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ELLIPTIC WP-BAILEY TRANSFORM AND THETA HYPERGEOMETRIC SERIES

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Abstract. In this paper, idea of WP-Bailey transform has been extended to elliptic WP-Bailey transform and it has been applied to establish certain interesting summation and transformation formulas for elliptic and theta hypergeometric series.

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1. Introduction, Notations and Definitions

Bailey in 1947 established a remarkable lemma which has been widely used for obtaining transformation formulas for ordinary hypergeometric series as well as for basic hypergeometric series. Andrews [1] generalized Bailey pair and introduced WP-Bailey pair, WP-Bailey chain and WP-Bailey tree. Making use of WP-Bailey pairs, several Mathematicians attempted to establish new transformations and identities for basic hypergeometric series. The elliptic analogues of hypergeometric series were introduced by Frenkel and Turaev [3] in their study of elliptic 6j-symbols. These 6j-symbols, which correspond to certain elliptic solution of the Yang-Baxter equation found by Baxter [2] can be expressed in terms of elliptic generalizations of terminating, balanced, very-well-poised ${}_{10}\phi_9$ series. Warnaar [12] introduced the elliptic analogue of the WP-Bailey transform and used it to establish new transformation formulas for theta and elliptic hypergeometric series. Such type of works have been done in [1][5][6][7][8] and [9]. In this paper, using elliptic WP-Bailey transform, we have established more transformation and summation formulas for elliptic and theta hypergeometric series.

A modified Jacobi's theta function is defined by [4,(11.2),p.303]

$$\theta(x;p)=(x;p)_\infty\left(\frac{p}{x};p\right)_\infty$$

and $$\theta(x_1,x_2,...,x_m;p)=\theta(x_1;p)\theta(x_2;p)...\theta(x_m;p).$$

where $x_1, x_2, ..., x_m \neq 0.$

An elliptic or theta shifted factorial is defined by,

$$(a;q,p)_n = \begin{cases} 1, & n=0 \\ \theta(a;p)\theta(aq;p)...\theta(aq^{n-1};p), & for \;\; n \geq 1, \end{cases}$$

and $$(a;q,p)_{-n} = \frac{(-1)^n q^{n(n+1)/2}}{a^n \left(\frac{q}{a};q,p\right)_n}.$$

Also, $(a_1, a_2, ..., a_r; q, p)_n = (a_1;q,p)_n (a_2;q,p)_n ... (a_1;q,p)_n$,

where $a_1, a_2, ..., a_r \neq 0.$

Following Spiridonov [10] an elliptic hypergeometric series with base q and nome p is defined by

$${}_{r+1}E_r \left[\begin{matrix} a_1, a_2, ..., a_{r+1}; q, p; z \\ b_1, b_2, ..., b_r \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_{r+1}; q, p)_n z^n}{(q, b_1, b_2, ..., b_r; q, p)_n}$$

and very well-poised theta hypergeometric series is defined by

$${}_{r+1}V_r \left[a_1; a_6, a_7, ..., a_{r+1}; q, p; z \right] = \sum_{n=0}^{\infty} \frac{\theta(a_1 q^{2n}; p)(a_1, a_6, ..., a_{r+1}; q, p)_n (zq)^n}{\theta(a_1; p)\left(q, \frac{a_1 q}{a_6}, \frac{a_1 q}{a_7}, ..., \frac{a_1 q}{a_{r+1}}; q, p\right)_n}, \tag{1}$$

where for convergence reasons we require the ${}_{r+1}V_r$ to terminate. The rationale behind the above labeling of the ${}_{r+1}V_r$ series is that [17]

$$\frac{\theta(a_1 q^{2n}; p)}{\theta(a_1; p)} = \frac{\left(q\sqrt{a_1}, -q\sqrt{a_1}, q\sqrt{\frac{a_1}{p}}, -q\sqrt{a_1 p}; q, p\right)_n (-q)^n}{\left(\sqrt{a_1}, -\sqrt{a_1}, -\sqrt{\frac{a_1}{p}}, \sqrt{a_1 p}; q, p\right)_n}. \tag{2}$$

A ${}_{r+1}V_r$ series is called balanced if $a_6 ... a_{r+1} q = (a_1 q)^{\frac{(r-5)}{2}}$. All known identities for elliptic hypergeometric series are both balanced and very-well-poised.

If the argument z in ${}_{r+1}V_r$ is 1 then we suppress 1 and it is denoted by,

$${}_{r+1}V_r\left[a_1;a_6,a_7,...,a_{r+1};q,p\right]. \tag{3}$$

We shall make use of following summation formula in our analysis,

$${}_{10}V_9\left[a;b,c,d,e,q^{-n};q,p\right]=\frac{\left(aq,\frac{aq}{bc},\frac{aq}{bd},\frac{aq}{cd};q,p\right)_n}{\left(\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\frac{aq}{bcd};q,p\right)_n}, \tag{4}$$

provided that $bcde=a^2q^{n+1}$. [4;(11.2.25),p.307]

Elliptic Extension of WP-Bailey Lemma:

Following Warnaar [12] elliptic extension of WP-Bailey pair is defined as,

A pair of sequences $\langle\alpha_n(a,k;q,p),\beta_n(a,k;q,p)\rangle$ is said to be elliptic WP-Bailey pair if

$$\beta_n(a,k;q,p)=\sum_{r=0}^{n}\frac{\left(\frac{k}{a};q,p\right)_{n-r}(k;q,p)_{n+r}}{(q;q,p)_{n-r}(aq;q,p)_{n+r}}\alpha_r(a,k;q,p). \tag{5}$$

Similarly, a pair of sequences $\langle\gamma_n(a,k;q,p),\delta_n(a,k;q,p)\rangle$ is said to be elliptic conjugate WP-Bailey pair if

$$\gamma_n(a,k;q,p)=\sum_{r=0}^{\infty}\frac{\left(\frac{k}{a};q,p\right)_r(k;q,p)_{2n+r}}{(q;q,p)_r(aq;q,p)_{2n+r}}\delta_{r+n}(a,k;q,p), \tag{6}$$

provided the infinite series is convergent.

Again, following Bailey lemma we have,

If $\langle\alpha_n(a,k;q,p),\beta_n(a,k;q,p)\rangle$ is an elliptic WP-Bailey pair and $\langle\gamma_n(a,k;q,p),\delta_n(a,k;q,p)\rangle$ is corresponding elliptic conjugate WP-Bailey pair then under suitable convergence conditions,

$$\sum_{n=0}^{\infty}\alpha_n(a,k;q,p)\gamma_n(a,k;q,p)=\sum_{n=0}^{\infty}\beta_n(a,k;q,p)\delta_n(a,k;q,p). \tag{7}$$

Warnaar Lemma:

(Elliptic WP Bailey transform) For a and k indeterminates the following two equations are equivalent.

$$\beta_n(a,k;q,p)=\sum_{r=0}^{n}\frac{\left(\frac{k}{a};q,p\right)_{n-r}(k;q,p)_{n+r}}{(q;q,p)_{n-r}(aq;q,p)_{n+r}}\alpha_r(a,k;q,p), \tag{8}$$

$$\alpha_n(a,k;q,p)=\frac{\theta(aq^{2n};p)}{\theta(a;p)}\sum_{r=0}^{n}\frac{\theta(aq^{2r};p)\left(\frac{a}{k};q,p\right)_{n-r}(a;q,p)_{n+r}}{\theta(k;p)(q;q,p)_{n-r}(kq;q,p)_{n+r}}\left(\frac{k}{a}\right)^{n-r}\beta_r(a,k;q,p). \tag{9}$$

Spiridonov's [10] theorems for constructing elliptic WP-Bailey pairs:

Theorem 1: If $\langle\alpha_n(a,k;q,p),\beta_n(a,k;q,p)\rangle$ is an elliptic WP Bailey pair, then so is the pair $\langle\alpha_n'(a,k;q,p),\beta_n'(a,k;q,p)\rangle$ given by,

$$\alpha_n'(a,k;q,p)=\frac{(b,c;q,p)_n}{\left(\frac{aq}{b},\frac{aq}{c};q,p\right)_n}\left(\frac{aq}{bc}\right)^n\alpha_n\left(a,\frac{bck}{aq};q,p\right), \tag{10}$$

$$\beta_n'(a,k;q,p)=\frac{\left(\frac{ck}{a},\frac{bk}{a};q,p\right)_n}{\left(\frac{aq}{b},\frac{aq}{c};q,p\right)_n}\sum_{r=0}^{n}\frac{\theta\left(\frac{bck}{aq}q^{2r};p\right)(b,c;q,p)_r\left(\frac{aq}{bc};q,p\right)_{n-r}}{\theta\left(\frac{bck}{aq};p\right)\left(\frac{ck}{a},\frac{bk}{a};q,p\right)_r(q;q,p)_{n-r}}$$

$$\times\frac{(k;q,p)_{n+r}}{\left(\frac{bck}{a};q,p\right)_{n+r}}\left(\frac{aq}{bc}\right)^r\beta_r\left(a,\frac{bck}{aq};q,p\right). \tag{11}$$

Theorem 2: If $\langle\alpha_n(a,k;q,p),\beta_n(a,k;q,p)\rangle$ is an elliptic WP Bailey pair, then so is the pair $\langle\alpha_n'(a,k;q,p),\beta_n'(a,k;q,p)\rangle$ given by,

$$\alpha_n'(a^2,k;q^2,p^2)=\alpha_n\left(a,\frac{k}{aq};q,p\right), \tag{12}$$

and

$$\beta_n'(a^2,k;q^2,p^2)=\frac{\left(-\frac{k}{a};q,p\right)_{2n}}{(-aq;q,p)_{2n}}\sum_{r=0}^{n}\frac{\theta\left(\frac{k}{aq}q^{2r};p\right)\left(\frac{a^2q^2}{k};q^2,p^2\right)_{n-r}}{\theta\left(\frac{k}{aq};p\right)(q^2;q^2,p^2)_{n-r}}$$

$$\times \frac{\left(k;q^2,p^2\right)_{n+r}}{\left(\frac{k^2}{a^2};q^2,p^2\right)_{n+r}} \left(\frac{k}{a^2 q}\right)^{n-r} \beta_r\left(a,\frac{k}{aq};q,p\right). \tag{13}$$

Theorem 3: If $\langle \alpha_n\left(a,k;q,p\right),\beta_n\left(a,k;q,p\right)\rangle$ is an elliptic WP Bailey pair, then so is the pair $\langle \alpha_n^{'}\left(a,k;q,p\right),\beta_n^{'}\left(a,k;q,p\right)\rangle$ given by,

$$\alpha_{2n}^{'}\left(a,k;q,p\right)=\alpha_n\left(a,\frac{k^2}{a};q^2,p\right), \tag{14}$$

and

$$\beta_n^{'}\left(a,k;q,p\right)=\frac{\left(\frac{k^2 q}{a};q^2,p\right)_n}{\left(aq;q^2,p\right)_n}\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{\theta\left(\frac{k^2}{a}q^{2r};p\right)\left(\frac{a}{k};q,p\right)_{n-2r}}{\theta\left(\frac{k^2}{a};p\right)\left(q;q,p\right)_{n-2r}}$$

$$\times \frac{\left(k;q,p\right)_{n+2r}}{\left(\frac{k^2 q}{a};q,p\right)_{n+2r}}\left(-\frac{k}{a}\right)^{n-2r}\beta_r\left(a,\frac{k^2}{a};q^2,p\right). \tag{15}$$

2. Elliptic WP-Bailey pairs and Elliptic conjugate WP-Bailey pairs:

In this section we shall obtain certain elliptic WP-Bailey pairs and elliptic conjugate WP-Bailey pairs.

(a) From (8) we find that

$$\beta_n\left(a,k;q,p\right)=\frac{\left(k,\frac{k}{a};q,p\right)_n}{\left(q,aq;q,p\right)_n} \tag{16}$$

and

$$\alpha_n\left(a,k;q,p\right)=\delta_{n,0} \tag{17}$$

is an elliptic WP-Bailey pair.

(b) From (9) we find that

$$\beta_n\left(a,k;q,p\right)=\delta_{n,0} \tag{18}$$

and

$$\alpha_n(a,k;q,p)=\frac{\theta(aq^{2n};p)\left(a,\frac{a}{k};q,p\right)_n}{\theta(a;p)(q,kq;q,p)_n}\left(\frac{k}{a}\right)^n \tag{19}$$

is an elliptic WP-Bailey pair.

(c) Taking
$$\alpha_r(a,k;q,p)=\frac{\theta(aq^{2r};p)\left(a,b,c,\frac{a^2q}{bck};q,p\right)_r}{\theta(a;p)\left(q,\frac{aq}{b},\frac{aq}{c},\frac{bck}{a};q,p\right)_r}\left(\frac{k}{a}\right)^r \tag{20}$$

in (8) and using the summation formula (4) we have

$$\beta_n(a,k;q,p)=\frac{\left(k,\frac{aq}{bc},\frac{bk}{a},\frac{ck}{a};q,p\right)_n}{\left(q,\frac{aq}{b},\frac{aq}{c},\frac{bck}{a};q,p\right)_n}. \tag{21}$$

$\alpha_n(a,k;q,p)$ and $\beta_n(a,k;q,p)$ given in (20) and (21) form an elliptic WP-Bailey pair.

(d) If we choose

$$\delta_r(a,k;q,p)=\frac{\theta(kq^{2r};p)\left(b,c,\frac{ak}{bc}q^{1+N};q,p\right)_r\left(\frac{1}{k};q,p\right)_{-N-r}}{\theta(k;p)\left(\frac{kq}{b},\frac{kq}{c},\frac{bc}{a}q^{-N};q,p\right)_r(q;q,p)_{N-r}\left(kq^{2N+1}\right)^r} \tag{22}$$

in (6) we get,

$$\gamma_n(a,k;q,p)=\frac{(-k)^N q^{N(N+1)/2}\left(b,c,\frac{ak}{bc}q^{1+N},q^{-N};q,p\right)_n\left(\frac{kq}{bc},\frac{aq}{b},\frac{aq}{c};q,p\right)_N\left(\frac{a}{k}\right)^n}{\left(\frac{aq}{b},\frac{aq}{c},\frac{bc}{a}q^{-N},aq^{1+N};q,p\right)_n\left(q,\frac{kq}{b},\frac{kq}{c},\frac{aq}{bc},aq;q,p\right)_N}. \tag{23}$$

Thus $\gamma_n(a,k;q,p)$ and $\delta_n(a,k;q,p)$ given in (23) and (22) form an elliptic WP-Bailey pair.

Putting these elliptic conjugate WP-Bailey pair in (7) we have the following result.

If $\langle\alpha_n(a,k;q,p),\ \beta_n(a,k;q,p)\rangle$ is an elliptic WP-Bailey pair then

$$\frac{\left(kq,\frac{kq}{bc},\frac{aq}{b},\frac{aq}{c};q,p\right)_N}{\left(aq,\frac{aq}{bc},\frac{kq}{b},\frac{kq}{c};q,p\right)_N}\sum_{n=0}^{N}\frac{\left(b,c,\frac{ak}{bc}q^{1+N},q^{-N};q,p\right)_n\left(\frac{aq}{k}\right)^n}{\left(\frac{aq}{b},\frac{aq}{c},\frac{bc}{a}q^{-N},aq^{1+N};q,p\right)_n}\alpha_n\left(a,k;q,p\right)$$

$$=\sum_{n=0}^{N}\frac{\theta\left(kq^{2n};p\right)\left(b,c,\frac{ak}{bc}q^{1+N},q^{-N};q,p\right)_n}{\theta\left(k;p\right)\left(\frac{kq}{b},\frac{kq}{c},\frac{bc}{a}q^{-N},kq^{1+N};q,p\right)_n}\beta_n\left(a,k;q,p\right). \tag{24}$$

Application of (24):

If we make use of elliptic WP-Bailey pair due to Warnaar [12] we have the following summation formula,

$$\sum_{n=0}^{\infty}\frac{\theta\left(aq^{2n};p\right)\left(a,\frac{a}{k},b,c,\frac{ak}{bc}q^{N+1},q^{-N};q,p\right)_n}{\theta\left(a;p\right)\left(q,kq,\frac{aq}{b},\frac{aq}{c},\frac{bc}{k}q^{-N},aq^{N+1};q,p\right)_n}={}_{10}V_9\left[a;\frac{a}{k},b,c,\frac{ak}{bc}q^{N+1},q^{-N};q,p\right]$$

$$=\frac{\left(aq,\frac{aq}{bc},\frac{kq}{b},\frac{kq}{c};q,p\right)_N}{\left(kq,\frac{kq}{bc},\frac{aq}{b},\frac{aq}{c};q,p\right)_N}. \tag{25}$$

Again replacing a,q,p by a^2,q^2,p^2 respectively in (24) it takes the form,

$$\frac{\left(kq^2,\frac{kq^2}{bc},\frac{a^2q^2}{b},\frac{a^2q^2}{c};q^2,p^2\right)_N}{\left(a^2q^2,\frac{a^2q^2}{bc},\frac{kq^2}{b},\frac{kq^2}{c};q^2,p^2\right)_N}\sum_{n=0}^{N}\frac{\left(b,c,\frac{a^2k}{bc}q^{2+2N},q^{-2N};q^2,p^2\right)_n\left(\frac{a^2q^2}{k}\right)^n}{\left(\frac{a^2q^2}{b},\frac{a^2q^2}{c},\frac{bc}{k}q^{-2N},a^2q^{2+2N};q^2,p^2\right)_n}\alpha_n\left(a^2,k;q^2,p^2\right)$$

$$=\sum_{n=0}^{N}\frac{\theta\left(kq^{4n};p^2\right)\left(b,c,\frac{a^2k}{bc}q^{2+2N},q^{-2N};q^2,p^2\right)_n}{\theta\left(k;p^2\right)\left(\frac{kq^2}{b},\frac{kq^2}{c},\frac{bc}{a^2}q^{-2N},kq^{2+2N};q^2,p^2\right)_n}\beta_n\left(a^2,k;q^2,p^2\right). \tag{26}$$

Using the elliptic WP-Bailey pair due to Warnaar [12],

$$\alpha_n\left(a^2,k;q^2,p^2\right)=\frac{\theta\left(aq^{2n};p\right)\left(a,\frac{a^2q}{k};q,p\right)_n}{\theta\left(a;p\right)\left(q,\frac{k}{a};q,p\right)_n}\left(\frac{k}{a^2q}\right)^n$$

$$\beta_n\left(a^2,k;q^2,p^2\right)=\frac{\left(-\frac{k}{a};q,p\right)_{2n}\left(k,\frac{a^2q^2}{k};q^2,p^2\right)_n}{\left(-aq;q,p\right)_{2n}\left(q^2,\frac{k^2}{a^2};q^2,p^2\right)_n}\left(\frac{k}{a^2q}\right)^n$$

in (26) we get the transformation formula,

$$\frac{\left(kq^2,\frac{kq^2}{bc},\frac{a^2q^2}{b},\frac{a^2q^2}{c};q^2,p^2\right)_N}{\left(a^2q^2,\frac{a^2q^2}{bc},\frac{kq^2}{b},\frac{kq^2}{c};q^2,p^2\right)_N}\sum_{n=0}^{N}\frac{\left(b,c,\frac{a^2k}{bc}q^{2+2N},q^{-2N};q^2,p^2\right)_n\theta\left(aq^{2n};p\right)\left(a,\frac{a^2q}{k};q,p\right)_n}{\left(\frac{a^2q^2}{b},\frac{a^2q^2}{c},\frac{bc}{k}q^{-2N},a^2q^{2+2N};q^2,p^2\right)_n\theta\left(a;p\right)\left(q,\frac{k}{a};q,p\right)_n}q^{-n}$$

$$=\sum_{n=0}^{N}\frac{\theta\left(kq^{4n};p^2\right)\left(k,\frac{a^2q^2}{k},b,c,\frac{a^2q}{bc}q^{2+2N},q^{-2N};q^2,p^2\right)_n\left(-\frac{k}{a};q,p\right)_{2n}}{\theta\left(k;p^2\right)\left(q^2,\frac{k^2}{a^2},\frac{kq^2}{b},\frac{kq^2}{c},\frac{bc}{a^2}q^{-2N},kq^{2+2N};q^2,p^2\right)_n\left(-aq;q,p\right)_{2n}}\left(\frac{k}{a^2q}\right)^n. \quad (27)$$

Some more elliptic WP-Bailey pairs:

We shall use the elliptic WP-Bailey pairs of {(16), (17)}, {(18), (19)}, {(20), (21)} in theorem 1,2 and 3 to obtain new elliptic WP-Bailey pairs.

(e) Using the elliptic WP-Bailey pair of (16) and (17) in theorem 1 we have another elliptic WP-Bailey pair.

$$\alpha_n^{'}\left(a,k;q,p\right)=1 \quad (28)$$

$$\beta_n^{'}\left(a,k;q,p\right)=\frac{\left(\frac{ck}{a},\frac{bk}{a};q,p\right)_n}{\left(\frac{aq}{b},\frac{aq}{c};q,p\right)_n}\sum_{r=0}^{n}\frac{\theta\left(\frac{bck}{aq}q^{2r};p\right)\left(b,c;q,p\right)_r\left(\frac{aq}{bc};q,p\right)_{n-r}}{\theta\left(\frac{bck}{aq};p\right)\left(\frac{ck}{a},\frac{bk}{a};q,p\right)_r\left(q;q,p\right)_{n-r}}$$

$$\times\frac{(k;q,p)_{n+r}}{\left(\frac{bck}{a};q,p\right)_{n+r}}\left(\frac{aq}{bc}\right)^r\frac{\left(a,\frac{bc}{a^2q};q,p\right)_n}{(q,aq;q,p)_n}. \tag{29}$$

(f) Using the elliptic WP-Bailey pair of (18) and (19) in theorem 1 we obtain another elliptic WP-Bailey pair,

$$\alpha_n'(a,k;q,p)=\frac{(b,c;q,p)_n\,\theta(aq^{2n};p)\left(a,\frac{a^2q}{bc};q,p\right)_n}{\left(\frac{aq}{b},\frac{aq}{c};q,p\right)_n\theta(a;p)\left(q,\frac{bc}{a};q,p\right)_n}\left(\frac{k}{a}\right)^n \tag{30}$$

$$\beta_n'(a,k;q,p)=\frac{\left(\frac{ck}{a},\frac{bk}{a};q,p\right)_n\left(\frac{aq}{bc};q,p\right)_n(k;q,p)_n}{\left(\frac{aq}{b},\frac{aq}{c};q,p\right)_n(q;q,p)_n\left(\frac{bck}{a};q,p\right)_n}. \tag{31}$$

(g) Using the elliptic WP-Bailey pair of (20) and (21) in theorem 1 we obtain another elliptic WP-Bailey pair,

$$\alpha_n'(a,k;q,p)=\frac{(b,c;q,p)_n\,\theta(aq^{2n};p)\left(a,b,c,\frac{a^3q^2}{b^2c^2k};q,p\right)_n}{\left(\frac{aq}{b},\frac{aq}{c};q,p\right)_n\theta(a;p)\left(q,\frac{aq}{b},\frac{aq}{c},\frac{b^2c^2k}{a^2q};q,p\right)_n}\left(\frac{bck}{a^2q}\right)^n, \tag{32}$$

$$\beta_n'(a,k;q,p)=\frac{\left(\frac{ck}{a},\frac{bk}{a};q,p\right)_n}{\left(\frac{aq}{b},\frac{aq}{c};q,p\right)_n}\sum_{r=0}^{n}\frac{\theta\left(\frac{bck}{aq}q^{2r};p\right)(b,c;q,p)_r\left(\frac{aq}{bc};q,p\right)_{n-r}}{\theta\left(\frac{bck}{aq};p\right)\left(\frac{ck}{a},\frac{bk}{a};q,p\right)_r(q;q,p)_{n-r}}$$

$$\times\frac{(k;q,p)_{n+r}\left(\frac{bck}{aq},\frac{aq}{bc},\frac{b^2ck}{a^2q},\frac{bc^2k}{a^2q};q,p\right)_n}{\left(\frac{bck}{a};q,p\right)_{n+r}\left(q,\frac{aq}{b},\frac{aq}{c},\frac{b^2c^2k}{a^2q};q,p\right)_n}. \tag{33}$$

(h) Making use of elliptic WP-Bailey pair of (16) and (17) in theorem 2 we obtain another elliptic WP-Bailey pair,

$$\alpha_n'\left(a^2,k;q^2,p^2\right)=1 \tag{34}$$

$$\beta_n'\left(a^2,k;q^2,p^2\right)=\frac{\left(-\frac{k}{a};q,p\right)_{2n}}{\left(-aq;q,p\right)_{2n}}\sum_{r=0}^{n}\frac{\theta\left(\frac{k}{aq}q^{2r};p\right)\left(\frac{a^2q^2}{k};q^2,p^2\right)_{n-r}}{\theta\left(\frac{k}{aq};p\right)\left(q^2;q^2,p^2\right)_{n-r}}$$

$$\times\frac{\left(k;q^2,p^2\right)_{n+r}\left(k,\frac{k}{a^2q};q,p\right)_n}{\left(\frac{k^2}{a^2};q^2,p^2\right)\left(q,aq;q,p\right)_n}\left(\frac{k}{a^2q}\right)^{n-r}. \tag{35}$$

(i) Making use of elliptic WP-Bailey pair of (18) and (19) in theorem 2 we obtain another elliptic WP-Bailey pair,

$$\alpha_n'\left(a^2,k;q^2,p^2\right)=\frac{\theta\left(aq^{2n};p\right)\left(a,\frac{a^2q}{k};q,p\right)_n}{\theta\left(a;p\right)\left(q,\frac{k}{a};q,p\right)_n}\left(\frac{k}{aq}\right)^n, \tag{36}$$

$$\beta_n'\left(a^2,k;q^2,p^2\right)=\frac{\left(-\frac{k}{a};q,p\right)_{2n}\left(\frac{a^2q^2}{k};q^2,p^2\right)_n\left(k;q^2,p^2\right)_n}{\left(-aq;q,p\right)_{2n}\left(q^2;q^2,p^2\right)_n\left(\frac{k^2}{a^2};q^2,p^2\right)_n}\left(\frac{k}{a^2q}\right)^n. \tag{37}$$

(j) Making use of elliptic WP-Bailey pair of (20) and (21) in theorem 2 we obtain another elliptic WP-Bailey pair,

$$\alpha_n'\left(a^2,k;q^2,p^2\right)=\frac{\theta\left(aq^{2n};p\right)\left(a,b,c,\frac{a^3q^2}{bck};q,p\right)_n}{\theta\left(a;p\right)\left(q,\frac{aq}{b},\frac{aq}{c},\frac{bck}{a^2q};q,p\right)_n}\left(\frac{k}{a^2q}\right)^n, \tag{38}$$

$$\beta_n'\left(a^2,k;q^2,p^2\right)=\frac{\left(-\frac{k}{a};q,p\right)_{2n}}{\left(-aq;q,p\right)_{2n}}\sum_{r=0}^{n}\frac{\theta\left(\frac{k}{aq}q^{2r};p\right)\left(\frac{a^2q^2}{k};q^2,p^2\right)_{n-r}}{\theta\left(\frac{k}{aq};p\right)\left(q^2;q^2,p^2\right)_{n-r}}$$

$$\times\frac{\left(k;q^2,p^2\right)_{n+r}\left(\frac{k}{aq},\frac{aq}{bc},\frac{bk}{a^2q},\frac{ck}{a^2q};q,p\right)_n}{\left(\frac{k^2}{a^2};q^2,p^2\right)_{n+r}\left(q,\frac{aq}{b},\frac{aq}{c},\frac{bck}{a^2q};q,p\right)_n}\left(\frac{k}{a^3q^2}\right)^{n-r}. \tag{39}$$

(k) Making use of elliptic WP-Bailey pair of (16) and (17) in theorem 3 we obtain another elliptic WP-Bailey pair,

$$\alpha_{2n}^{'}\left(a,k;q,p\right)=1 \tag{40}$$

$$\beta_n^{'}\left(a,k;q,p\right)=\frac{\left(\frac{k^2q}{a};q^2,p\right)_n}{\left(aq;q^2,p\right)_n}\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{\theta\left(\frac{k^2}{a}q^{2r};p\right)\left(\frac{a}{k};q,p\right)_{n-2r}}{\theta\left(\frac{k^2}{a};p\right)\left(q;q,p\right)_{n-2r}}$$

$$\times\frac{\left(k;q,p\right)_{n+2r}\left(\frac{k^2}{a},\frac{k^2}{a^2};q^2,p\right)_n}{\left(\frac{k^2q}{a};q,p\right)_{n+2r}\left(q,aq;q^2,p\right)_n}\left(-\frac{k}{a}\right)^{n-2r}. \tag{41}$$

(l) Making use of elliptic WP-Bailey pair of (18) and (19) in theorem 3 we obtain another elliptic WP-Bailey pair,

$$\alpha_{2n}^{'}\left(a,k;q,p\right)=\frac{\theta\left(aq^{2n};p\right)\left(a,\frac{a^2}{k^2};q^2,p\right)_n}{\theta\left(a;p\right)\left(q,\frac{k^2q}{a};q^2,p\right)_n}\left(\frac{k^2}{a^2}\right)^n, \tag{42}$$

$$\beta_n^{'}\left(a,k;q,p\right)=\frac{\left(\frac{k^2q}{a};q^2,p\right)_n\left(\frac{a}{k};q,p\right)_n\left(k;q,p\right)_n}{\left(aq;q^2,p\right)_n\left(q;q,p\right)_n\left(\frac{k^2q}{a};q,p\right)_n}\left(-\frac{k}{a}\right)^n.$$

$$\times\frac{\left(k;q,p\right)_{n+2r}}{\left(\frac{k^2q}{a};q,p\right)_{n+2r}}\left(-\frac{k}{a}\right)^{n-2r}\beta_r\left(a,\frac{k^2}{a};q^2,p\right). \tag{43}$$

(m) Making use of elliptic WP-Bailey pair of (20) and (21) in theorem 3 we obtain another elliptic WP-Bailey pair,

$$\alpha_{2n}'(a,k;q,p)=\frac{\theta\left(aq^{2n};p\right)\left(a,b,c,\frac{a^3q}{bck^2};q^2,p\right)_n}{\theta(a;p)\left(q,\frac{aq}{b},\frac{aq}{c},\frac{bck^2}{a^2};q^2,p\right)_n}\left(\frac{k^2}{a^2}\right)^n, \tag{44}$$

$$\beta_n'(a,k;q,p)=\frac{\left(\frac{k^2q}{a};q^2,p\right)_n}{\left(aq;q^2,p\right)_n}\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{\theta\left(\frac{k^2}{a}q^{2r};p\right)\left(\frac{a}{k};q,p\right)_{n-2r}}{\theta\left(\frac{k^2}{a};p\right)(q;q,p)_{n-2r}}$$

$$\times\frac{(k;q,p)_{n+2r}\left(\frac{k^2}{a},\frac{aq}{bc},\frac{bk^2}{a^2},\frac{ck^2}{a^2};q^2,p\right)_n}{\left(\frac{k^2q}{a};q,p\right)_{n+2r}\left(q,\frac{aq}{b},\frac{aq}{c},\frac{bck^2}{a^2};q^2,p\right)_n}\left(-\frac{k^2}{a^2}\right)^{n-2r}. \tag{45}$$

3. Main Results

In this section we shall establish certain transformation and summation formulas for theta hypergeometric series.

(i) Putting the elliptic WP-Bailey pair given in (28), (29) in (8) we find,

$${}_{10}V_9\left[\frac{bck}{aq};b,c,kq^n,q^{-n},q;q,p\right]=\frac{\left(\frac{aq}{b},\frac{aq}{c},\frac{bck}{a},q,\frac{k}{a};q,p\right)_n}{\left(\frac{ck}{a},\frac{bk}{a},\frac{bc}{a^2q},a,\frac{aq}{bc};q,p\right)_n}$$

$$\times\ {}_3E_2\left[\begin{matrix}q^{-n},kq^n;q;q,p;\frac{aq}{k}\\ \frac{a}{k}q^{1-n},aq^{n+1}\end{matrix}\right] \tag{46}$$

(ii) Putting the elliptic WP-Bailey pair given in (30), (31) in (8) we find,

$$ {}_{10}V_9\left[\frac{ck}{a};\frac{bk}{a},\frac{aq}{bc},\frac{bc}{a},b,q^{-n};q,p\right] = {}_3E_2\left[\begin{matrix} q^{-n},kq^{n};q;q,p;\dfrac{aq}{k} \\ \dfrac{a}{k}q^{1-n},aq^{n+1} \end{matrix}\right] \tag{47} $$

(iii) Putting the elliptic WP-Bailey pair given in (32), (33) in (8) we get

$$ {}_{10}V_9\left[\frac{bck}{aq};b,c,kq^{n},q^{-n},q;q,p\right] = \frac{\left(\frac{bck}{a};q,p\right)_n \left(\frac{k}{a};q,p\right)_n (b,c;q,p)_n}{\left(\frac{aq}{bc};q,p\right)_n (aq;q,p)_n \left(\frac{ck}{a},\frac{bk}{a};q,p\right)_n} $$
$$ \times \frac{\left(a,b,c,\frac{a^3q^2}{b^2c^2k},q,\frac{k}{a};q,p\right)_n}{\left(\frac{bck}{aq},\frac{aq}{bc},\frac{b^2ck}{a^2q},\frac{bc^2k}{a^2q};q,p\right)_n} \times {}_3E_2\left[\begin{matrix} q^{-n},kq^{n};q;q,p;\dfrac{aq}{k} \\ \dfrac{a}{k}q^{1-n},aq^{n+1} \end{matrix}\right] \tag{48} $$

Similarly, we can find another summation formula.

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