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SOME PROPERTIES OF ANALYTIC FUNCTIONS DEFINED BY GEGENBAUR POLYNOMIALS

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Abstract. In this work, we introduce and study a new subclass of analytic functions defined by a differential operator and obtained coefficient estimates, growth and distortion theorems, radii of starlikeness, convexity and close-to-convexity are obtained. Furthermore, we obtained integral means inequalities for the function.

Keywords: analytic, coefficient bounds, starlike, distortion.

2010 Mathematics subject classification: 30C45.

1. Introduction

Let A denote the class of functions f of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1}$$

which are analytic in the open unit disk $E=\left\{z\in\square \qquad \right\}$.

A function f in the class A is said to be in the class $ST(\alpha)$ of starlike functions of order α in *E*, if it satisfy the inequality

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\ (0\le\alpha<1),\ (z\in E) \tag{2}$$

Note that $ST(0)=ST$ is the class of starlike functions.

Denote by T the subclass of A consisting of functions *f* of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n\ \ (a_n\ge 0)\,. \tag{3}$$

This subclass was introduced and extensively studied by Silverman [6].

The class $T(\lambda), \lambda \geq 0$ were introduced and investigated by Szynal [10] as the subclass of A consisting of functions of the form

$$f(z) = \int_{-1}^{1} k(z,m) d\mu(m). \tag{4}$$

where

$$k(z,m) = \frac{z}{(1-2mz+z^2)^{\lambda}} \quad m \in [-1,1], \ (z \in E). \tag{5}$$

And μ is a probability measure on the interval [-1,1]. The collection of such measure on [a,b] is denoted by $P_{[a,b].}$.

The Taylor series expansion of the function in (5) gives

$$k(z,t) = z + c_1^{\lambda}(m)z^2 + c_2^{\lambda}(m)z^3 + ... \tag{6}$$

and the coefficients for (6) were given below :

$$c_0^{\lambda}(m) = 1, \ c_1^{\lambda}(m) = 2\lambda m, \ c_2^{\lambda}(m) = 2\lambda(\lambda+1)m^2 - \lambda,$$

$$c_3^{\lambda}(m) = \frac{4}{3}\lambda(\lambda+1)(\lambda+2)m^3 - 2\lambda(\lambda+1)m, ... \tag{7}$$

Where $c_n^{\lambda}(m)$ denotes the Gegenbauer polynomial of degree n. Varying the parameter λ in (6), we obtain the class of typically real functions studied by [1], [4], [5] and [9].

For $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \ (z \in E).$$

Let $\mathcal{G}_{\lambda}^{m} : A \to A$ defined in terms of the convolution by

$$\mathcal{G}_{\lambda}^{m} f(z) = k(z,m) * f(z),$$

We have $\mathcal{G}_{\lambda}^{m} f(z) = z + \sum_{n=2}^{\infty} \phi_n(\lambda, m) a_n z^n$, (8)

where $\phi_n(\lambda, m) = c_{n-1}^{\lambda}(m)$.

In this paper, using the operator $\mathcal{G}_{\lambda}^{m} f(z)$, we define the following new class motivated by Murugusunderamoorthy and Magesh [3].

Definition 1. The function $f(z)$ of the form (1) is in the class $S_\lambda^m(\mu,\gamma)$ if it satisfies the inequality

$$\operatorname{Re}\left\{\frac{z\left(\mathcal{G}_\lambda^m f(z)\right)'}{(1-\mu)z+\mu\mathcal{G}_\lambda^m f(z)}-\gamma\right\}>\varsigma\left|\frac{z\left(\mathcal{G}_\lambda^m f(z)\right)'}{(1-\mu)z+\mu\mathcal{G}_\lambda^m f(z)}-1\right|$$

for $0\le\lambda\le 0$, and $0\le\gamma<1$.

Further we define $TS_\lambda^m(\mu,\gamma)=S_\lambda^m(\mu,\gamma)\cap T$.

The aim of this paper is to study the coefficient bounds, radii of close-to-convex and starlikeness convex linear combinations for the class $TS_\lambda^m(\mu,\gamma)$.Furthermore, we obtained integral means inequalities for the functions in $TS_\lambda^m(\mu,\gamma)$.

Theorem 1: A function *f(z)* of the form (1) is in $S_\lambda^m(\mu,\gamma)$

$$\sum_{n=2}^{\infty}[2n-\mu(\gamma+1)]\ \phi_n(\lambda,m)|a_n|\le 1-\gamma \tag{9}$$

where $0\le\mu\le 1,\ 0\le\gamma<1,$ and $\phi_n(\lambda,m)$ is given by (8).

Proof: It suffices to show that

$$\left|\frac{z\left(\mathcal{G}_\lambda^m f(z)\right)'}{(1-\mu)z+\mu\mathcal{G}_\lambda^m f(z)}-1\right|-\operatorname{Re}\left\{\frac{z\left(\mathcal{G}_\lambda^m f(z)\right)'}{(1-\mu)z+\mu\mathcal{G}_\lambda^m f(z)}-1\right\}\le 1-\gamma\ -$$

We have

$$\left|\frac{z\left(\mathcal{G}_\lambda^m f(z)\right)'}{(1-\mu)z+\mu\mathcal{G}_\lambda^m f(z)}-1\right|-\operatorname{Re}\left\{\frac{z\left(\mathcal{G}_\lambda^m f(z)\right)'}{(1-\mu)z+\mu\mathcal{G}_\lambda^m f(z)}-1\right\}$$

$$\le 2\left|\frac{z\left(\mathcal{G}_\lambda^m f(z)\right)'}{(1-\mu)z+\mu\mathcal{G}_\lambda^m f(z)}-1\right|$$

$$\leq 2\frac{\sum_{n=2}^{\infty}(n-\mu)\phi_n(\lambda,m)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)|a_n||z|^{n-1}}$$

$$\leq 2\frac{\sum_{n=2}^{\infty}(n-\mu)\phi_n(\lambda,m)|a_n|}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)|a_n|}$$

The last expression is bounded above by $(1-\gamma)$ if

$$\sum_{n=2}^{\infty}[2n-\mu(\gamma+1)]\ \phi_n(\lambda,m)|a_n|\leq 1-\gamma$$

and the proof is complete.

Theorem 2: Let $0\leq\mu\leq 1,$ and $0\leq\gamma<1$ then a function *f* of the form (3) to be in the class $TS_{\lambda}^{m}(\mu,\gamma)$ if and only if

$$\sum_{n=2}^{\infty}[2n-\mu(\gamma+1)]\ \phi_n(\lambda,m)\leq 1-\gamma \tag{10}$$

where $\phi_n(\lambda,m)$ are given by (5)

Proof: In view of Theorem 1, we need only to prove the necessity. If $f\in TS_{\lambda}^{m}(\mu,\gamma)$ and *z* is real then

$$\mathrm{Re}\left\{\frac{1-\sum_{n=2}^{\infty}n\phi_n(\lambda,m)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)a_n z^{n-1}}-\gamma\right\}>\varsigma\left|\frac{\sum_{n=2}^{\infty}(n-\mu)\phi_n(\lambda,m)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)a_n z^{n-1}}\right|$$

Letting z → 1 along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[2n-\mu(\gamma+1)]\ \ \phi_n(\lambda,m)|a_n|\leq 1-\gamma\,,$$

where $0\leq\mu<1,\ 0\leq\gamma<1,$ and $\phi_n(\lambda,m)$ are given by (6).

Corollary 1. If *f(z)* $\in$ $TS_{\lambda}^{m}(\mu,\gamma)$, then

$$|a_n| \le \frac{1-\gamma}{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)} \tag{11}$$

where $0 \le \mu < 1$, $0 \le \gamma < 1$, and $\phi_n(\lambda,m)$ are given by (5). Equality holds for the function

$$f(z) = z - \frac{1-\gamma}{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)} z^n \tag{12}$$

Theorem 3. Let

$f_1(z) = z$ and

$$f_n(z) = z - \frac{1-\gamma}{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)} z^n, \; n \ge 2. \tag{13}$$

Then *f(z)* ∈ $TS_\lambda^m(\mu,\gamma)$, if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} w_n f_n(z) \;,\; w_n \ge 0, \sum_{n=1}^{\infty} w_n = 1 \;. \tag{14}$$

Proof. Suppose *f(z)* can be written as in (14).Then

$$f(z) = z - \sum_{n=2}^{\infty} w_n \frac{1-\gamma}{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)} z^n \;.$$

Now,

$$\sum_{n=2}^{\infty} w_n \frac{(1-\gamma)[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)[2n-\mu(\gamma+1)]\phi_n(\lambda,m)} = \sum_{n=2}^{\infty} w_n = 1 - w_1 \le 1.$$

Thus *f(z)* ∈ $TS_\lambda^m(\mu,\gamma)$. Conversely, let us have *f(z)* ∈ $TS_\lambda^m(\mu,\gamma)$. Then by using (11), we get

$$w_n = \frac{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)} a_n \;,\; n \ge 2$$

and $w_1 = 1 - \sum_{n=2}^{\infty} w_n$. Then we have $f(z) = \sum_{n=1}^{\infty} w_n f_n(z)$ and hence this completes the proof of Theorem.

Theorem 4. The class $TS_\lambda^m(\mu,\gamma)$ is a convex set.

Proof. Let the function

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \ , \ a_{n,j} \geq 0, \ \mathrm{j} = 1,2 \tag{15}$$

be in the class $TS_{\lambda}^{m}(\mu,\gamma,\varsigma)$. It sufficient to show that the function $h(z)$ defined by

$h(z) = \xi f_1(z) + (1-\xi) f_2(z) \ , \ 0 \leq \xi < 1$,

is in the class $TS_{\lambda}^{m}(\mu,\gamma)$. Since

$h(z) = z - \sum_{n=2}^{\infty} \left[\xi a_{n,1} + (1-\xi) a_{n,2} \right] z^n$,

An easy compution with the aid of of Theorem 2, gives

$$\sum_{n=2}^{\infty} [2n - \mu(\gamma+1)] \xi \phi_n(\lambda,m) a_{n,1} + \sum_{n=2}^{\infty} [2n - \mu(\gamma+1)] \, (1-\xi) \phi_n(\lambda,m) a_{n,2}$$

$$\leq \xi(1-\gamma) + (1-\xi)(1-\gamma)$$

$$\leq (1-\gamma),$$

which implies that $h \in TS_{\lambda}^{m}(\mu,\gamma)$.

Hence $TS_{\lambda}^{m}(\mu,\gamma)$ is convex.

Next we obtain the radii of close-to-convexity, starlikeness and convexity for the class $TS_{\lambda}^{m}(\mu,\gamma)$.

Theorem 5. Let the function $f(z)$ defined by (3) belong to the class $TS_{\lambda}^{m}(\mu,\gamma)$. Then $f(z)$ is close-to-convex of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_1$, where

$$r_1 = \inf_{n \geq 2} \left[\frac{(1-\delta) \sum_{n=2}^{\infty} [2n - \mu(\gamma+1)] \phi_n(\lambda,m)}{n(1-\gamma)} \right]^{1/n-1} , \ n \geq 2. \tag{16}$$

The result is sharp, with the extremal function $f(z)$ is given by (13)

Proof. Given $f \in T$, and f is close-to-convex of order δ, we have

$$|f'(z) - 1| < 1 - \delta \tag{17}$$

For the left hand side of (17) we have

$$|f'(z)-1| \le \sum_{n=2}^{\infty} na_n |z|^{n-1}$$

The last expression is less than $1-\delta$

$$\sum_{n=2}^{\infty} \frac{n}{1-\delta} a_n |z|^{n-1} \le 1.$$

Using the fact, that *f(z)* ∈ $TS_{\lambda}^{m}(\mu,\gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)} a_n \le 1,$$

Thus, (17) is true if

$$\frac{n}{1-\delta}|z|^{n-1} \le \frac{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)}$$

or, equivalently,

$$|z| \le \left\{ \frac{(1-\delta)[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{n(1-\gamma)} \right\}^{1/n-1}$$

which completes the proof.

Theorem 6. Let the function $f(z)$ defined by (3) belong to the class $TS_{\lambda}^{m}(\mu,\gamma)$. Then $f(z)$ is starlike of order of order δ $(0 \le \delta < 1)$ in the disc $|z| < r_2$, where

$$r_2 = \inf_{n\ge 2} \left[\frac{(1-\delta)\sum_{n=2}^{\infty}[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{(n-\delta)(1-\gamma)} \right]^{1/n-1} \tag{18}$$

The result is sharp, with extremal function $f(z)$ is given by (13).

Proof. Given $f \in T$, and f is starlike of order δ, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1-\delta \tag{19}$$

For the left hand side of (19) we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \sum_{n=2}^{\infty}\frac{(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n|z|^{n-1}}$$

The last expression is less than 1-δ if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n|z|^{n-1}<1.$$

Using the fact that *f(z)* ∈ $TS_{\lambda}^{m}(\mu,\gamma)$ if and if

$$\sum_{n=2}^{\infty}\frac{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)}a_n \le 1,$$

We can say (2.11) is true if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}|z|^{n-1} \le \frac{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)}$$

or equivalently

$$|z|^{n-1} \le \frac{(1-\delta)[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{(n-\delta)(1-\gamma)}$$

which yields the starlikeness of the family.

Integral Means Inequalities

In [6], Silverman found that the function $f_2(z)=z-\frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality conjuctured [7] and settled in [8], that

$$\int_0^{2\pi}\left|f(re^{i\varphi})\right|^{\eta}d\varphi \le \int_0^{2\pi}\left|f_2(re^{i\varphi})^{\eta}\right|d\varphi \ ,$$

for all $f \in T$, $\eta > 0$ and $0 < r < 1$. In [6], he also proved his conjucture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T.

Now, we prove Silverman ‘s conjecture for the class of functions $TS_{\lambda}^{m}(\mu,\gamma)$.

We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [2].

Two functions f and g, which are analytic in E, the function f is said to be subordinate to g in E if there exists a function w analytic in E with

$w(0)=0$, $|w(z)|<1$, $(z\in E)$ Such that $f(z)=g(w(z))$, $(z\in E)$.

We denote this subordination by $f(z)\prec$. ($\prec$ denotes subordination).

Lemma 1. If the functions f and g are analytic in E with $f(z)\prec$,then for $\eta>0$ and $z=re^{i\varphi}$ $0<r<1$,

$$\int_0^{2\pi}\left|g(re^{i\varphi})\right|^{\eta}d\varphi\le\int_0^{2\pi}\left|f\,(re^{i\varphi})\right|^{\eta}d\varphi$$

Now, we discuss the integral means inequalities for functions f in $TS_{\lambda}^{m}(\mu,\gamma)$

$$\int_0^{2\pi}\left|g(re^{i\varphi})\right|^{\eta}d\varphi\le\int_0^{2\pi}\left|f\,(re^{i\varphi})\right|^{\eta}d\varphi$$

Theorem 7. Let $f\in TS_{\lambda}^{m}(\mu,\gamma)$, $0\le\mu<1$, $0\le\gamma<1$, and $f_2(z)$ be defined by

$$f_2(z)=z-\frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\gamma)}z^2 \tag{20}$$

Proof. For $f(z)=z-\sum_{n=2}^{\infty}a_nz^n$, (20) is equivalent to

$$\int_0^{2\pi}\left|1-\sum_{n=2}^{\infty}a_nz^{n-1}\right|^{\eta}d\varphi\le\int_0^{2\pi}\left|1-\frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\gamma)}z\right|^{\eta}d\varphi$$

By Lemma 1, it is enough to prove that

$$1-\sum_{n=2}^{\infty}a_nz^{n-1}\prec\qquad\frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\gamma)}z$$

Assuming

$$1-\sum_{n=2}^{\infty}a_nz^{n-1}\prec\qquad\frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\gamma)}w(z)\ ,$$

and using (10) we obtain

$$|w(z)| = \left| \sum_{n=2}^{\infty} \frac{\varphi_2(\lambda, m, \mu, \gamma)}{1-\gamma} a_n z^{n-1} \right| \le |z| \sum_{n=2}^{\infty} \frac{\varphi_2(\lambda, m, \mu, \gamma)}{1-\gamma} a_n \le |z|$$

where $\varphi_n(\lambda, m, \mu, \gamma) = [2n - \mu(\gamma+1)]\phi_n(\lambda, m)$

This completes the proof.

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References

[1] Duren, P.L. (1983). Univalent Functions, A series of Comprehensive Studies in Mathematics, **259**, Springer, New York.

[2] Littlewood, J.E. (1925). On inequalities in the theory of functions, Proc. London Math. Soc., **23**(2), 481-519.

[3] Murugusundarmoorthy, G. and Magesh, N. (2010). Certain sub-classes of starlike functions of complex order involving generalized hypergeometric functions. Int. J. Math. Math. Sci., art ID 178605, **12**.

[4] Pommerenke, C. (1975). Univalent Functions, Vandenhoeck and Ruprecht, G¨ottingen.

[5] Schober, G. (1975). Univalent Functions, Selected topics Lecture Nptes in Mathematics, **478**, Springer, New York.

[6] Silverman, H. (1975). Univalent functions with negative coefficients, Proc. Amer. Math. Soc. **51**, 109-116.

[7] Silverman, H. (1991). A survey with open problems on univalent functions whose coefficient are negative, Rocky Mountain J. Math., **21**(3), 1099-1125.

[8] Silverman, H. (1997). Integral means for univalent functions with negative coefficient, Houston J. Math., **23**(1), 169-174.

[9] Sobczak-Knec, M. and Zaprawa, P. (2007). Covering domains for classes of functions with real coefficients, Complex var. Elliptic Equ. **52**(6), 519-535.

[10] Szynal, J. (1994). An extension of typically real functions, Ann. Univ. Mariae Curie-Sklodowska, sect. A. **48**, 193-201.