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# AN ANALYTICAL APPROACH TO THE FRACTIONAL BIOLOGICAL POPULATION MODEL VIA EXPONENTIAL LAW AND MITTAG-LEFFLER KERNEL

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**Abstract:** In this article, analytical approximate solution of time fractional non-linear biological population model which arises as a result of spatial diffusion is proposed. Considering the fractional derivatives in Atangana-Baleanu-Caputo and Caputo-Fabrizio-Caputo sense, the Laplace transform technique has been employed in combination to the homotopy perturbation method. Examples corresponding to Malthusian and Verhulst laws are worked out and it is shown that in most of the cases the numerical solution converges to the exact solution. The numerical simulations are presented to depict the behavior of the solution corresponding to the variations in the fractional parameter and time.

**MSC 2010:** 35D99.

**Keywords:** biological population model, homotopy perturbation method, Atangana - Baleanu fractional operator, caputo - fabrizio fractional operator.

## 1. Introduction

To study the dynamic behaviour of various physical and biological systems which are expressible in the form of non-linear equations, fractional calculus has drawn the attention of many researchers [11, 17]. In this paper, we aim at finding an approximate solution to the time fractional biological population equation [18] which represents the diffusion of biological population. The equation is considered in the following form:

$$ {}^{ABC}_{0}D_t^{\nu}\, u = u_{xx}^2 + u_{yy}^2 + \chi(u) \tag{1}$$

$$ {}^{CFC}_{0}D_t^{\nu}\, u = u_{xx}^2 + u_{yy}^2 + \chi(u), 0 < \nu \leq 1, t \geq 0, x, y \in \mathbb{R} \tag{2}$$

With initial condition $u(x, y, 0) = \varphi(x, y)$.

Here $u(x, y, t)$ is the population density and $\chi(u)$ denotes the supply of population due to births and deaths.In the above equations the fractional operators are taken in Atangana-Baleanu-Caputo(ABC) [1] and Caputo-Fabrizio-Caputo(CFC) [2] sense respectively. ABC operator is non-local and non- singular, based on the Mittag-Leffler kernel whereas

CFC operator being local and non-singular is based on exponential kernel. Both the operators have an advantage over the Riemann-Liouville and Liouville-Caputo operator by overcoming the singularity of kernel at the end point of interval.

To solve the equation, a more general form of $\chi(u)$ is considered as $\chi(u) = \hbar u^{\xi}(1 - ru^{\eta})$. $\xi, \eta, \hbar, r$ being the real numbers, the population equation corresponds to Malthusian and Verhulst law when specific values are assigned to them.

The population model has been solved previously using VIM[18], ADM[3], HPM with power law kernel [13]. However we will be using an efficient technique LHPM[10] which couples the Laplace transform[5,14] with homotopy perturbation method (HPM), first proposed by He [6-9]. HPM has been successfully applied by the researchers to find the approximate series solution of various fractional problems[4,12,15,16,19].

The paper is organized as follows:

In **Section 2**, ABC and CFC fractional derivatives along with their Laplace transform are presented, **Section 3** is dedicated to LHPM technique, **Section 4** deals with solving the examples and **Section 5** provides the conclusion.

## 2. ABC AND CFC Fractional derivatives

### 2.1 Atangana-Baleanu Type Fractional order derivative

The Atangana-Baleanu fractional derivative [1] in the Liouville-Caputo sense (ABC) is defined as

$$ {}^{ABC}_{0}D_t^{\nu} f(t) = \frac{B(\nu)}{n-\nu}\int_0^t \frac{d^n}{dt^n} f(\xi) E_{\nu}\left[-\nu\frac{(t-\xi)^{\nu}}{n-\nu}\right] d\xi \ , n-1 < \nu < n. \tag{3} $$

Where $B(\nu)$ is a constant of normalization depending on $\nu$, which satisfies $B(0) = B(1) = 1$.

The Atangana - Baleanu fractional integral of order $\nu$ of a function $f(t)$ is defined as [1]

$$ {}^{AB}_{0}I_t^{\nu} f(t) = f(t) - f(0) = \frac{(1-\nu)}{B(\nu)} f(t) + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t f(\xi)(t-\xi)^{\nu-1} d\xi \ . \tag{4} $$

For $0 < \nu \leq 1$ and $n \in \mathbb{N}$, the Laplace transform of ABC type fractional derivative [1] is defined as

$$ \mathcal{L}\left[{}^{ABC}_{0}D_t^{(\nu+n)} f(t)\right](s) = \frac{B(\nu)}{(1-\nu)(s^{1-\nu})} \left[\frac{s^{n+1}\mathcal{L}[f(t)] - s^n f(0) - s^{n-1} f'(0) \ldots f^n(0)}{s^{\nu} + \frac{\nu}{(1-\nu)}}\right] \tag{5} $$

### 2.2 CAPUTO-FABRIZIO TYPE Fractional order derivative

The Caputo-Fabrizio fractional derivative [2] in the Liouville-Caputo sense (CFC) is defined as

$$ {}^{CFC}_{0}D_t^{\nu} f(t) = \frac{A(\nu)}{n-\nu}\int_0^t \frac{d^n}{dt^n} f(\xi) e^{-\frac{\nu}{n-\nu}(t-\xi)} d\xi \ , n-1 < \nu < n \tag{6} $$

where $A(\nu)$ is a constant of normalization depending on $\nu$,which satisfies $A(0) = A(1) = 1$.

The Caputo-Fabrizio fractional integral of order $\nu$ of a function $f(t)$ is defined as [2]

$${}_0^{CF}I_t^{\nu} f(t) = (1-\nu)f(t) + \nu \int_0^t f(\xi)d\xi \ . \tag{7}$$

For $0 < \nu \leq 1$ and $n \in \mathbb{N}$, the Laplace transform of CFC type fractional derivative[2] is defined as

$$\mathcal{L}\left[{}_0^{CFC}D_t^{(\nu+n)} f(t)\right](s) = \left[\frac{s^{n+1}\mathcal{L}[f(t)] - s^n f(0) - s^{n-1}f'(0) \ldots f^n(0)}{s+\nu(1-s)}\right] \tag{8}$$

## 3. Application of LHPM to Fractional Biological Population Model

As the first case (ABC fractional operator), consider the fractional biological population model as

$${}_0^{ABC}D_t^{\nu}\, u = u_{xx}^2 + u_{yy}^2 + \hbar u^{\xi}(1 - r u^{\eta}) \tag{9}$$

With the given initial condition $u(x, y, 0)$.

Using (5) with $n = 0$ and applying Laplace transform to equation (9), we get

$$\frac{B(\nu)}{1-\nu}.\frac{s^{\nu}\tilde{u}(x,y,s) - s^{\nu-1}u(x,y,0)}{s^{\nu}+\frac{\nu}{(1-\nu)}} = \mathcal{L}\{u_{xx}^2 + u_{yy}^2 + \hbar u^{\xi}(1 - r u^{\eta})\},$$

which on simplifying reduces to

$$\tilde{u}(x, y, s) = \frac{u(x,y,0)}{s} + \frac{(1-\nu)s^{\nu}+\nu}{B(\nu)s^{\nu}}\mathcal{L}\{u_{xx}^2 + u_{yy}^2 + \hbar u^{\xi}(1 - r u^{\eta})\} \tag{10}$$

Applying the inverse Laplace transform to (10), we get

$$u(x, y, t) = u(x, y, 0) + \mathcal{L}^{-1}\left[\frac{(1-\nu)s^{\nu}+\nu}{B(\nu)s^{\nu}}\mathcal{L}\{u_{xx}^2 + u_{yy}^2 + \hbar u^{\xi}(1 - r u^{\eta})\}\right]. \tag{11}$$

As HPM assumes the solution $u(x, y, t)$ to be of the form

$$u(x, y, t) = \sum_{n=0}^{\infty} p^n u_n(x, y, t),$$

We get

$$\sum_{n=0}^{\infty} p^n u_n(x, y, t) = u(x, y, 0) + p\mathcal{L}^{-1}\left[\frac{(1-\nu)s^{\nu}+\nu}{B(\nu)s^{\nu}}\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n u_n(x, y, t)\right)^2 + \frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{\infty} p^n u_n(x, y, t)\right)^2 + \hbar\left(\sum_{n=0}^{\infty} p^n u_n(x, y, t)\right)^{\xi}\left(1 - r\left(\sum_{n=0}^{\infty} p^n u_n(x, y, t)\right)^{\eta}\right)\right\}\right] \tag{12}$$

Comparison of coefficients of like powers of $p$ on both sides gives

$$p^0 : u_0(x, y, t) = u(x, y, 0)$$

$$p^1 : u_1(x, y, t) = \mathcal{L}^{-1}\left[\frac{(1-\nu)s^{\nu}+\nu}{B(\nu)s^{\nu}}\mathcal{L}\left\{\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + \hbar {u_0}^{\xi}(1 - r {u_0}^{\eta})\right\}\right]$$

$$p^2: u_1(x,y,t) = \mathcal{L}^{-1}\left[\frac{(1-\nu)s^\nu+\nu}{B(\nu)s^\nu}\mathcal{L}\left\{\frac{\partial^2(2u_0u_1)}{\partial x^2}+\frac{\partial^2(2u_0u_1)}{\partial y^2}+\hbar\xi {u_0}^{\xi-1}u_1-\hbar r(\xi+\eta){u_0}^{\xi+\eta-1}u_1\right\}\right] \quad (13)$$

and so on.

For the second case (CFC fractional operator), consider the fractional biological population model as

$${}_0^{CFC}D_t^{\nu}\, u = u_{xx}^2 + u_{yy}^2 + \hbar u^{\xi}(1-ru^{\eta}) \quad (14)$$

With the given initial condition $u(x,y,0)$.

Using (8) with $n=0$ and applying Laplace transform to equation (14), we get

$$\frac{s\tilde{u}(x,y,s)-u(x,y,0)}{s+\nu(1-s)} = \mathcal{L}\{u_{xx}^2+u_{yy}^2+\hbar u^{\xi}(1-ru^{\eta})\}, \quad (15)$$

which on simplifying reduces to

$$\tilde{u}(x,y,s) = \frac{u(x,y,0)}{s}+\frac{s+\nu(1-s)}{s}\mathcal{L}\{u_{xx}^2+u_{yy}^2+\hbar u^{\xi}(1-ru^{\eta})\} \quad (16)$$

Applying the inverse Laplace transform to (16), we get

$$u(x,y,t) = u(x,y,0)+\mathcal{L}^{-1}\left[\frac{s+\nu(1-s)}{s}\mathcal{L}\{u_{xx}^2+u_{yy}^2+\hbar u^{\xi}(1-ru^{\eta})\}\right]. \quad (17)$$

As HPM assumes the solution $u(x,y,t)$ to be of the form

$$u(x,y,t) = \sum_{n=0}^{\infty} p^n u_n(x,y,t),$$

We get

$$\sum_{n=0}^{\infty} p^n u_n(x,y,t) = u(x,y,0) + p\mathcal{L}^{-1}\left[\frac{s+\nu(1-s)}{s}\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right)^2+\frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right)^2+\hbar\left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right)^{\xi}\left(1-r\left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right)^{\eta}\right)\right\}\right] \quad (18)$$

Comparison of coefficients of like powers of $p$ on both sides gives

$$p^0: u_0(x,y,t) = u(x,y,0)$$

$$p^1: u_1(x,y,t) = \mathcal{L}^{-1}\left[\frac{s+\nu(1-s)}{s}\mathcal{L}\left\{\frac{\partial^2 u_0^2}{\partial x^2}+\frac{\partial^2 u_0^2}{\partial y^2}+\hbar {u_0}^{\xi}(1-r{u_0}^{\eta})\right\}\right]$$

$$p^2: u_2(x,y,t) = \mathcal{L}^{-1}\left[\frac{s+\nu(1-s)}{s}\mathcal{L}\left\{\frac{\partial^2(2u_0u_1)}{\partial x^2}+\frac{\partial^2(2u_0u_1)}{\partial y^2}+\hbar\xi {u_0}^{\xi-1}u_1-\hbar r(\xi+\eta){u_0}^{\xi+\eta-1}u_1\right\}\right] \quad (19)$$

and so on.

## 4. Solution of Biological Population Model

This section includes the application of iterative scheme of LHPM described in Section 3 to solve four examples for the population model and numerical simulation of the obtained solution.

### Example 4.1

In this example, we consider equation (9) with $\xi = 1, r = 0$ and $u(x, y, 0) = \sqrt{xy}$ (verification of Malthusian law).The exact solution of this equation is $u(x, y, t) = \sqrt{xy}\ e^{\hbar t}$.

Applying the scheme described in equation (13) and comparing the coefficients of like powers of $p$, we get the successive approximation as

$$p^0: u_0(x, y, t) = u(x, y, 0) = \sqrt{xy},$$

$$p^1: u_1(x, y, t) = \mathcal{L}^{-1}\left[\frac{(1-\nu)s^\nu+\nu}{B(\nu)s^\nu}\mathcal{L}\left\{\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + \hbar u_0\right\}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{(1-\nu)s^\nu + \nu}{B(\nu)s^\nu}\mathcal{L}\{\hbar\sqrt{xy}\}\right]$$

$$= \hbar\sqrt{xy}\left[\frac{1}{B(\nu)}\left\{(1- \nu) + \frac{\nu t^\nu}{\Gamma(\nu+1)}\right\}\right],$$

$$p^2: u_2(x, y, t) = \mathcal{L}^{-1}\left[\frac{(1-\nu)s^\nu + \nu}{B(\nu)s^\nu}\mathcal{L}\left\{\frac{\partial^2(2u_0u_1)}{\partial x^2} + \frac{\partial^2(2u_0u_1)}{\partial y^2} + \hbar u_1\right\}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{(1-\nu)s^\nu + \nu}{B(\nu)s^\nu}\mathcal{L}\left\{\hbar^2\sqrt{xy}\left[\frac{1}{B(\nu)}\left\{(1-\nu) + \frac{\nu t^\nu}{\Gamma(\nu+1)}\right\}\right]\right\}\right]$$

$$= \hbar^2\sqrt{xy}\left[\frac{1}{B(\nu)^2}\left\{(1-\nu)^2 + 2\nu(1-\nu)\frac{t^\nu}{\Gamma(\nu+1)} + \nu^2\frac{t^{2\nu}}{\Gamma(2\nu+1)}\right\}\right]$$

Similarly $u_3(x, y, t) = \hbar^3\sqrt{xy}\left[\frac{1}{B(\nu)^3}\left\{(1-\nu)^3 + 3\nu(1-\nu)^2\frac{t^\nu}{\Gamma(\nu+1)} + 3\nu^2(1-\nu)\frac{t^{2\nu}}{\Gamma(2\nu+1)} + \nu^3\frac{t^{3\nu}}{\Gamma(3\nu+1)}\right\}\right]$ (20)

And so on.

Accordingly, the approximate series solution of the population model (9) is

$$u(x, y, t) = \sum_{n=0}^{\infty} u_n(x, y, t) = \sqrt{xy} + \hbar\sqrt{xy}\left[\frac{1}{B(\nu)}\left\{(1-\nu) + \frac{\nu t^\nu}{\Gamma(\nu+1)}\right\}\right] +$$

$$\hbar^2\sqrt{xy}\left[\frac{1}{B(\nu)^2}\left\{(1-\nu)^2 + 2\nu(1-\nu)\frac{t^\nu}{\Gamma(\nu+1)} + \nu^2\frac{t^{2\nu}}{\Gamma(2\nu+1)}\right\}\right] + \hbar^3\sqrt{xy}\left[\frac{1}{B(\nu)^3}\left\{(1-\nu)^3 + 3\nu(1-\nu)^2\frac{t^\nu}{\Gamma(\nu+1)} + 3\nu^2(1-\nu)\frac{t^{2\nu}}{\Gamma(2\nu+1)} + \nu^3\frac{t^{3\nu}}{\Gamma(3\nu+1)}\right\}\right] + \cdots \quad (21)$$

For $\nu \to 1, B(\nu) = 1$,we have $u(x,y,t) = \sqrt{xy}\sum_{n=0}^{\infty}\frac{(\hbar t)^n}{\Gamma(n+1)} = \sqrt{xy}\ e^{\hbar t}$, (22)

which is the exact solution.

Now considering equation (14) with $\xi = 1, r = 0$ and $u(x,y,0) = \sqrt{xy}$ (verification of Malthusian law).

Applying the scheme described in equation (19) and comparing the coefficients of like powers of $p$, we get the successive approximation as

$p^0$: $u_0(x,y,t) = u(x,y,0) = \sqrt{xy}$

$$p^1: u_1(x,y,t) = \mathcal{L}^{-1}\left[\frac{s+\nu(1-s)}{s}\mathcal{L}\left\{\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + \hbar u_0\right\}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s+\nu(1-s)}{s}\mathcal{L}\left\{\frac{\partial^2 xy}{\partial x^2} + \frac{\partial^2 xy}{\partial y^2} + \hbar\sqrt{xy}\right\}\right]$$

$$= \hbar\sqrt{xy}\mathcal{L}^{-1}\left[\frac{s+\nu(1-s)}{s^2}\right] = \hbar\sqrt{xy}\{(1-\nu)+\nu t\},$$

$$p^2: u_2(x,y,t) = \mathcal{L}^{-1}\left[\frac{s+\nu(1-s)}{s}\mathcal{L}\left\{\frac{\partial^2(2u_0u_1)}{\partial x^2} + \frac{\partial^2(2u_0u_1)}{\partial y^2} + \hbar u_1\right\}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s+\nu(1-s)}{s}\mathcal{L}\left\{\hbar^2\sqrt{xy}\,\{(1-\nu)+\nu t\}\right\}\right]$$

$$= \hbar^2\sqrt{xy}\left[(1-\nu)^2 + 2\nu(1-\nu)t + \nu^2\frac{t^2}{2}\right],$$

Similarly $u_3(x,y,t) = \hbar^3\sqrt{xy}\left[(1-\nu)^3 + 3\nu(1-\nu)^2 t + 3\nu^2(1-\nu)\frac{t^2}{2} + \nu^3\frac{t^3}{6}\right]$, (23)

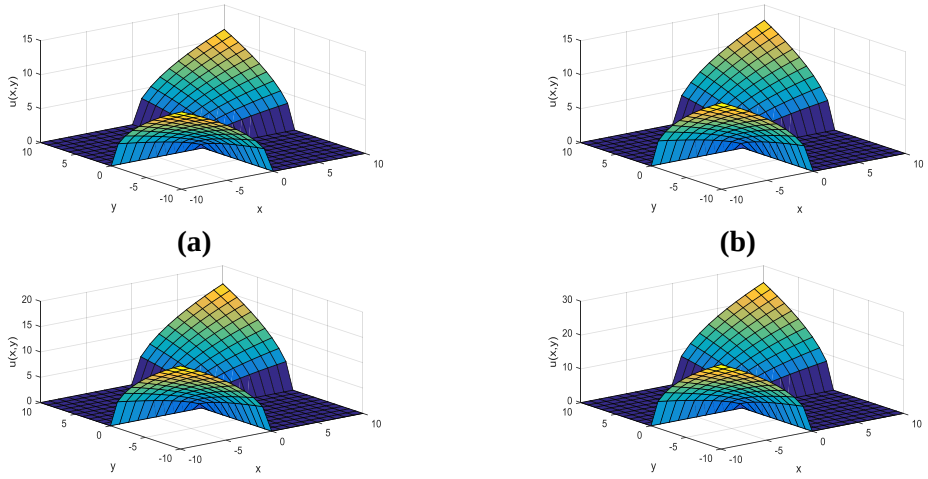
and so on.

Accordingly, the approximate series solution of the population model (14) is

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t) = \sqrt{xy} + \hbar\sqrt{xy}\{(1-\nu)+\nu t\} + \hbar^2\sqrt{xy}\left[(1-\nu)^2 + 2\nu(1-\nu)t + \nu^2\frac{t^2}{2}\right] + \hbar^3\sqrt{xy}\left[(1-\nu)^3 + 3\nu(1-\nu)^2 t + 3\nu^2(1-\nu)\frac{t^2}{2} + \nu^3\frac{t^3}{6}\right] + \cdots \quad (24)$$

For $\nu \to 1$,we have $u(x,y,t) = \sqrt{xy}\sum_{n=0}^{\infty}\frac{(\hbar t)^n}{\Gamma(n+1)} = \sqrt{xy}\ e^{\hbar t}$, (25)

which is the exact solution.




**(a)** **(b)**

**(c)** **(d)**

**FIGURE 1: Approximate solution $u(x, y, t)$ for equation (21) with $\hbar$ =.1, $t = 10$: (a) $\nu = .25$; (b) $\nu = .5$; (c) $\nu = .75$; (d) $\nu = 1$.**

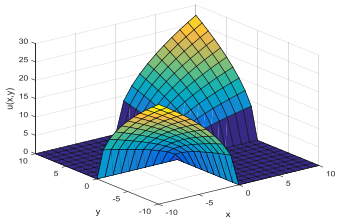
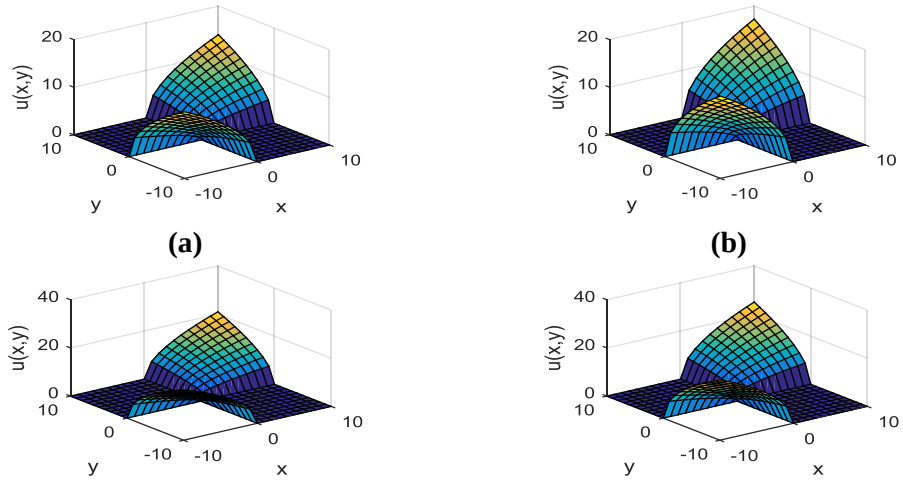



**FIGURE 2: Exact solution $u(x, y, t)$ given by equation (25)**




**(a)** **(b)**

**(c)** **(d)**

**FIGURE 3: Approximate solution $u(x, y, t)$ for equation.(24) with $\hbar$ =.1, $t = 10$: (a) $\nu = .25$; (b) $\nu = .5$; (c) $\nu = .75$; (d) $\nu = 1$.**

**Example 4.2** As second example, we solve equation (9) with $\xi = 1, \eta = 1$ and initial condition being $u(x, y, 0) = e^{\sqrt{\frac{\hbar r}{8}}(x+y)}$ (Verhulst law). The exact solution of this equation is $u(x, y, t) = e^{\sqrt{\frac{\hbar r}{8}}(x+y)+\hbar t}$.

Applying the scheme described in equation (13) and comparing the coefficients of like powers of $p$, we get the successive approximation as

$$p^0: u_0(x, y, t) = u(x, y, 0) = e^{\sqrt{\frac{\hbar r}{8}}(x+y)},$$

$$p^1: u_1(x, y, t) = \mathcal{L}^{-1}\left[\frac{(1-\nu)s^\nu+\nu}{B(\nu)s^\nu}\mathcal{L}\left\{\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + \hbar u_0 - \hbar r u_0^2\right\}\right]$$

$$= \hbar e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\mathcal{L}^{-1}\left[\frac{(1-\nu)s^\nu + \nu}{B(\nu)s^{\nu+1}}\right] = \hbar e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\left[\frac{1}{B(\nu)}\left\{(1-\nu) + \frac{\nu t^\nu}{\Gamma(\nu+1)}\right\}\right]$$

$$p^2: u_2(x, y, t) = \mathcal{L}^{-1}\left[\frac{(1-\nu)s^\nu + \nu}{B(\nu)s^\nu}\mathcal{L}\left\{\frac{\partial^2(2u_0u_1)}{\partial x^2} + \frac{\partial^2(2u_0u_1)}{\partial y^2} + \hbar u_1 - 2\hbar r u_0 u_1\right\}\right]$$

$$= \hbar^2 e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\left[\frac{1}{B(\nu)^2}\left\{(1-\nu)^2 + 2\nu(1-\nu)\frac{t^\nu}{\Gamma(\nu+1)} + \nu^2\frac{t^{2\nu}}{\Gamma(2\nu+1)}\right\}\right]$$

Similarly $u_3(x, y, t) = \hbar^3 e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\left[\frac{1}{B(\nu)^3}\left\{(1-\nu)^3 + 3\nu(1-\nu)^2\frac{t^\nu}{\Gamma(\nu+1)} + 3\nu^2(1-\nu)\frac{t^{2\nu}}{\Gamma(2\nu+1)} + \nu^3\frac{t^{3\nu}}{\Gamma(3\nu+1)}\right\}\right]$ (26)

and so on.

Accordingly, the approximate series solution of the population model (9) is

$$u(x, y, t) = \sum_{n=0}^{\infty} u_n(x, y, t) = e^{\sqrt{\frac{\hbar r}{8}}(x+y)} + \hbar e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\left[\frac{1}{B(\nu)}\left\{(1-\nu) + \frac{\nu t^\nu}{\Gamma(\nu+1)}\right\}\right] +$$
$$\hbar^2 e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\left[\frac{1}{B(\nu)^2}\left\{(1-\nu)^2 + 2\nu(1-\nu)\frac{t^\nu}{\Gamma(\nu+1)} + \nu^2\frac{t^{2\nu}}{\Gamma(2\nu+1)}\right\}\right] +$$
$$\hbar^3 e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\left[\frac{1}{B(\nu)^3}\left\{(1-\nu)^3 + 3\nu(1-\nu)^2\frac{t^\nu}{\Gamma(\nu+1)} + 3\nu^2(1-\nu)\frac{t^{2\nu}}{\Gamma(2\nu+1)} + \nu^3\frac{t^{3\nu}}{\Gamma(3\nu+1)}\right\}\right] + \cdots$$ (27)

For $\nu \to 1, B(\nu) = 1$,we have

$$u(x, y, t) = e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\sum_{n=0}^{\infty}\frac{(\hbar t)^n}{\Gamma(n+1)} = e^{\sqrt{\frac{\hbar r}{8}}(x+y)+\hbar t}.$$ (28)

Now considering equation (14) with $\xi = 1, \eta = 1$ and initial condition being $u(x, y, 0) = e^{\sqrt{\frac{\hbar r}{8}}(x+y)}$ (Verhulst law).

Applying the scheme described in equation (19) and comparing the coefficients of like powers of $p$, we get the successive approximation as

$$p^0: u_0(x, y, t) = u(x, y, 0) = e^{\sqrt{\frac{\hbar r}{8}}(x+y)}.$$

$$p^1: u_1(x, y, t) = \mathcal{L}^{-1}\left[\frac{s + v(1-s)}{s}\mathcal{L}\left\{\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + \hbar u_0 - \hbar r u_0^2\right\}\right]$$

$$= \hbar e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\mathcal{L}^{-1}\left[\frac{s + v(1-s)}{s^2}\right]$$

$$= \hbar e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\{(1-v) + vt\}$$

$$p^2: u_2(x, y, t) = \mathcal{L}^{-1}\left[\frac{s + v(1-s)}{s}\mathcal{L}\left\{\frac{\partial^2 (2u_0 u_1)}{\partial x^2} + \frac{\partial^2 (2u_0 u_1)}{\partial y^2} + \hbar u_1 - 2\hbar r u_0 u_1\right\}\right]$$

$$= \hbar^2 e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\left[(1-v)^2 + 2v(1-v)t + v^2\frac{t^2}{2}\right],$$

Similarly $u_3(x, y, t) = \hbar^3 e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\left[(1-v)^3 + 3v(1-v)^2 t + 3v^2(1-v)\frac{t^2}{2} + v^3\frac{t^3}{6}\right]$ (29)

and so on.

Accordingly, the approximate series solution of the population model (14) is

$$u(x, y, t) = \sum_{n=0}^{\infty} u_n(x, y, t) = e^{\sqrt{\frac{\hbar r}{8}}(x+y)} + \hbar e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\{(1-v) + vt\} + \hbar^2 e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\left[(1-v)^2 + 2v(1-v)t + v^2\frac{t^2}{2}\right] + \hbar^3 e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\left[(1-v)^3 + 3v(1-v)^2 t + 3v^2(1-v)\frac{t^2}{2} + v^3\frac{t^3}{6}\right] + \cdots \quad (30)$$

For $v \to 1$,we have $u(x, y, t) = e^{\sqrt{\frac{\hbar r}{8}}(x+y)}\sum_{n=0}^{\infty}\frac{(\hbar t)^n}{\Gamma(n+1)} = e^{\sqrt{\frac{\hbar r}{8}}(x+y)+\hbar t}$, (31)

which is the exact solution.

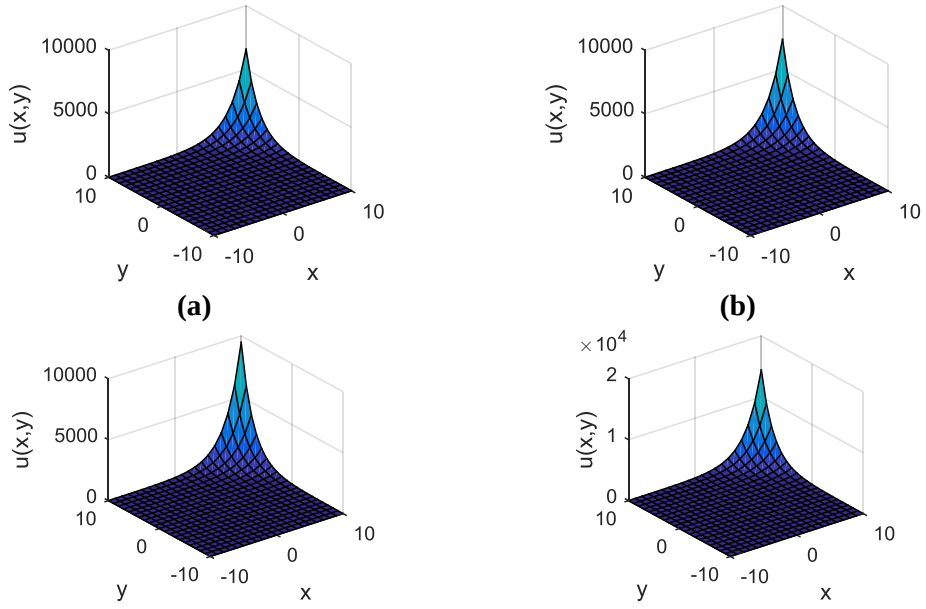



**FIGURE 4: Approximate solution $u(x, y, t)$ for equation (27) with $\hbar$ =.1, $t = 10, r = 15$: (a) $\nu = .25$; (b) $\nu = .5$; (c) $\nu = .75$; (d) $\nu = 1$.**

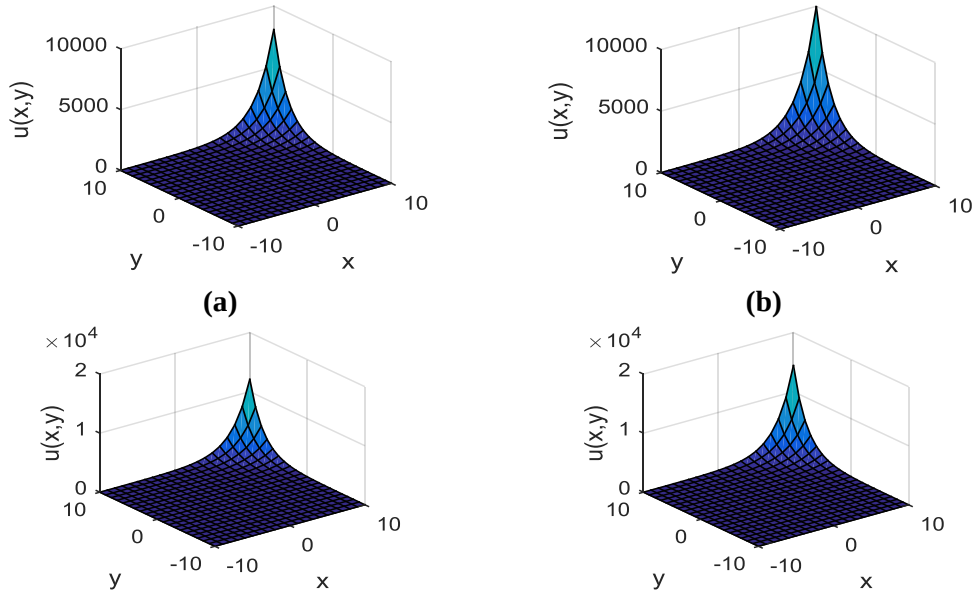



**FIGURE 5: Approximate solution $u(x, y, t)$ for equation (30) with $\hbar$ =.1, $t = 10, r = 15$: (a) $\nu = .25$; (b) $\nu = .5$; (c) $\nu = .75$; (d) $\nu = 1$.**

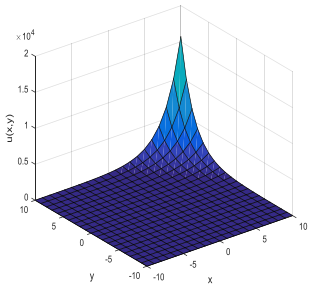



**FIGURE 6: Exact solution $u(x, y, t)$ given by equation (31)**

**Example 4.3**

In third example, we solve equation (9) with $\xi = -1, \eta = 1$ with the initial condition being $(x, y, 0) = \sqrt{\frac{\hbar r}{4}x^2 + \frac{\hbar r}{4}y^2 + y + 5}$. .

The exact solution of this equation is $u(x, y, t) = \sqrt{\frac{\hbar r}{4}x^2 + \frac{\hbar r}{4}y^2 + y + 2\hbar t + 5}$ [18]. (32)

Applying the scheme described in eq. (13) and comparing the coefficients of like powers of $p$, we get the successive approximation as

$$p^0: u_0(x, y, t) = u(x, y, 0) = \sqrt{\frac{\hbar r}{4}x^2 + \frac{\hbar r}{4}y^2 + y + 5}.$$

$$p^1: u_1(x, y, t) = \mathcal{L}^{-1}\left[\frac{(1-\nu)s^\nu + \nu}{B(\nu)s^\nu}\mathcal{L}\left\{\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + \hbar {u_0}^{-1} - \hbar r\right\}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{(1-\nu)s^\nu+\nu}{B(\nu)s^\nu}\mathcal{L}\{\hbar {u_0}^{-1}\}\right] = \hbar {u_0}^{-1}\left[\frac{1}{B(\nu)}\left\{(1-\nu) + \frac{\nu t^\nu}{\Gamma(\nu+1)}\right\}\right],$$

$$p^2: u_2(x, y, t) = \mathcal{L}^{-1}\left[\frac{(1-\nu)s^\nu+\nu}{B(\nu)s^\nu}\mathcal{L}\left\{\frac{\partial^2(2u_0u_1)}{\partial x^2} + \frac{\partial^2(2u_0u_1)}{\partial y^2} - \hbar {u_0}^{-2}u_1\right\}\right],$$

$$= -\hbar^2 {u_0}^{-3}\left[\frac{1}{B(\nu)^2}\left\{(1-\nu)^2 + 2\nu(1-\nu)\frac{t^\nu}{\Gamma(\nu+1)} + \nu^2\frac{t^{2\nu}}{\Gamma(2\nu+1)}\right\}\right]$$

Similarly $u_3(x, y, t) = \hbar^3 {u_0}^{-5}\left[\frac{1}{B(\nu)^3}\left\{(1-\nu)^3 + 3\nu(1-\nu)^2\frac{t^\nu}{\Gamma(\nu+1)} + 3\nu^2(1-\nu)\frac{t^{2\nu}}{\Gamma(2\nu+1)} + \nu^3\frac{t^{3\nu}}{\Gamma(3\nu+1)}\right\}\right]$, (33)

and so on.

Accordingly, the approximate series solution of the population model (9) is

$u(x, y, t) =$

$\sum_{n=0}^{\infty} u_n(x, y, t) = u_0 + \hbar {u_0}^{-1}\left[\frac{1}{B(\nu)}\left\{(1-\nu) + \frac{\nu t^\nu}{\Gamma(\nu+1)}\right\}\right] - \hbar^2 {u_0}^{-3}\left[\frac{1}{B(\nu)^2}\left\{(1-\nu)^2 +\right.\right.$

$2\nu(1-\nu)\frac{t^{\nu}}{\Gamma(\nu+1)}+\nu^2\frac{t^{2\nu}}{\Gamma(2\nu+1)}\Big\}\Big]+\hbar^3{u_0}^{-5}\Big[\frac{1}{B(\nu)^3}\Big\{(1-\nu)^3+3\nu(1-\nu)^2\frac{t^{\nu}}{\Gamma(\nu+1)}+$
$3\nu^2(1-\nu)\frac{t^{2\nu}}{\Gamma(2\nu+1)}+\nu^3\frac{t^{3\nu}}{\Gamma(3\nu+1)}\Big\}\Big]+\cdots$ (34)

For $\nu\to 1, B(\nu)=1$, we have

$u(x,y,t)=u_0+\hbar t{u_0}^{-1}-\frac{\hbar^2t^2}{2}{u_0}^{-3}+\frac{\hbar^3t^3}{6}{u_0}^{-5}-\cdots.$

Now considering equation (14) with $\xi=-1,\eta=1$ and initial condition being $(x,y,0)=\sqrt{\frac{\hbar r}{4}x^2+\frac{\hbar r}{4}y^2+y+5}$ .

Applying the scheme described in equation (19) and comparing the coefficients of like powers of $p$, we get the successive approximation as

$p^0$: $u_0(x,y,t)=u(x,y,0)=\sqrt{\frac{\hbar r}{4}x^2+\frac{\hbar r}{4}y^2+y+5},$

$p^1$: $u_1(x,y,t)=\mathcal{L}^{-1}\Big[\frac{s+\nu(1-s)}{s}\mathcal{L}\Big\{\frac{\partial^2u_0^2}{\partial x^2}+\frac{\partial^2u_0^2}{\partial y^2}+\hbar{u_0}^{-1}-\hbar r\Big\}\Big]$

$=\hbar{u_0}^{-1}\mathcal{L}^{-1}\left[\frac{s+\nu(1-s)}{s^2}\right]=\hbar{u_0}^{-1}\{(1-\nu)+\nu t\}$

$p^2$: $u_2(x,y,t)=\mathcal{L}^{-1}\Big[\frac{s+\nu(1-s)}{s}\mathcal{L}\Big\{\frac{\partial^2(2u_0u_1)}{\partial x^2}+\frac{\partial^2(2u_0u_1)}{\partial y^2}-\hbar{u_0}^{-2}u_1\Big\}\Big]$

$=-\hbar^2{u_0}^{-3}\ \Big[(1-\nu)^2+2\nu(1-\nu)t+\nu^2\frac{t^2}{2}\Big],$

Similarly $u_3(x,y,t)=\hbar^3{u_0}^{-5}\Big[(1-\nu)^3+3\nu(1-\nu)^2t+3\nu^2(1-\nu)\frac{t^2}{2}+\nu^3\frac{t^3}{6}\Big]$, (35)

and so on.

Accordingly, the approximate series solution of the population model (14) is

$u(x,y,t)=\sum_{n=0}^{\infty}u_n(x,y,t)=u_0+\hbar{u_0}^{-1}\{(1-\nu)+\nu t\}-\hbar^2{u_0}^{-3}\ \Big[(1-\nu)^2+$
$2\nu(1-\nu)t+\nu^2\frac{t^2}{2}\Big]+\hbar^3{u_0}^{-5}\Big[(1-\nu)^3+\ 3\nu(1-\nu)^2t+3\nu^2(1-\nu)\frac{t^2}{2}+\nu^3\frac{t^3}{6}\Big]+\cdots.$
(36)

For $\nu\to 1$,we have $u(x,y,t)=u_0+\hbar t{u_0}^{-1}-\frac{\hbar^2t^2}{2}{u_0}^{-3}+\frac{\hbar^3t^3}{6}{u_0}^{-5}-\cdots.$

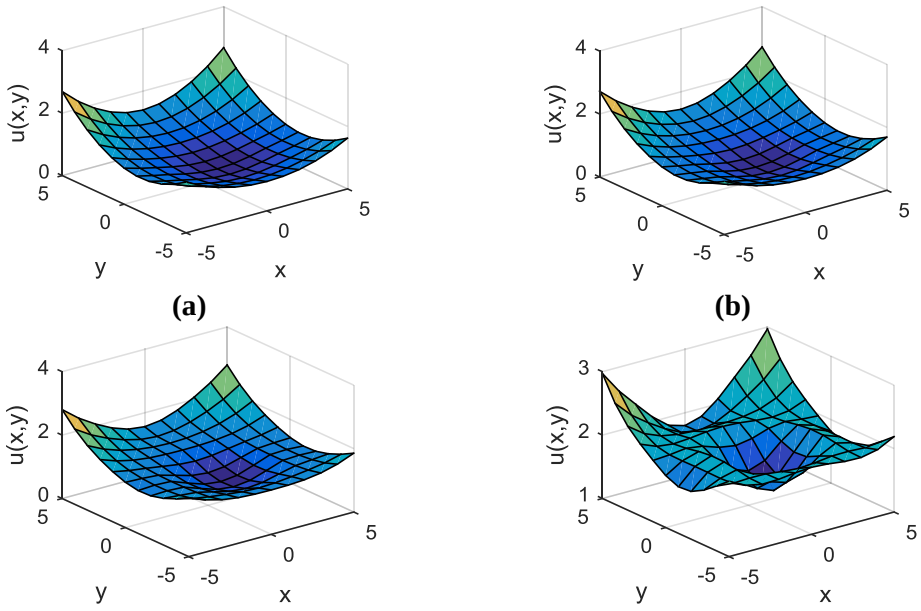



**FIGURE 7: Approximate solution $u(x, y, t)$ for equation(34) with $\hbar$ =.1, $t = 10, r = 15$: (a) $v = .25$; (b) $v = .5$; (c) $v = .75$; (d) $v = 1$.**

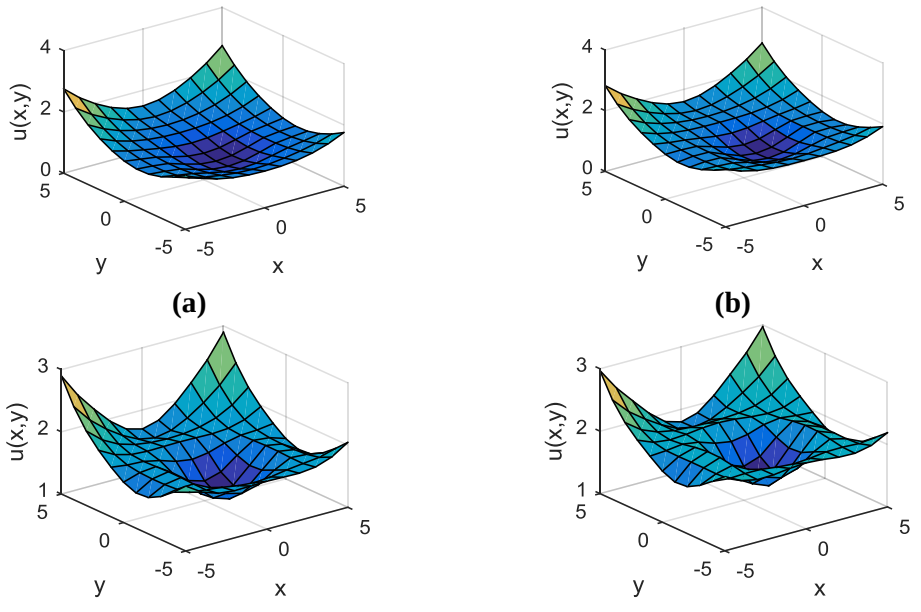



**FIGURE 8: Approximate solution $u(x, y, t)$ for equation (36) with $\hbar$ =.1, $t = 10, r = 15$: (a) $v = .25$; (b) $v = .5$; (c) $v = .75$; (d) $v = 1$.**

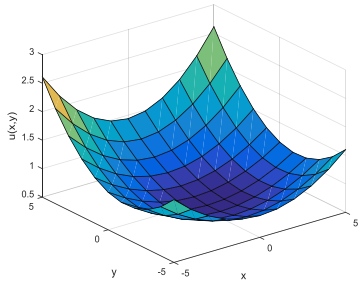



**FIGURE 9: Exact solution $u(x, y, t)$ given by equation (32)**

## 5. Conclusion

In this paper an efficient technique LHPM has been used to approximate the solution of fractional biological population model, fractional derivative being taken in Atangana - Baleanu and Caputo - Fabrizio in Liouville - Caputo sense. The solutions obtained exhibit that for the same model the solutions corresponding to the CFC operator are the particular cases of the solutions for the ABC operator. The solution behaviour for the various forms of the model is numerically analyzed corresponding to the parameters $\nu, \hbar, \xi, \eta$ and $r$ and compared with the exact solution .Thus Homotopy perturbation method is an easy to use, powerful and effective technique for linear as well as non-linear partial differential equations. Matlab R 2015a has been used for the graphs presented.

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