β, the result [30-34] in view of the result [22] reduces to a result involving Lagrange polynomial $g_n{}^{(\alpha,\beta)}(x,y)$, the above replacement in (22) offers us

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(\alpha+\beta)_n} F_2\,[p+n;\lambda_1,\lambda_2{:}\,\mu_1,\mu_2;y,z]\;\; g_n{}^{(\alpha,\beta)}(X,X+1)t^n$$

$$= (\omega)^{-p} F_A^{(3)}\left[p;\lambda_1,\lambda_2,\alpha;\mu_1,\mu_2,\alpha+\beta;\frac{y}{\omega},\;\frac{z}{\omega},\frac{-t}{\omega}\right], \tag{24}$$

where $\omega = 1-(1+X)\,t$.

Further, equation $(20a)$, on taking $x_1 = y\ and\ x_2 = 0$ gives:

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(-\alpha-\beta)_n(1+p-q)_n}\frac{[(g_G)]_n}{[(h_H)]_n}\,{}_{1+A+C}F_{1+D+F}\left[\begin{matrix}p+n,(a_A),(c_C);\\ q,(d_D),(f_F);\end{matrix}\,y\right] H_n{}^{(\alpha-n,\beta-n)}\,[\nu,\sigma,x]\,t^n$$
$$=$$

$$= F^{(3)}\left[\begin{matrix}p\ ::-\,;(g_G)\,;-:(a_A)\,,(c_C)\,;\nu\,;-\alpha\,;\\ -::-\,;1+p-q\,,(\square_H)\,;-:q,(d_D)\,,(f_F)\,;\sigma;-\alpha-\beta\,;\end{matrix}\,y,xt,-t\right] \tag{25}$$

which on putting $C=D=F=G=H=0,\ A=1$ such that on replacing $a_1 by\,\gamma$, the equation (25) in view of result (14) reduces to a generating relation for special Jacobi polynomials:

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(-\alpha-\beta)_n(1+p-q)_n}\,\frac{[(g_G)]_n}{[(h_H)]_n}\,{}_2F_1\left[\begin{matrix}p+n,\gamma;\\ q;\end{matrix}\,y\right]\boldsymbol{P_n}{}^{(\alpha-n,\beta-n)}(x)\,t^n =$$

$$= F^{(3)}\left[\begin{matrix}p::-;\ (g_G);\,-{:}\gamma;\,\sigma;-\alpha;\\ -::-;1+p-q;(h_H);-{:}q;\sigma;-\alpha-\beta\,;\end{matrix}\ y,\frac{(1-x)}{2}t,-t\right]. \tag{26}$$

Again, equation (26) in the light of the result (14) gives an elegant bilateral generating relation:

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(-\alpha-\beta)_n(1+p-q)_n}\left(\frac{1-x}{2}\right)^n\frac{[(g_G)]_n}{[(h_H)]_n}\;{}_2F_1\left[\begin{matrix}p+n,\gamma;\\ q;\end{matrix}\,y\right]\boldsymbol{P_n}{}^{(-\alpha-\beta-1,\beta-n)}\left(\frac{x+3}{x-1}\right)t^n$$

$$= F^{(3)}\left[\begin{matrix}p::-;(g_G);-\,{:}\gamma;-;-\alpha;\\ -::-;1+p-q,(h_H);-{:}\,q;-;-\alpha-\beta;\end{matrix}\ y,\frac{(1-x)}{2}t,-t\right]. \tag{27}$$

Further by means of the relations (9) to (11) equation (26) reduces to generating functions involving Ultraspherical, Legendre and generalized Laguerre polynomials respectively.

Case (II) Consider the generating relation [34, p.145, eq. (31)]:

$$\sum_{n=0}^{\infty} \frac{[(g_G)]_n}{(\alpha+1)_n(\beta+1)_n[(h_H)]_n}\,\boldsymbol{P_n}{}^{(\alpha,\beta)}(x)\,t^n$$

$$= F^{(2)}\left[\begin{matrix}(g_G)\,{:}-;-;\\ (h_H){:}\ \alpha+1;\beta+1;\end{matrix}\ \frac{1}{2}(x-1)t,\frac{1}{2}(x+1)t\right], \tag{28}$$

where $P_n{}^{(\alpha,\beta)}(x)$ are a Jacobi polynomials, defined by equation (8).

In (3), we take $F(x,t) = F^{(2)}\left[\begin{smallmatrix} (g_G):-;-; \\ (h_H):\alpha+1;\beta+1; \end{smallmatrix} \frac{1}{2}(x-1)t,\ \frac{1}{2}(x+1)t\right]$, combining (28) with (3) in Theorem 2 and then taking $y = H = O, G = 2$ such that on replacing g_1 by γ and g_2 by δ, we get a bilateral generating relation:

$$\sum_{n=0}^{\infty} \frac{(b)_n(\gamma)_n(\delta)_n}{(\alpha+1)_n(\beta+1)_n}\ {}_2F_1\left[\begin{smallmatrix} b+n,a; \\ d; \end{smallmatrix} z\right] \boldsymbol{P_n}^{(\alpha,\beta)}(x)\ t^n =$$

$$= F^{(3)}\left[\begin{smallmatrix} b\ ::\ -;\ \gamma,\delta;-:\ a;-;-; \\ -\ ::\ -;-;-:\ d;\ \alpha+1;\beta+1; \end{smallmatrix} z, \frac{1}{2}(x-1)t, \frac{1}{2}(x+1)t\right]. \tag{29}$$

Again in (29) putting $z = 0$ and then replacing $by\frac{t}{k}$, multiplying by $e^k k^{-b}$ and evaluating the result obtained with the help of Hankel's contour integral for Gamma function, given in [11,p.32, eq. (1.5.1.5)]:

$$\frac{1}{2\pi i} \oint e^t t^{-a-m} dt = \frac{1}{\Gamma(a+m)} \tag{30}$$

where m is a non-negative integer and a does not take non-positive integer values, we obtain a linear generating relation:

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n(\delta)_n}{(\alpha+1)_n(\beta+1)_n}\ \boldsymbol{P_n}^{(\alpha,\beta)}(x)\ t^n =$$

$$= F_4\left[\gamma,\delta;\alpha+1,\beta+1;\ \frac{1}{2}(x-1)t, \frac{1}{2}(x+1)t\ \right], \tag{31}$$

which is a known result of Brafman [3].

Now in (29) using results (16) and (17) for Shebyshev polynomials of first and second kind respectively, we obtain two more bilateral generating relations:

$$\textstyle\sum_{n=0}^{\infty} \frac{(b)_n(\gamma)_n(\delta)_n}{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n} \binom{n-\frac{1}{2}}{n}\ {}_2F_1\ [b+n,a;d;z]\ \boldsymbol{T_n}(x)\ t^n =$$

$$= F^{(3)}\left[\begin{smallmatrix} b::-;\gamma,\delta;-:a;-;-; \\ -\ ::\ -;\ -;\ -:d;\frac{1}{2};\frac{1}{2}; \end{smallmatrix}\ z, \frac{1}{2}(x-1)t\ , \frac{1}{2}(x+1)t\right], \tag{32}$$

and

$$\sum_{n=0}^{\infty} 2\binom{n+\frac{1}{2}}{n+1} \frac{(b)_n(\gamma)_n(\delta)_n}{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}\ {}_2F_1[b+n,a;d;z]\ \boldsymbol{U_n}(x)t^n$$

$$= F^{(3)}\left[\begin{smallmatrix} b\ ::\ -\ ;\ \gamma\ ,\delta;-:\ a;-;-; \\ -::\ -\ ;\ -\ ;-:\ d\ ;\frac{3}{2};\frac{3}{2}; \end{smallmatrix}\ z, \frac{1}{2}(x-1)t\ , \frac{1}{2}(x+1)t\right]. \tag{33}$$

Further in result (28) the application of Theorem 3, with $G = H = 0$ would give us:

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(\alpha+1)_n (\beta+1)_n} F_4\,[a+n, b+n; c, d; y, z]\, \boldsymbol{P_n}^{(\alpha,\beta)}(x)\, t^n = \\ = F_C{}^{(4)} \left[a, b; c, d, \alpha+1, \beta+1; y, z, \tfrac{1}{2}(x-1)t, \tfrac{1}{2}(x+1)t\right]. \tag{34}$$

which is known result of Chaudhary [5].

Now on taking $a = \lambda, b = \delta, c = 1+\mu$ and $z = 0$, equation (34) gives a known result of Manocha and Sharma [19, p.79, eq.(31)] in the form:

$$\textstyle\sum_{n=0}^{\infty} \frac{(\lambda)_n (\delta)_n}{(\alpha+1)_n (\beta+1)_n}\ {}_2F_1[\lambda+n, \delta+n; 1+\mu; y]\, \boldsymbol{P_n}^{(\alpha,\beta)}(x)\, t^n = F_C{}^{(3)} \left[\lambda, \delta; 1+\mu, \alpha+1, \beta+1; y, \frac{(x-1)t}{2}, \frac{(x+1)t}{2}\right], \tag{35}$$

which was originally obtained by using fractional derivative technique.

Case (III) Consider the generating relation [21]:

$$\textstyle\sum_{n=0}^{\infty} \frac{[(g_G)]_n}{[(h_H)]_n} \boldsymbol{P_n}^{(\alpha-n,\beta-n)}(x)\ t^n = F^{(2)} \left[{}^{(g_G):\,-\alpha\,;\,-\beta;}_{(h_H):\,-\,;\,-\,;} - \tfrac{1}{2}(x+1)t, -\tfrac{1}{2}(x-1)t\right], \tag{36}$$

where $\boldsymbol{P_n}^{(\alpha-n,\beta-n)}(x)$ are special Jacobi polynomials defined by the relation (12).

In (5), we take $F(x,t) = F^{(2)} \left[{}^{(g_G):-\alpha\,;\,-\beta;}_{(h_H):\,-\,;\,-\,;} -\tfrac{1}{2}(x+1)t\,, -\tfrac{1}{2}(x-1)t\right]$, combining (36) with (5) in Theorem 3 and then putting $z = 0$, we obtain a bilateral generating relation

$$\sum_{n=0}^{\infty} (a)_n (b)_n \frac{[(g_G)]_n}{[(h_H)]_n}\ {}_2F_1\left[{}^{a+n,b+n;}_{c;} y\right] \boldsymbol{P_n}^{(\alpha-n,\beta-n)}(x)\, t^n = \\ = F^{(3)} \left[{}^{a,b\,::-;\,(g_G)\,;\,-:-;-\alpha\,;\,-\beta\,;}_{-\,::\,-;\,(h_H);-:\,c\,;-\,;\,-\,;} y, -\tfrac{1}{2}(x+1)t, -\tfrac{1}{2}(x-1)t\right]. \tag{37}$$

Now in (37), putting $G = H = 0$, replacing t by $\frac{t}{pq}$ and multiplying by $e^{p+q}(p)^{-\lambda}(q)^{-\mu}$ and then evaluating the result obtained with the help of Hankel's contour integral for Gamma function given by (30), we get

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(\lambda)_n (\mu)_n}\ {}_2F_1\,[a+n, b+n; c; y]\, \boldsymbol{P_n}^{(\alpha-n,\beta-n)}(x)\, t^n = \\ = F^{(3)} \left[{}^{a,b\,::\,-\,;\,-\,;\,-:-\,;\,-\alpha\,;\,-\beta\,;}_{-\,::\,-;\,\lambda\,,\mu\,;-:\,c\,;-\,;\,-\,;} y, -\tfrac{1}{2}(x+1)t, -\tfrac{1}{2}(x-1)t\right]. \tag{38}$$

Again on putting $y = 0$ and then replacing b by μ, in equation (38) we obtain a linear generating relation:

$$\textstyle\sum_{n=0}^{\infty} \frac{(a)_n}{(\lambda)_n} \boldsymbol{P_n}^{(\alpha-n,\beta-n)}(x)\, t^n = F_1\left[a, -\alpha, -\beta; \lambda; -\tfrac{1}{2}(x+1)t\,, -\tfrac{1}{2}(x-1)t\right], \tag{39}$$

which is a known result of Manocha [17] .

Also equation (39) on replacing a by λ and then using binomial theorem gives:

$$\textstyle\sum_{n=0}^{\infty} \boldsymbol{P_n}^{(\alpha-n,\beta-n)}(x)\, t^n = \left[1+\tfrac{1}{2}(x+1)t\right]^{\alpha}\left[1+\tfrac{1}{2}(x-1)t\right]^{\beta}, \qquad (40)$$

$$\left(|t| < \min\left\{\frac{2}{1+x}, \frac{2}{1-x}\right\}\right)$$

which is a known result of Chaundy [7, p.62, eq.(25), See also 34, p.205, eq.(10)].

Further in relation (36), the application of the Corollary of Theorem 1, with $x_1 = y$ and $x_2 = 0$ gives:

$$\sum_{n=0}^{\infty} \frac{[(g_G)]_n}{[(h_H)]_n} \frac{(p)_n}{(1+p-q)_n}\ {}_{1+A+C}F_{1+D+F}\left[\begin{matrix} p+n, & (a_A), & (c_C); \\ q, & (d_D), & (f_F); \end{matrix}\ y\right] \boldsymbol{P_n}^{(\alpha-n,\beta-n)}(x)\, t^n =$$

$$= F^{(3)}\left[\begin{matrix} p :: -; (g_G) & ; -: (a_A), (c_C) & ; -\alpha; -\beta; \\ - :: -: (h_H), 1+p-q & ; -: q, (d_D), (f_F) & ; -; -; \end{matrix}\ y, -\tfrac{1}{2}(x+1)t,\ -\tfrac{1}{2}(x-1)t\right] \qquad (41)$$

Now in equation (41), putting G = H = C = D =F = 0, A = 1 such that letting $a_1 = \gamma$ and then replacing 1+p-q by μ, we obtain bilateral generating function for Saran's function F_G in the form:

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(\mu)_n}\ {}_2F_1\left[\begin{matrix} p+n, & \gamma; \\ q & ; \end{matrix}\ y\right] \boldsymbol{P_n}^{(\alpha-n,\beta-n)}(x)\, t^n =$$

$$= F_G[p,p,p;\ \gamma,-\alpha,-\beta;\ q,\mu,\mu;\ y,-\tfrac{1}{2}(x+1)t,\ -\tfrac{1}{2}(x-1)t] \qquad (42)$$

which is known result of Chaudhary et al. [6, p.48, eq. (3.12)].

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References

[1] Al-Salam, W.A. (1964). Operational representations for the Laguerre and other polynomials Duke Math. J, **31**,127-142.

[2] Appell, P. and Kampé de Fériet, J. (1926). Fonctions Hypergéométriques et Hypersphériques: Polynômers d' Hermite, Gauthier–Villars, Paris.

[3] Brafman, F. (1951). Generating functions of Jacobi and related polynomials, Proc. Amer. Math. Soc. **2**, 942-949.

[4] Burchnall, J.L. and Chaundy, T.W. (1941). Expansions of Appell's double hypergeometric functions-II, Quart. J. Math. (Oxford) Ser. **12**, 112-128.

[5] Chaudhary, Wali Mohd. (1997). Bilinear and bilateral generating functions of generalized Polynomials, J. Austral Math. Soc. Ser. B**39**, 257-270.

[6] Chaudhary, Wali Mohd., Qureshi, M.I., Quraishi, K.A. and Ram Pal. (2014). Some bilinear and bilateral hypergeometric generating relations, J. of Ramanujan Society of Math. and Math. Sc., **3**(2), 39-54.

[7] Chaundy, T.W. (1943). An extension of hypergeometric functions (I), Quart. J. Math. Oxford Ser. **14**, 55-78.

[8] Chhabra, S.P. and Rusia, K.C. (1980). A transformation formula for general hypergeometric functions of three variables, Jñãnãbha **9/10**, 155-159.

[9] Deshpande, V.L. (1981). Certain formulas associated with hypergeometric functions of three variables, Pure and Applied Mathematika Sciences. **14**,39-45.

[10] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G. (1953). Higher Transcen- dental Functions, Vol. I and II. Mc Graw- Hill, Book Co. Inc., New York, Toronto and London.

[11] Exton, H. (1976). Multiple Hypergeometric Functions and Applications, John Wiley and Sons (Halsted Press), New York; Ellis Horwood Ltd, Chichester, UK.

[12] Jain, R.N. (1966). The confluent hypergeometric functions of three variables, Pro.Nat.Acad. Sci. India Sect. A **36**, 395-408.

[13] Khandekar, P.R. (1964). On a generalization of Rice's polynomial I, Proc. Nat. Acad.,Sci., India, Sect. A. **34**, 157-162.

[14] Khan, I.A., (1972). Generating functions for Jacobi and related polynomials, Proc. Amer. Math. Soc.**32**, 179-186.

[15] Khan, I.A. (1972). On generalized Rice's polynomials, Indian J. Pure Appl. Math. **3**, 1203-1215.

[16] Lauricella, G. (1893). Sulle funzioni ipergeometriche a Più Variabili, Rend. Circ. Mat. Palermo**.7**, 111-158.

[17] Manocha, H.L. (1967). Some bilinear generating functions for Jacobi polynomials, Proc. Camb. Philos. Soc, **63**, 457-459.

[18] Manocha, H.L. (1968). Some formulae for generalized Rice polynomials, Proc. Camb. Philos. Soc., **64**, 431-434.

[19] Manocha, H.L. and Sharma, B.L. (1966). Infinite series of hypergeometric functions, Ann. Soc. Sci. Bruxelles Sér. I**80**, 73-86.

[20] Mathur, B.L. (1978). On some results involving Lauricella functions, Bull. Calcutta. Math. Soc. **70**, 221-227.

[21] Munot, P.C., Mathur, B.L. and Kushwaha, R.S. (1975). On generating functions for classical polynomials, Proc. Nat. Acad. Sci. India, Sect. A, **45**, 187-192.

[22] Rainville, E.D. (1960). Special Functions, The Macmillen Co. Inc., New York, reprinted by Chelsea Publ. Co., Bronx, New York.(1971).

[23] Saran, S. (1954). Hypergeometric functions of three variables, Ganita**, 5**(2), 71-91, Corrigendum. Ibid. **7** (1956), 65.

[24] Saran, S. (1970). A general theorem for bilinear generating functions, Pacific, J. Math. **35**, 783-786.

[25] Saran, S. (1972). Theorems on bilinear generating functions, Indian J. Pure Appl.Math., **3**(1), 12-20.

[26] Sharma, B.L. (1970). Some theorems for Appell's functions, Proc. Camb. Philos. Soc. **67**, 613 -618.

[27] Sharma, B.L. and Mittal, K. C. (1968). Some new generating functions for Jacobi polynomials, Proc. Camb. Philos. Soc. **64**, 691-694.

[28] Srivastava, H.M. (1964). Hypergeometric functions of three variables, Ganita, **15** (2), 97-108.

[29] Srivastava, H.M. (1967). Generalized Neumann expansions involving hypergeometric functions, Proc. Camb. Philos. Soc., **63**, 425-429.

[30] Srivastava, H.M. (1967). Some integrals representing triple hypergeometric functions, Rend. Circ. Mat. Palermo(2), **16**, 1-3, 99-115.

[31] Srivastava, H.M. (1969). On a generating functions for the Jacobi polynomials, J. Math. Sci. **4**, 61-68.

[32] Srivastava, H.M. (1972). Certain formulas associated with generalized Rice polynomials II., Ann. Polon. Math. **27**, 73-83.

[33] Srivastava, H.M. and Daoust, M. C. (1971). Some generating functions for the Jacobi Polynomials, Comment. Math. Univ. St. Paul., **20**, fasc.1, 15-20.

[34] Srivastava, H.M. and Manocha, H.L. (1984). A Treatise on Generating Functions, John Wiley and Sons, (Halsted Press), New York; Ellis Horwood Ltd, Chichester, U.K.

[35] Szegö, G. (1975). Orthogonal Polynomials, Fourth edition, Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc, Providence, Rhode Island.