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THE GUMBEL – EXPONENTIAL DISTRIBUTION

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Abstract: Here we introduce Gumbel-Exponential Distribution (three-parameter model), which have wider applications in the field of order statistics, engineering and others as it has more number of parameters. Expressions for its characteristic function, moments, hazard rate function, survivor function, and quantile function have been derived. We use 'Mathematica', software to plot graphs for probability density function (pdf) of the distribution. In the following sections we investigate the variation of the skewness, kurtosis and estimation by the method of maximum-likelihood.

Keywords: Gumbel-Exponential Distribution, Characteristic Function, Moments, Survivor Function, Hazard Rate Function, Quantile Function, Estimation.

1. Introduction

Many lifetime distributions have been formed because of their applications in several areas, like reliability engineering, survival analysis, demography, hydrology, actuarial study, and others. So we need extended forms of classical distributions, which are more complaint to model real data in these areas. Ingeneralized continuous univariate distributions additional parameters like inequality, scale, shape, and locationare introduced and then we see changes in their shapes. To explore other properties of the distribution and the tails, the introduction of shape parameters is important. Using these generalized distribution, applied statisticians are now able to obtain more improved results because these distributions provides better goodness-of-fit to the data sets rather than the old. The founders in this area were Azzalini[5], Azzalini and Capitania[6], Marshall and Olkin [14] and Gupta et al. [11]. After introducing the beta generator, defined by Eugene et al. [9] and subsequently by Jones [13], the revolutionary work starts. Other competing generators were introduced by Cordeiro and de-Castro[8] and Alexander et al.[4]

In this present paper, we propose and study a new three-parameter model called the "Gumbel-Exponential" Distribution.We organized the rest of the paper as - In Section 2, we give some basic definitions which will be required in subsequent sections. In Section 3, we introduce the Gumbel-Exponential Distribution. We also obtain the moments, characteristic function, hazard rate function and the survivor function for the newly defined distribution. In Section 4, we discuss estimation by the method of maximum-likelihood. In Section 5, we provide conclusion.

2. Preliminaries

2.1 Gumbel-X Family

Being applied for problems in engineering, the Gumbel Distribution[7] is an important distribution of type I. It applied in areas like network engineering, flood frequency analysis, risk based engineering, nuclear engineering, offshore engineering, space engineering, structural engineering etc.

The Gumbel-X family is proposed by Al-Aqtash [1] and Al-Aqtash et al.([2],[3]) as

$$F(x) = \exp\left\{-e^{\frac{\mu}{\sigma}}\left[\frac{G(x)}{1-G(x)}\right]^{-\frac{1}{\sigma}}\right\} \tag{1}$$

Where $0 < x < \infty, \quad \sigma, \lambda > 0, \quad -\infty < \mu < \infty$

Setting $\lambda = e^{\frac{\mu}{\sigma}}$ throughout the manuscript to avoid non-identifiability problem, the cdf (1) reduces to

$$F(x) = \exp\left\{-\lambda\left[\frac{G(x)}{1-G(x)}\right]^{-\frac{1}{\sigma}}\right\} \tag{2}$$

The pdf corresponding to (2) is given by

$$f(x) = \frac{\lambda h(x)}{\sigma G(x)}\left[\frac{G(x)}{1-G(x)}\right]^{-\frac{1}{\sigma}} \exp\left\{-\lambda\left[\frac{G(x)}{1-G(x)}\right]^{-\frac{1}{\sigma}}\right\} \tag{3}$$

Where X is any continuous random variable with cdf F(x) and hazard rate function (hrf) h(x)

2.2 Exponential Distribution

In the theory and practice of statistics the Exponential Distribution [12] is an extreme value distribution. Its most notably statistical property is its characterization through memory-less property. In a homogeneous Poisson process it is used to describe the lengths of the inter-arrival times. It shows great mathematical controllability. This distribution is used to analysis continuous-time stochastic processes to model the time until something happens in the process.

A random variable has the Exponential Distribution with parameter α if its cdf is given by

$$G_\alpha(x) = 1 - e^{-\alpha x} \tag{4}$$

The pdf corresponding to (4) is

$$g_\alpha(x) = \alpha e^{-\alpha x} \tag{5}$$

where $\alpha > 0, \quad x \in [0, \infty)$

2.3 The Composition of cdf's

Let F(x) and f(x) be the cdf and pdf of a statistical distribution referred as the first and having the support $[0,a]$, where $1 \le a < \infty$ and G(x) and g(x) be the cdf and pdf of the second distribution with the support as $[b,c]$, where $b,c \in R$ or $[0,\infty)$ or $(-\infty,\infty)$. We define the new distribution with cdf as given by

$$F_G(x) = \frac{F(G(x))}{F(1)} \tag{6}$$

and the corresponding pdf as

$$f_G(x) = \frac{F'(G(x))}{F(1)} g(x) \tag{7}$$

Clearly the support of the new distribution is same as that of the second distribution. Also for different choices of first and second distributions we can construct a large number of new distributions. The additional advantage of the new distribution is that it has more parameters to have a better control.

3. The Gumbel - Exponential Distribution

Inserting (4) and (5) in (2) and (3), using (6) and (7) the cdf and pdf of the three-parametric Gumbel - Exponential Distribution are given by

$$F_{\alpha,\sigma,\lambda}(x) = \exp\left\{-\lambda(e^{\alpha x} - 1)^{\frac{-1}{\sigma}}\right\} \tag{8}$$

$$f_{\alpha,\sigma,\lambda}(x) = \frac{\alpha\lambda}{\sigma} e^{\alpha x}(e^{\alpha x} - 1)^{\left(\frac{-1}{\sigma}-1\right)} \exp\left\{-\lambda(e^{\alpha x} - 1)^{\frac{-1}{\sigma}}\right\} \tag{9}$$

where $\alpha, \sigma, \lambda > 0, \qquad x \in [0,\infty)$

Remarks

1. On taking $\alpha = 1$ and $\sigma = 1$, in (8) and (9), the cdf and pdf are reduced to the following forms respectively

$$F(x,\lambda,1,1) = \exp\{-\lambda(e^x - 1)^{-1}\}$$

and

$$f(x,\lambda,1,1) = \lambda e^x (e^x - 1)^{(-2)} \exp\{-\lambda(e^x - 1)^{-1}\}$$

The above distribution may be referred as the standard Gumbel-Exponential Distribution.

3.1 Graphs

Some graphs illustrating the effects of various parameters on the shape of the pdf have been drawn in support to the study.

Here, we have drawn three sets of graphs using the software ‘Mathematica’. In the first set (Fig 1.1-1.3), we fix the parameters $\lambda = 1$, $\sigma = 1$ and plot graphs of Gumbel, Exponential and the Gumbel-Exponential pdf for varying values of α. In the second set (Fig. 2.1 – Fig 2.3), we fix the parameters $\lambda = 1$, $\alpha = 1$ and plot the graphs for varying values of σ. In the third set (Fig. 3.1 – Fig 3.3), we fix the parameters $\alpha = 1, \sigma = 1$ and plot the graphs for varying values of λ.

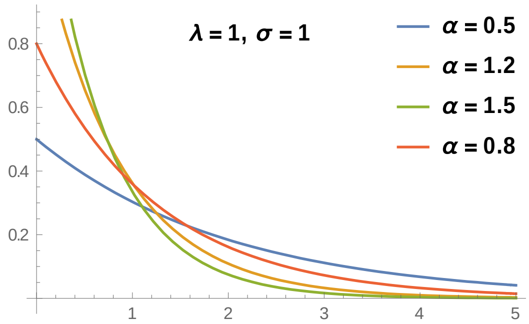

Fig: 1.1 The Exponential pdf f(x)

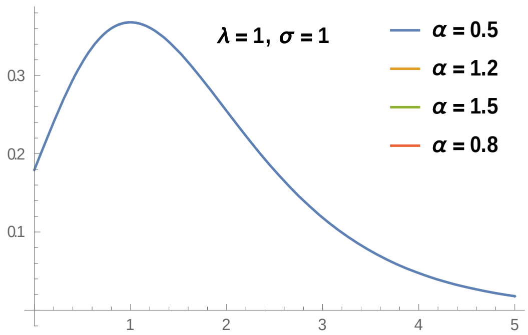

Fig: 1.2 The Gumbelpdf f(x)

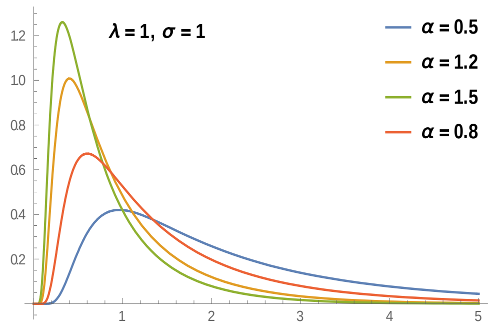

Fig: 1.3 The Gumbel- Exponential pdf f(x)

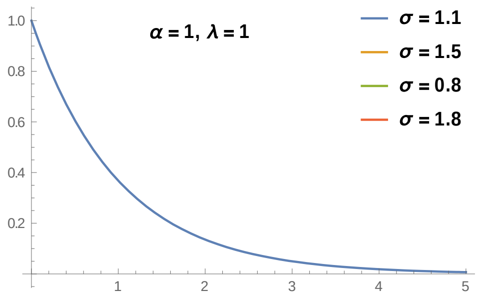

Fig: 2.1 The Exponential pdf f(x)

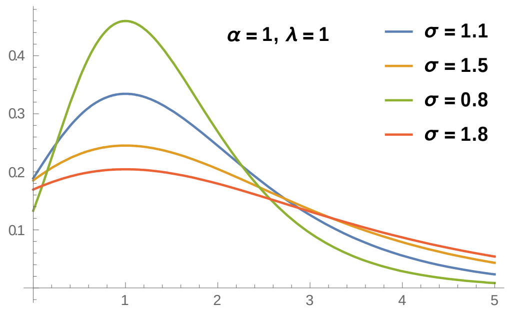

Fig:2.2 The Gumbelpdf f(x)

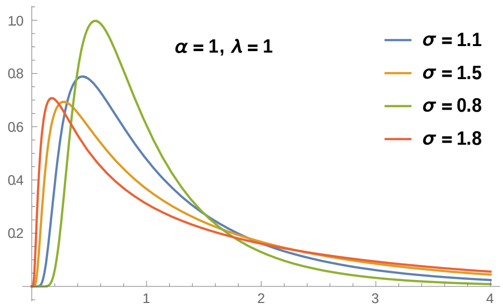

Fig: 2.3 The Gumbel-Exponential pdf f(x)

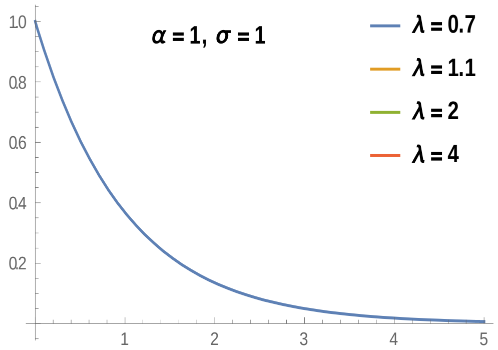

Fig: 3.1The Exponential pdf f(x)

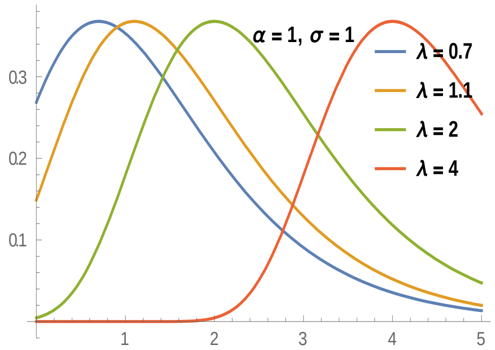

Fig: 3.2The Gumbelpdf f(x)

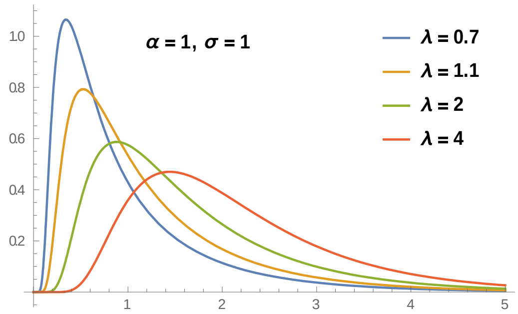

Fig: 3.3 The Gumbel-Exponential pdf f(x)

3.2 The Characteristic Function

The Characteristic Function of pdf f(x) is given by

$$\phi(t)=E(\exp(itx))=\int_{-\infty}^{\infty}\exp(itx)\,f(x)dx \qquad ,i=\sqrt{-1} \tag{10}$$

Substituting the value of f(x) from (9)

$$\phi(t)=\int_0^\infty e^{itx}\,\frac{\alpha\lambda}{\sigma}\,e^{\alpha x}(e^{\alpha x}-1)^{\left(\frac{-1}{\sigma}-1\right)}\exp\left\{-\lambda(e^{\alpha x}-1)^{\frac{-1}{\sigma}}\right\}dx$$

using binomial expansion and then evaluating the integral we get

$$\phi(t)=\frac{\alpha\lambda}{\sigma}\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\int_0^\infty e^{(it+\alpha)x}(e^{\alpha x}-1)^{\left(\frac{-1}{\sigma}-1\right)}(e^{\alpha x}-1)^{\left(\frac{-n}{\sigma}\right)}dx \tag{11}$$

$$=\frac{\alpha\lambda}{\sigma}\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\int_0^\infty e^{(it+\alpha)x}(e^{\alpha x}-1)^{-\left(\frac{n+1}{\sigma}\right)-1}dx=$$
$$\frac{\alpha}{\sigma}\sum_{n=0}^{\infty}\frac{(-1)^n(\lambda)^{n+1}}{n!}\int_0^\infty e^{(it+\alpha)x}e^{-\alpha\left\{\frac{n+1}{\sigma}+1\right\}x}(1-e^{-\alpha x})^{-\left\{\frac{n+1}{\sigma}+1\right\}}dx$$

Again using binomial expansion and then evaluating the integral we get

$$=\frac{\alpha}{\sigma}\sum_{n=0}^{\infty}\frac{(-1)^n(\lambda)^{n+1}}{n!}\sum_{m=0}^{\infty}\frac{\left(\frac{n+\sigma+1}{\sigma}\right)_m}{m!}\int_0^\infty e^{(it+\alpha)x}e^{-\alpha\left\{\frac{n+1}{\sigma}+1\right\}x}e^{-\alpha m x}\,dx$$

$$=\frac{\alpha}{\sigma}\sum_{n=0}^{\infty}\frac{(-1)^{n}(\lambda)^{n+1}}{n!}\sum_{m=0}^{\infty}\frac{\left(\frac{n+\sigma+1}{\sigma}\right)_{m}}{m!}\sum_{s=0}^{\infty}\frac{(it+\alpha)^{s}}{s!}\int_{0}^{\infty}x^{s}e^{-\alpha\left\{\frac{n+1}{\sigma}+1+m\right\}x}dx$$

Evaluating the integral using a property of definite integral ([10] p.340, (3.351.3)), we get

$$=\frac{\alpha}{\sigma}\sum_{n,m,s=0}^{\infty}\frac{(-1)^{n}(\lambda)^{n+1}}{n!}\frac{\left(\frac{n+\sigma+1}{\sigma}\right)_{m}}{m!}\frac{(it+\alpha)^{s}s!}{s!}\left[\left\{\frac{n+1}{\sigma}+1+m\right\}\alpha\right]^{-(s+1)}$$

with $\mathrm{Re}\left[\left\{\frac{n+1}{\sigma}+1+m\right\}\alpha\right]>0$

$$\phi(t)=\frac{\alpha}{\sigma}\sum_{n,m,s=0}^{\infty}\frac{(-1)^{n}(\lambda)^{n+1}}{n!}\frac{\left(\frac{n+\sigma+1}{\sigma}\right)_{m}}{m!}(it+\alpha)^{s}\left[\left\{\frac{n+1}{\sigma}+1+m\right\}\alpha\right]^{-(1+s)} \qquad (12)$$

3.3 The Moments

The r^{th} moment of the pdf f(x) about the origin is given by

$$\mu'_{r}=E(x^{r})=\int_{-\infty}^{\infty}x^{r}f(x)dx \qquad , \qquad r=1,2,3,\ldots\ldots$$

Substituting the value of f(x) from (9)

$$\mu'_{r}=\int_{0}^{\infty}x^{r}\frac{\alpha\lambda}{\sigma}e^{\alpha x}(e^{\alpha x}-1)^{\left(\frac{-1}{\sigma}-1\right)}\exp\left\{-\lambda(e^{\alpha x}-1)^{\frac{-1}{\sigma}}\right\}dx$$

using binomial expansion and then evaluating the integral we get

$$=\frac{\alpha\lambda}{\sigma}\sum_{n=0}^{\infty}\frac{(-\lambda)^{n}}{n!}\int_{0}^{\infty}x^{r}e^{\alpha x}(e^{\alpha x}-1)^{\left(\frac{-1}{\sigma}-1-\frac{n}{\sigma}\right)}dx$$

$$=\frac{\alpha\lambda}{\sigma}\sum_{n=0}^{\infty}\frac{(-\lambda)^{n}}{n!}\int_{0}^{\infty}x^{r}e^{\alpha x}(e^{\alpha x}-1)^{-\left(\frac{n+1}{\sigma}\right)-1}dx=$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{n}(\lambda)^{n+1}}{n!}\int_{0}^{\infty}x^{r}e^{\alpha x}e^{-\alpha\left\{\frac{n+1}{\sigma}+1\right\}x}(1-e^{-\alpha x})^{-\left\{\frac{n+1}{\sigma}+1\right\}}dx=$$

$$\frac{\alpha}{\sigma}\sum_{n=0}^{\infty}\frac{(-1)^{n}(\lambda)^{n+1}}{n!}\int_{0}^{\infty}x^{r}e^{\alpha x-\alpha\left\{\frac{n+1}{\sigma}+1\right\}x}(1-e^{-\alpha x})^{-\left\{\frac{n+1}{\sigma}+1\right\}}dx$$

$$=\frac{\alpha}{\sigma}\sum_{n=0}^{\infty}\frac{(-1)^{n}(\lambda)^{n+1}}{n!}\int_{0}^{\infty}x^{r}e^{\alpha x-\alpha\left\{\frac{n+1}{\sigma}\right\}x-\alpha x}(1-e^{-\alpha x})^{-\left\{\frac{n+1}{\sigma}+1\right\}}dx$$

$$=\frac{\alpha}{\sigma}\sum_{n=0}^{\infty}\frac{(-1)^{n}(\lambda)^{n+1}}{n!}\int_{0}^{\infty}x^{r}e^{-\alpha\left\{\frac{n+1}{\sigma}\right\}x}(1-e^{-\alpha x})^{-\left\{\frac{n+1}{\sigma}+1\right\}}dx$$

Again using binomial expansion and then evaluating the integral we get

$$=\frac{\alpha}{\sigma}\sum_{n=0}^{\infty}\frac{(-1)^{n}(\lambda)^{n+1}}{n!}\sum_{m=0}^{\infty}\frac{\left(\frac{n+\sigma+1}{\sigma}\right)_{m}}{m!}\int_{0}^{\infty}x^{r}e^{-\alpha\left\{\frac{n+1}{\sigma}\right\}x}e^{-\alpha mx}dx$$

$$=\frac{\alpha}{\sigma}\sum_{n=0}^{\infty}\frac{(-1)^{n}(\lambda)^{n+1}}{n!}\sum_{m=0}^{\infty}\frac{\left(\frac{n+\sigma+1}{\sigma}\right)_{m}}{m!}\int_{0}^{\infty}x^{r}e^{-\alpha\left\{\frac{n+1}{\sigma}+m\right\}x}dx$$

Evaluating the integral using a property of definite integral ([10], p.340, (3.351.3)), we get

$$\mu_r' = \frac{\alpha}{\sigma}\sum_{n,m=0}^{\infty}\frac{(-1)^n(\lambda)^{n+1}}{n!}\frac{\left(\frac{n+\sigma+1}{\sigma}\right)_m}{m!}r!\left[\left\{\frac{n+1}{\sigma}+m\right\}\alpha\right]^{-(r+1)} \quad (13)$$

with $\mathrm{Re}\left[\left\{\frac{n+1}{\sigma}+m\right\}\alpha\right] > 0$

In particular, for $r = 1, 2, 3, \ldots\ldots$ we get the values of various moments as follows

$$\mu_1' = \frac{\alpha}{\sigma}\sum_{n,m=0}^{\infty}\frac{(-1)^n(\lambda)^{n+1}}{n!}\frac{\left(\frac{n+\sigma+1}{\sigma}\right)_m}{m!}\left[\left\{\frac{n+1}{\sigma}+m\right\}\alpha\right]^{-2}$$

$$\mu_2' = \frac{\alpha}{\sigma}\sum_{n,m=0}^{\infty}\frac{(-1)^n(\lambda)^{n+1}}{n!}\frac{2!\left(\frac{n+\sigma+1}{\sigma}\right)_m}{m!}\left[\left\{\frac{n+1}{\sigma}+m\right\}\alpha\right]^{-3}$$

$$\mu_3' = \frac{\alpha}{\sigma}\sum_{n,m=0}^{\infty}\frac{(-1)^n(\lambda)^{n+1}3!}{n!}\frac{\left(\frac{n+\sigma+1}{\sigma}\right)_m}{m!}\left[\left\{\frac{n+1}{\sigma}+m\right\}\alpha\right]^{-4}$$

$$\mu_4' = \frac{\alpha}{\sigma}\sum_{n,m=0}^{\infty}\frac{(-1)^n(\lambda)^{n+1}4!}{n!}\frac{\left(\frac{n+\sigma+1}{\sigma}\right)_m}{m!}\left[\left\{\frac{n+1}{\sigma}+m\right\}\alpha\right]^{-5}$$

The variance, skewness, and kurtosis measures can now be calculated using the following relations

$$\mathrm{Var}(x) = \mu_2' - \mu_1' \quad (14)$$

$$\mathrm{Skewness}(x) = \frac{\mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3}{\mathrm{Var}^{\frac{3}{2}}(x)} \quad (15)$$

$$\mathrm{Kurtosis}(x) = \frac{\mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4}{\mathrm{Var}^{2}(x)} \quad (16)$$

3.4 Survivor Function

The Survivor Function S(x) is given by

$$S(x) = P(X > x) = \int_x^{\infty} f(u)du = 1 - F(x)$$

Substituting the value of F(x) from (3.1), we get

$$S(x) = 1 - \exp\left\{-\lambda(e^{\alpha x} - 1)^{\frac{-1}{\sigma}}\right\} \quad (17)$$

3.5 Hazard Rate Function

The Hazard Rate Function h(x) is given by

$$h(x) = \frac{f(x)}{1 - F(x)}$$

Substituting the values of f(x) and F(x) from (9) and (8) respectively, and simplifying we get

$$h(x) = \frac{\alpha\lambda}{\sigma}\frac{e^{\alpha x}(e^{\alpha x}-1)^{\left(\frac{-1}{\sigma}-1\right)}\exp\left\{-\lambda(e^{\alpha x}-1)^{\frac{-1}{\sigma}}\right\}}{1-\exp\left\{-\lambda(e^{\alpha x}-1)^{\frac{-1}{\sigma}}\right\}} \tag{18}$$

Figure 3.4 display plot of the hrf of the Gumbel-Exponential Distribution for some parameter values.

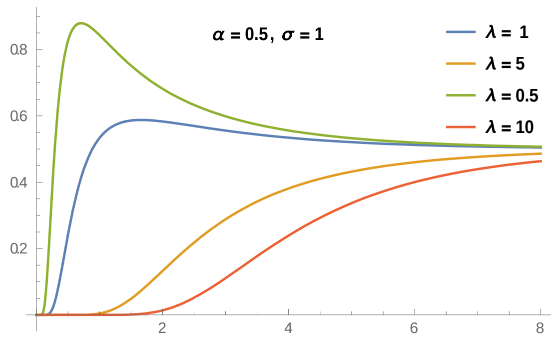

Fig: 3.4 The Gumbel-Exponential hrf f(x)

3.6 Quantile function

The quantile function of a distribution has many uses in both the theory and applications in Statistics. The qf of X is obtained by inverting (8) as

$$x = Q(u) = \frac{1}{\alpha}\log\left[1+\left(\frac{-1}{\lambda}\log u\right)^{-\sigma}\right] \qquad \text{where } 0 < u < 1 \tag{19}$$

If uhas a uniform distribution in (0,1), then x= Q(u) has the density (9).

4. Estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used when constructing confidence intervals and regions and also in test statistics. The normal approximation for these estimators in large sample distribution theory is easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters for the Gumbel-Exponential model by maximum likelihood.

Let $x1,\ldots,xn$ be a sample of size n from the Gumbel-Exponential Distribution given by (9)

$$\log L(\alpha,\sigma,\lambda) = \sum_{i=1}^{n}\log\left[\frac{\alpha\lambda}{\sigma}e^{\alpha x_i}(e^{\alpha x_i}-1)^{\left(\frac{-1}{\sigma}-1\right)}e^{\left\{-\lambda(e^{\alpha x_i}-1)^{\frac{-1}{\sigma}}\right\}}\right] \tag{20}$$

$$=\sum_{i=1}^{n}\left[\log\frac{\alpha\lambda}{\sigma}+\log e^{\alpha x_i}+\log(e^{\alpha x_i}-1)^{\left(\frac{-1}{\sigma}-1\right)}+\log e^{\left\{-\lambda(e^{\alpha x_i}-1)^{\frac{-1}{\sigma}}\right\}}\right]$$

$$= \sum_{i=1}^{n}\left[\log\frac{\alpha\lambda}{\sigma} + \alpha x_i - \left(\frac{1}{\sigma}+1\right)\log(e^{\alpha x_i}-1) - \lambda(e^{\alpha x_i}-1)^{\frac{-1}{\sigma}}\right]$$

$$= \sum_{i=1}^{n}\left[\log\alpha + \log\lambda - \log\sigma + \alpha x_i - \left(\frac{1}{\sigma}+1\right)\log(e^{\alpha x_i}-1) - \lambda(e^{\alpha x_i}-1)^{\frac{-1}{\sigma}}\right]$$

$$= \sum_{i=1}^{n}\log\alpha + \sum_{i=1}^{n}\log\lambda - \sum_{i=1}^{n}\log\sigma + \sum_{i=1}^{n}\alpha x_i - \sum_{i=1}^{n}\left(\frac{1}{\sigma}+1\right)\log(e^{\alpha x_i}-1) - \sum_{i=1}^{n}\lambda(e^{\alpha x_i}-1)^{\frac{-1}{\sigma}}$$

$$= n\log\alpha + n\log\lambda - n\log\sigma + \alpha\sum_{i=1}^{n}x_i - \left(\frac{1}{\sigma}+1\right)\sum_{i=1}^{n}\log(e^{\alpha x_i}-1) - \lambda\sum_{i=1}^{n}(e^{\alpha x_i}-1)^{\frac{-1}{\sigma}} \qquad (21)$$

The first order partial derivatives of (21) with respect to the three parameters are

$$\frac{\partial\log L}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}x_i - \left(\frac{1}{\sigma}+1\right)\sum_{i=1}^{n}\frac{x_i.e^{\alpha x_i}}{(e^{\alpha x_i}-1)} + \frac{\lambda}{\sigma}\sum_{i=1}^{n}(e^{\alpha x_i}-1)^{-\left(\frac{1}{\sigma}+1\right)}e^{\alpha x_i}.x_i \qquad (22)$$

$$\frac{\partial\log L}{\partial\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n}\log(e^{\alpha x_i}-1) + \frac{\lambda}{\sigma^2}\sum_{i=1}^{n}(e^{\alpha x_i}-1)^{\frac{-1}{\sigma}}\log(e^{\alpha x_i}-1) \qquad (23)$$

$$\frac{\partial\log L}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}(e^{\alpha x_i}-1)^{\frac{-1}{\sigma}}(23)$$

Setting these expressions to zero and solving them simultaneously yields the maximum-likelihood estimates of the three parameters.

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5. Conclusion

In this paper we introduce a new three-parameter model called the Gumbel-Exponential Distribution. We derive expressions for its characteristic function, moments, survivor function, hazard rate function, and quantile function. We plot some graphs for its probability density function (pdf) using the software 'Mathematica'. We investigate the variation of the skewness and kurtosis and discuss estimation by the method of maximum-likelihood.

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