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CONFORMAL β-CHANGE OF FINSLER METRIC BY h-VECTOR

H.S. Shukla, Manmohan Pandey and **Neelam Mishra**
Department of Mathematics and Statistics
D.D.U. Gorakhpur University, Gorakhpur 273009, India
Email: profhsshuklagkp@rediffmail.com, manmohanp752@gmail.com, pneelammishra@gmail.com

Abstract: In this paper we investigate the necessary and sufficient conditions for conformal β-change of Finsler metric by h-vector to be a projective change. We also search out the condition under which this change converts a Douglas space into a Douglas space.

Key Words: Finsler metric, conformal β-change, h-vector, projective change, Douglas space.

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1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space on the differentiable manifold M^n equipped with the fundamental function L(x,y). Shibata [14], Prasad and Bindu Kumari[13] have studied the general case of β-change, that is, $L^*(x, y) = f(L, \beta)$, where f is a positively homogeneous function of degree one in L and β, and β given by$\beta(x,y) = b_i(x)y^i$ is a one-form on M^n. The β-change of special Finsler spaces has been studied by Shukla, Pandey and Khageshwar Mandal[19].

The conformal theory of Finsler space was initiated by Knebelman[8] in 1929 and has been investigated in detail by many authors (Hashiguchi[4], Izumi[5,6] and Kitayama[7]). The conformal change is defined as $L'(x, y) = e^{\sigma(x)}L(x, y)$, where σ(x) is a function of position only and known as conformal factor. In 2008, Abed [1,2] introduced the change $L''(x, y) = e^{\sigma(x)}L(x, y) + \beta(x, y)$,which he called a β-conformal change. In 2009 and 2010, Nabil L. Youssef et al. [20,21] introduced the transformation$L'''(x, y) = f(e^{\sigma}L, \beta)$, which is β-change of conformally changed Finsler metric L. They have not only established the relationships between some important tensors of (M^n, L) and the corresponding tensors of (M^n, L'''), but have also studied several properties of this change.

The order of combination of the above two changes has been reversed in paper [18], where authors have applied β-change first and conformal change afterwards, i.e.,

$$L^{**}(x, y) = e^{\sigma(x)}f(L(x, y), \beta(x, y))$$

where$\sigma(x)$ is a function of x, $\beta(x, y) = b_i(x)y^i$ is a 1-form. They have called this change as conformal β-change of Finslermetric. In this paper they have investigated the condition under which a conformal β-change of Finsler metric leads a Douglas space into a Douglas space. They have also found the necessary and sufficient conditions for this change to be a projective change. Some properties of the conformal β-change have been studied in [17].

Now we assume that b_i in β is a function of x and y both and satisfies the condition $\frac{\partial b_i}{\partial y^j} = h_{ij}$, where ρ is a scalar function of x, y and h_{ij} is the angular metric tensor of (M^n, L). Thus we have supposed that b_i in β is an h-vector and with this supposition we deal with the change

$$\bar{L}(x, y) = e^{\sigma(x)} f(L(x, y), \beta(x, y)). \tag{1}$$

We call this change as conformal β-change of Finsler metric by h-vector. In this paper we investigate the necessary and sufficient condition for this change to be a projective change.We also search out the condition under which this change converts a Douglas space into a Douglas space.

The Finsler space equipped with the metric $\bar{L}$ as given by (1) will be denoted by $\bar{F}^n$. Throughout the paper the quantities corresponding to $\bar{F}^n$ will be denoted by putting bar on the top of them. Homogeneiyof f gives

$$Lf_1 + \beta f_2 = f \tag{2}$$

where subscripts 1 and 2 denote the partial derivatives with respect to L and β respectively. Differentiating above equations with respect to L and β respectively, we get

$$Lf_{12} + \beta f_{22} = 0, \ Lf_{11} + \beta f_{21} = 0. \tag{3}$$

Hence we have

$$f_{11}/\beta^2 = -f_{12}/L\beta = f_{22}/L^2, \tag{4}$$

which gives

$$f_{11} = \beta^2 w, \ f_{12} = -L\beta w, \ f_{22} = L^2 w, \tag{5}$$

where Weierstrass function w is positively homogeneous of degree –3 in L and β. Therefore

$$Lw_1 + \beta w_2 + 3w = 0, \tag{6}$$

where w_1 and w_2 are positively homogeneous of degree -4 in L and β. Throughout the paper we frequently use the above equations without quoting them. Also we have assumed that f is not linear function of L and β so that $w \neq 0$.

The fundamental metric tensor g_{ij}, the normalized element of support l_i and angular metric tensor h_{ij} of F^n are given by

$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$, $l_i = \frac{\partial L}{\partial y^i}$ and $h_{ij} = L\frac{\partial^2 L}{\partial y^i \partial y^j} = g_{ij} - l_i l_j$.

We shall denote the partial derivative with respect to x^i and y^i by ∂_i and $\dot{\partial}_i$ respectively and write

$L_i = \dot{\partial}_i L$, $L_{ij} = \dot{\partial}_i \dot{\partial}_j L$, $L_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L$.

Then

$L_i = l_i$, $L^{-1}h_{ij} = L_{ij}$.

The geodesics of F^n are given by the system of differential equations

$$\frac{d^2x^i}{ds^2} + 2G^i\left(x\frac{dx}{ds}\right) = 0,$$

where $G^i(x, y)$ are positively homogeneous of degree 2 in y^i and are given by

$$2G^i = g^{ij}\left(y^r \dot{\partial}_j \partial_r F - \partial_j F\right), F = \frac{L^2}{2},$$

where g^{ij} are the inverse of g_{ij} .

The well known Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of a Finsler space is constructed from the quantity G^i appearing in the equation of geodesic and is given by [10]:

$$G^i_j = \dot{\partial}_j G^i, \qquad G^i_{jk} = \dot{\partial}_k G^i_j .$$

The Cartan's connection $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$ is constructed from the metric function L by the following five axioms[10]:

(i) $g_{ij|k} = 0$, (ii) $g_{ij}|_k = 0$, (iii) $F^i_{jk} = F^i_{kj}$ (iv) $F^i_{0k} = G^i_k$, (v) $C^i_{jk} = C^i_{kj}$,

where $_{|k}$ and $|_k$ denote h- and v-covariant derivatives with respect to $C\Gamma$. It is clear that the h-covariant derivative of L with respect to $B\Gamma$ and $C\Gamma$ is the same and vanishes identically. Furthermore, the h-covariant derivatives of L_i and L_{ij} with respect to $C\Gamma$ are also zero.

We shall use the following result obtained by Shukla, Pandey and Joshi [15]:

$$\dot{\partial}_k \rho = -\frac{\rho}{L} l_k, \ n > 2. \tag{7}$$

We shall write

$$2r_{ij} = b_{i|j} + b_{j|i}, 2s_{ij} = b_{i|j} - b_{j|i}. \tag{8}$$

2. Fundamental Quantities of $(\mathbf{M^n}, \bar{\mathbf{L}})$

To find the relation between fundamental quantities of (M^n, L) and $(M^n, \bar{L})$, we use the following results:

$$\dot{\partial}_i \beta = b_i,\ \dot{\partial}_i L = l_i,\ \dot{\partial}_j l_i = L^{-1} h_{ij}\ . \tag{9}$$

The successive differentiation of (1) with respect to y^i, y^j and y^k gives respectively:

$$\bar{L}_i = e^{\sigma}(f_1 L_i + f_2 b_i)\ : \tag{10a}$$

$$\bar{L}_{ij} = e^{\sigma}[\beta^2 w L_i L_j - \beta L w\left(L_i b_j + L_j L_i\right) + (f_1 + \rho L f_2) L_{ij} + L^2 w b_i b_j], \tag{10b}$$

where $\frac{\partial b^i}{\partial y^j} = \rho h_{ij} = \rho L L_{ij}$;

$$\begin{aligned}\bar{L}_{ijk} = e^{\sigma}[&(f_1 + \rho L f_2) L_{ijk} + \left(\beta^2 w - \rho\beta L^2 w\right)(L_i L_{jk} + L_j L_{ik} + L_k L_{ij}) \\ &+(-\beta L w + \rho L^3 w)(b_i L_{jk} + b_j L_{ik} + b_k L_{ij}) \\ &+\left(2\beta w + \beta^2 w_2\right)(L_i L_j b_k + L_i L_k b_j + L_j L_k b_i) \\ &-(Lw + L\beta w_2)(L_i b_j b_k + L_j b_i b_k + L_k b_i b_j) \\ &+\beta^2 w_1 L_i L_j L_k - L^2 w_2 b_i b_j b_k]\ .\end{aligned} \tag{10c}$$

The differentiation of (10 a) with respect to x^j and (10 b) with respect to x^k gives respectively

$$\begin{aligned}\partial_j \bar{L}_i = e^{\sigma}[&(f_1 L_i + f_2 b_i)\,\partial_j \sigma + (\beta^2 w L_i - \beta L w b_i)\ \partial_j L \\ &+ (-\beta L w L_i + L^2 w b_i)\,\partial_j \beta + f_1\,\partial_j L_i + f_2\,\partial_j b_i],\end{aligned} \tag{11a}$$

$$\begin{aligned}\partial_k \bar{L}_{ij} = e^{\sigma}[&\{(f_1 + \rho L f_2) L_{ij} + \beta^2 w L_i L_j - \beta L w\left(L_i b_j + L_j b_i\right) \\ &+ L^2 w b_i b_j\}\sigma_k + (f_1 + \rho L f_2)\,\partial_k L_{ij} + \{\left(\beta^2 w + \rho f_2 - \rho\beta L^2 w\right) L_{ij} \\ &+\beta^2 w_1 L_i L_j - (\beta w + \beta L w_1)\left(L_i b_j + L_j b_i\right) + (2Lw + L^2 w_1) b_i b_j\}\,\partial_k L \\ &+\{(-\beta L w + \rho L^3 w) L_{ij} + (2\beta w + \beta^2 w_2) L_i L_j - (Lw + \beta L w_2)\left(L_i b_j + L_j L_i\right) \\ &+ L^2 w_2 b_i b_j\}\,\partial_k \beta + (\beta^2 w L_j - \beta L w b_j)\,\partial_k L_i + (\beta^2 w L_i - \beta L w b_i)\,\partial_k L_j \\ &+(L^2 w b_j - \beta L w L_j)\,\partial_k b_i + (L^2 w b_i - \beta L w L_i)\,\partial_k b_j + L f_2 L_{ij}\,\partial_k \rho\ ]\ .\end{aligned} \tag{11b}$$

3. Difference Tensor of Conformal β-change of Finsler Metric by h-Vector

We may put

$$\bar{G}^i = G^i + D^i\ . \tag{12}$$

Then $\bar{G}^i_j = G^i_j + D^i_j$ and $\bar{G}^i_{jk} = G^i_{jk} + D^i_{jk}$, where $D^i_j = \dot{\partial}_j D^i$ and $D^i_{jk} = \dot{\partial}_k D^i_j$. The tensors D^i, D^i_j and D^i_{jk} are positively homogeneous in y^i of degree two, one and zero respectively.

To find difference tensor D^i we deal with equation $L_{ij|k} = 0$ [16], i.e.

$$\partial_k L_{ij} - L_{ijr} G^r_k - L_{rj} F^r_{ik} - L_{ir} F^r_{jk} = 0. \tag{13}$$

Since $\bar{L}_{ij|\bar{}k} = 0$ in $\bar{F}^n$, after using (12), we have

$$\partial_k \bar{L}_{ij} - \bar{L}_{ijr}(G^r_k + D^r_k) - \bar{L}_{rj}(F^r_{ik} + {}^cD^r_{ik}) - \bar{L}_{ir}(F^r_{jk} + {}^cD^r_{jk}) = 0, \tag{14}$$

where $\bar{F}^i_{jk} - F^i_{jk} = {}^cD^i_{jk}$ [9].

Substituting in equation (14) the values of $\partial_k \bar{L}_{ij}, \bar{L}_{ir}$ and $\bar{L}_{ijr}$ from (11 b), (10 b) and (10 c) respectively, using (13) and then contracting the equation thus obtained with y^k we get

$$\begin{aligned}
&\{(f_1 + \rho L f_2)L_{ij} + \beta^2 w L_i L_j - \beta L w(L_i b_j + L_j L_i) + L^2 w b_i b_j\}\sigma_0 \qquad (15)\\
&- 2\{(f_1 + \rho L f_2)L_{ijr} + (\beta^2 w - \rho\beta L^2 w)(L_i L_{jr} + L_j L_{ri} + L_r L_{ij})\\
&+(-\beta L w + \rho L^3 w)(b_i L_{jr} + b_j L_{ir} + b_r L_{ij}) + (2\beta w + \beta^2 w_2)(L_i L_j b_r + L_i L_r b_j + L_j L_r b_i)\\
&- (Lw + L\beta w_2)(L_i b_j b_r + L_j b_r b_i + L_r b_i b_j) + \beta^2 w_1 L_i L_j L_r\\
&+ L^2 w_2 b_i b_j b_r\}D^r\\
&+\{(f_1 + \rho L f_2)L_{jr} + \beta^2 w L_r L_j - \beta L w(L_j b_r + L_r b_j) + L^2 w b_r b_j\}D^r_i\\
&+\{(f_1 + \rho L f_2)L_{ir} + \beta^2 w L_r L_i - \beta L w(L_i b_r + L_r b_i) + L^2 w b_r b_i\}D^r_j\\
&- (L^2 w b_j - \beta L w L_j)(r_{0i} + s_{0i}) - (L^2 w b_i - \beta L w L_i)(r_{0j} + s_{0j})\\
&-\{(-\beta L w + \rho L^3 w)L_{ij} + (2\beta w + \beta^2 w_2)L_i L_j\\
&-(Lw + \beta L w_2)(L_i b_j + L_j b_i) + L^2 w_2 b_i b_j\}r_{00}\\
&-L f_2 y^k L_{ij}\, \partial_k \rho - 2\rho f_2 L_r L_{ij} G^r = 0,
\end{aligned}$$

where we have used the fact that$D^i_{jk} y^k = {}^cD^i_{jk} y^k = D^i_j$ [9], and ‘0’ stands for contraction with respect to y^i,viz. $r_{0k} = r_{ik} y^i, r_{00} = r_{ij} y^i y^j$.

Next, we deal with $\bar{L}_{i|\bar{}j} = 0$, i.e. $\partial_j \bar{L}_i - \bar{L}_{ir}\bar{G}^r_j - \bar{L}_r \bar{F}^r_{ij} = 0$ to get

$$\partial_j \bar{L}_i - \bar{L}_{ir}(G^r_j + D^r_j) - \bar{L}_r(F^r_{ij} + {}^cD^r_{ij}) = 0. \tag{16}$$

Putting the values of $\partial_j \bar{L}_i$, $\bar{L}_{ir}$ and $\bar{L}_r$ from (11 a), (10 b), (10 a) respectively in (16), using equation $\partial_j L_i - L_{ir} G^r_j - L_r F^r_{ij} = 0$ and then rearranging the terms, we get

$$\begin{aligned}
f_2 b_{i|J} &= (f_1 L_i + f_2 b_i)\sigma_j + \{(f_1 + \rho L f_2)L_{ir} + \beta^2 w L_r L_i - \beta L w(L_i b_r + L_r b_i) +\\
&L^2 w b_r b_i\}D^r_j + (\beta L w L_i - L^2 w b_i)(r_{0j} + s_{0j}) + (f_1 L_r + f_2 b_r)D^r_{ij} - \rho L f_2 L_{ir} F^r_{ij},
\end{aligned}$$

which, after using (8), gives

$$2f_2 r_{ij} = -(f_1 L_i + f_2 b_i)\sigma_j - \left(f_1 L_j + f_2 b_j\right)\sigma_i$$
$$+ \left\{(f_1 + \rho L f_2)L_{ir} + \beta^2 w L_r L_i - \beta L w(L_i b_r + L_r b_i) + L^2 w b_r b_i\right\}D^r_j$$
$$+ \left\{(f_1 + \rho L f_2)L_{jr} + \beta^2 w L_r L_j - \beta L w\left(L_j b_r + L_r b_j\right) + L^2 w b_r b_i\right\}D^r_i$$
$$+ (\beta L w L_i - L^2 w b_i)\left(r_{0j} + s_{0j}\right)$$
$$+\left(\beta L w L_j - L^2 w b_j\right)(r_{0i} + s_{0i}) + 2(f_1 L_r + f_2 b_r)D^r_{ij}, \tag{17}$$

and

$$2f_2 s_{ij} = -(f_1 L_i + f_2 b_i)\sigma_j + \left(f_1 L_j + f_2 b_j\right)\sigma_i + \left\{(f_1 + \rho L f_2)L_{ir} + \beta^2 w L_r L_i - \right.$$
$$\left.\beta L w(L_i b_r + L_r b_i) + L^2 w b_r b_i\right\}D^r_j - \{(f_1 + \rho L f_2)L_{jr} + \beta^2 w L_r L_j$$
$$-\beta L w\left(L_j b_r + L_r b_j\right) + L^2 w b_r b_j\}D^r_i + (\beta L w L_i - L^2 w b_i)\left(r_{0j} + s_{0j}\right)$$
$$-\left(\beta L w L_j - L^2 w b_j\right)(r_{0i} + s_{0i}). \tag{18}$$

Subtracting (17) from (15) and contracting the resulting equation with y^i, we obtain

$$(f_1 L_j + f_2 b_j)\sigma_0 + (f_1 L + f_2 \beta)\sigma_j + \{-(f_1 + \rho L f_2)L_{jr} \tag{19}$$
$$+L\beta w L_j b_r + L\beta w L_r b_j - \beta^2 w L_j L_r - L^2 w b_j b_r\}D^r$$
$$-(L\beta w L_j - L^2 w b_j)\frac{r_{00}}{2} - (f_1 L_r + f_2 b_r)D^r_j + f_2 r_{0j} = 0.$$

Contracting (19) with y^j, we get

$$f\sigma_0 + f_2 r_{00} = 2(f_1 L_r + f_2 b_r)D^r\ . \tag{20}$$

Adding (15) and (18) and contracting the resulting equation with y^j, we get

$$\left\{(f_1 + \rho L f_2)L_{ir} + L\beta w(L_i b_r + L_r b_i) + \beta^2 w L_i L_r - L^2 w b_i b_r\right\}D^r \tag{21}$$
$$= (L^2 w b_i - L\beta w L_i)\frac{r_{00}}{2} + f_2 s_{i0} + L\beta w(L_i \beta - L b_i)\sigma_0$$
$$+\frac{1}{2}(f_1 L_i + f_2 b_i)\sigma_0 - \frac{1}{2}\sigma_i f.$$

In view of $LL_{ir} = g_{ir} - L_i L_r$, the equation (21) can be written as

$$f_2 s_{i0} - (L^2 w b_i - L\beta w L_i)\frac{r_{00}}{2} + \frac{1}{2}(f_1 L_i + f_2 b_i)\sigma_0 - \frac{1}{2}\sigma_i f \tag{22}$$
$$= \left(\rho f_2 + \frac{f_1}{L}\right)g_{ir}D^r + \left\{\left(-\frac{f_1}{L} + \beta^2 w - \rho f_2\right)L_i - L\beta w b_i\right\}L_r D^r$$
$$(L^2 w b_i - L\beta w L_i)b_r D^r.$$

Contracting (22) with $b^i = g^{ij}b_j$, we get

$$\left\{-\frac{f_1\beta}{L^2} - L\beta w - \rho L f_2\right\}L_r D^r + \left(\frac{f_1}{L} + L^2 w\Delta + \rho f_2\right)b_r D^r$$

$$=\frac{L^2w}{2}r_{00}+f_2s_0+\frac{1}{2}\left(\frac{f_1\beta}{L}+f_2b^2\right)-\frac{1}{2}f\sigma_1, \tag{23}$$

where $\Delta = b^2-\frac{\beta^2}{L^2}$ and $s_0 = s_{r0}b^r$.

The equations (20) and (23) constitute the system of algebraic equations in L_rD^r and b_rD^r whose solution is given by

$$b_rD^r = \frac{(fL^3w\Delta+f_1f_2\beta+\rho L\beta f_2^2)r_{00}+2f_1f_2L^2s_0+\{\beta(f_1+\rho Lf_2+L^3w\Delta)+L(f_1^2\beta+Lb^2f_1f_2)\}\sigma_0-f_1L^2\sigma_1}{2f(f_1+\rho Lf_2+L^3w\Delta)}, \tag{24}$$

and

$$L_rD^r = \frac{L(f_1+\rho Lf_2)f_2r_{00}-f_2^2L^2s_0+L\{f(f_1+\rho Lf_2+L^3w\Delta)+(Lf_2^2b^2+\beta f_1f_2)\}\sigma_0-ff_2L^2\sigma_1}{2f(f_1+\rho Lf_2+L^3w\Delta)}. \tag{25}$$

Contracting (22) by g^{ij} and putting the values of b_rD^r and L_rD^r from (24) and (25) respectively, we get

$$\begin{aligned} D^i = {} & [\frac{f_1f_2-fL\beta w+Lf_2^2\rho}{2f(f_1+\rho Lf_2+L^3w\Delta)}r_{00}-\frac{Lf_2(f_1f_2-fL\beta w+Lf_2^2\rho)}{(f_1+\rho Lf_2+L^3w\Delta)(f_1+\rho Lf_2)}s_0 \\ & +\frac{(f_1f_2-L\beta wf)\{Lf\sigma_1-(Lb^2f_2+\beta f_1)\sigma_0\}}{2f(f_1+\rho Lf_2+L^3w\Delta)}+\sigma_0]y^i+[\frac{L^3w}{2(f_1+\rho Lf_2+L^3w\Delta)}r_{00} \\ & -\frac{f_2L^4w}{(f_1+\rho Lf_2+L^3w\Delta)(f_1+\rho Lf_2)}s_0+\frac{Lf_2}{2(f_1+\rho Lf_2)}\sigma_0 \\ & +\frac{L^3w\{Lf\sigma_1-(Lb^2f_2+\beta f_1)\sigma_0\}}{2(f_1+\rho Lf_2+L^3w\Delta)(f_1+\rho Lf_2)}]b^i-\frac{Lf}{(f_1+\rho Lf_2)}\sigma^i+\frac{Lf_2}{f_1+\rho Lf_2}s_0^i. \end{aligned} \tag{26}$$

Proposition 3.1. The difference tensor $D^i = \overline{G}^i - G^i$ of the conformal β-change of Finsler metric by h-vector is given by (26).

4. Condition to be Projective Change

The Finsler space $\overline{F}^n$ is said to be projective to Finsler space F^n if every geodesic of F^n is transformed to a geodesic of $\overline{F}^n$ and vice-versa. It is well known that the change $L \to \overline{L}$ is projective iff $\overline{G}^i = G^i + P(x,y)y^i$, where $P(x,y)$ is a homogeneous scalar function of degree one in y^i, called projective factor [11]. Thus from (12) it follows that $L \to \overline{L}$ is projective iff $D^i = Py^i$. First we suppose that the conformal β-change of Finsler metric by h-vector is projective. Then from equation (26), we have

$$Py^i = [\frac{f_1f_2-fL\beta w+Lf_2^2\rho}{2f(f_1+\rho Lf_2+L^3w\Delta)}r_{00}-\frac{Lf_2(f_1f_2-fL\beta w+Lf_2^2\rho)}{(f_1+\rho Lf_2+L^3w\Delta)(f_1+\rho Lf_2)}s_0 \tag{27}$$

$$+\frac{(f_1f_2-L\beta wf)\{Lf\sigma_1-(Lb^2f_2+\beta f_1)\sigma_0\}}{2f(f_1+\rho Lf_2+L^3w\Delta)}+\sigma_0]y^i+[\frac{L^3w}{2(f_1+\rho Lf_2+L^3w\Delta)}r_{00}$$

$$-\frac{f_2L^4w}{(f_1+\rho Lf_2+L^3w\Delta)(f_1+\rho Lf_2)}s_0+\frac{Lf_2}{2(f_1+\rho Lf_2)}\sigma_0$$

$$+\frac{L^3w\{Lf\sigma_1-(Lb^2f_2+\beta f_1)\sigma_0\}}{2(f_1+\rho Lf_2+L^3w\Delta)(f_1+\rho Lf_2)}]b^i-\frac{Lf}{(f_1+\rho Lf_2)}\sigma^i+\frac{Lf_2}{f_1+\rho Lf_2}s_0^i.$$

Contracting (27) with $y_i(=g_{ij}y^j)$ and using the fact that $s_0^iy_i=0$ and $y_iy^i=L^2$, we get

$$P=\frac{f_1f_2+Lf_2^2\rho}{2f(f_1+\rho Lf_2+L^3w\Delta)}r_{00}-\frac{Lf_2^2}{f(f_1+\rho Lf_2+L^3w\Delta)}s_0 \tag{28}$$

$$+\frac{Lff_2\sigma_1+\{f(f_1+\rho Lf_2+L^3w\Delta)-f_2(Lb^2f_2+\beta f_1)\sigma_0\}}{2f(f_1+\rho Lf_2+L^3w\Delta)}.$$

Putting the value of P from (28) in (27), we get

$$\beta w\big((f_1+\rho Lf_2)r_{00}-2Lf_2s_0\big)y^i+Lf\sigma_1-(Lb^2f_2+\beta f_1)\sigma_0$$

$$=L^2w\big((f_1+\rho Lf_2)r_{00}-2Lf_2s_0\big)b^i+2f_2(f_1+\rho Lf_2+L^3w\Delta)s_0^i$$

$$-Lf(f_1+\rho Lf_2+L^3w\Delta)\sigma^i. \tag{29}$$

Tranvecting (29) by b_i, we get

$$r_{00}=\frac{(Lb^2f_2+\beta f_1)\sigma_0-Lf\sigma_1-2Lf_2s_0}{(f_1+\rho Lf_2)\Delta}. \tag{30}$$

Substituting the value of r_{00} from (30) in (28), we get

$$P=\frac{\{fL^3w\Delta+f_2(Lb^2f_2+\beta f_1)\}\sigma_0+Lff_2\sigma_1-2f_2^2s_0}{2f\Delta(f_1+\rho Lf_2)}. \tag{31}$$

Eliminating P and r_{00} from (31), (30) and (27), we get

$$s_0^i=\left(b^i-\frac{\beta}{L^2}y^i\right)\frac{s_0}{\Delta f(f_1+\rho Lf_2)}-\frac{\left(f_1y^i+f_2b^i\right)\sigma_0}{2Lf_2}+\frac{f}{2Lf_2}\sigma^i$$

$$+\frac{Lf\sigma_1-(Lb^2f_2+\beta f_1)\sigma_0+fL^3w\Delta\sigma_0}{2f_2L^4w}b^i. \tag{32}$$

The equations (30) and (32) give the necessary conditions under which the conformal β-change of Finsler metric by h-vector becomes a projective change.

Conversely, if conditions (30) and (32) are satisfied, then putting these conditions in (26), we get

$$D^i=[\frac{\{fL^3w\Delta+f_2(Lb^2f_2+\beta f_1)\}\sigma_0+Lff_2\sigma_1-2f_2^2s_0}{2f\Delta(f_1+\rho Lf_2)}]y^i,$$

i.e $D^i = Py^i$, where P is given by (31).

Thus $\bar{F}^n$ is projective to F^n.

Theorem (4.1). The conformal β-change of Finsler metric by h-vector is projective if and only if (30) and (32) hold.

Corollary (4.1). When $\sigma = 0$, the change (1) becomes general β-change of Finsler metric by h-vector. Then the conditions for projective change as narrated in Theorem (4.1) reduce to the conditions as obtained in [19]:

$$r_{00} = \frac{-2Lf_2 s_0}{(f_1+\rho Lf_2)\Delta}, s_0^i = \left(b^i - \frac{\beta}{L^2} y^i\right) \frac{s_0}{\Delta f(f_1+\rho Lf_2)}.$$

Corollary (4.2). When $\rho = 0$, the change (1) becomes simply conformal β-change. Then the conditions for projective change as stated in Theorem (4.1) reduce to the conditions as obtained in [18]:

$$r_{00} = \frac{(Lb^2 f_2+\beta f_1)\sigma_0 - Lf\sigma_1 - 2Lf_2 s_0}{\Delta f_1},$$

$$s_0^i = \left(b^i - \frac{\beta}{L^2} y^i\right) \frac{s_0}{\Delta f f_1} - \frac{(f_1 y^i + f_2 b^i)\sigma_0}{2Lf_2} + \frac{f}{2Lf_2}\sigma^i$$

$$+ \frac{Lf\sigma_1 - (Lb^2 f_2 + \beta f_1)\sigma_0 + fL^3 w\Delta\sigma_0}{2f_2 L^4 w} b^i.$$

5. The Conformal β-change of Finsler Metric by h-Vector of a Douglas Space

The notion of Douglas space has been introduced by the Basco and Matsumoto [3] as a generalization of Berwald space from the view-point of geodesic equation. It is remarkable that a Finsler space is a Douglas space or is of Douglas type iff the Douglas tensor vanishes identically.

The Finsler space F^n is called a Douglas space iff $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three in y^i [12]. We shall write hp(r) to denote a homogeneous polynomial in y^i of degree r. If we write $B^{ij} = D^i y^j - D^j y^i$, then from (26), we get

$$B^{ij} = [\frac{L^3 w(f_1+\rho Lf_2) r_{00} - 2f_2 L^4 w s_0}{2(f_1+\rho Lf_2)(f_1+L^3 w\Delta+\rho Lf_2)} + \frac{L^3 w\{Lf\sigma_1 - (Lb^2 f_2+\beta f_1)\sigma_0\}}{2f_1(f_1+L^3 w\Delta+\rho Lf_2)} + \frac{Lf_2}{zf_1+\rho Lf_2}\sigma_0]$$

$$(b^i y^j - b^j y^i) + \frac{Lf_2}{f_1+\rho Lf_2}(s_0^i y^j - s_0^j y^i). \tag{33}$$

If a Douglas space is transformed to a Douglas space by conformal β-change of Finsler metric by h-vector, then B^{ij} must be hp(3) and vice-versa.

Theorem (5.1). The conformal β-change of Finsler metric by h-vector transforms a Douglas space into a Douglas space iff B^{ij} given by (33) is hp(3).

Corollary (5.1). When $\sigma = 0$, then the condition under which a Douglas space is transformed to a Douglas space as obtained in Theorem (5.1) reduces to the condition as obtained in [19] for similar purpose.

Corollary (5.2). When $\rho = 0$, then the condition as obtained in Theorem (5.1) to lead a Douglas space to a Douglas space reduces to the condition as obtained in [18] for similar purpose.

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