Journal of Rajasthan Academy of Physical Sciences
ISSN : 0972-6306; URL : http://raops.org.in
Vol.17, No.1&2, January-June, 2018, 81-100

PERFORMANCE ANALYSIS OF A TWO-STATE QUEUEING MODEL WITH RETRIALS

Neelam Singla[1] and Sonia Kalra[2]
[1]Assistant Professor, Department of Statistics, Punjabi University
Patiala – 147002, E-mail: neelgagan2k3@yahoo.co.in
[2]Research Scholar (Corresponding Author), Department of Statistics,
Punjabi University, Patiala – 147002, E-mail: soniakalra276@gmail.com

Abstract

In this paper, a single server retrial queueing model is studied. The primary arrivals follow Poisson distribution. In case of blocking, the customer leaves the service area but returns after some random amount of time to try his luck again. The repeating calls also follow Poisson distribution when they retry for service from orbit (virtual queue). Service times are exponentially distributed. Time dependent probabilities of exact number of arrivals and departures at when the server is free or busy from the system are obtained by solving the difference-differential equations recursively. Some important performance measures of this model are evaluated. The numerical results are obtained and represented graphically.

Keywords: Retrial, arrivals, departures, queueing, probability.

Mathematics Subject Classification: 60K25; 90B22; 68M20.

1. Introduction

It is well known that a telephone subscriber who obtains a busy signal, usually repeats the call until the required connection is made. In data transmission, a packet transmitted from the source may not successfully reach the destination and returns back; the packet may retry for the transmission until it is finally transmitted.

A new class of queueing systems, systems with repeated calls has been introduced for the analysis of problems discussed above. These repeated attempts of jobs have been analyzed via retrial queueing model. This class of queues is characterized by the following feature: when an arriving customer finds all the servers (accessible for him) busy, leaves the service area but repeats his demand after some random period of time. In aviation, where a plane on finding the runway occupied and remakes its attempt of landing then it is said to be in orbit. Retrial queueing situations arise in many real time systems such as telecommunications systems, computer systems, aviation systems, manufacturing systems. For the stochastic modeling of such situations, retrial queueing models find a wide area of application.

Bunday [4] wrote a text on a “Basic Queueing Theory” for originated the subject of queueing theory as a very practical subject. One of the earliest papers in retrial queues was “On the Influence of Repeated Calls in the Theory of Probabilities of Blocking” by Kosten [10]. J.W. Cohen [5] published “Basic Problems of Telephone Traffic Theory and the Influence of Repeated Calls” in which he considered the more general M/M/C retrial queue with impatient customers. The detailed overviews of retrial queues are given in Falin [7], Falin and Templeton [6], Artalejo [2,3], Artalejo and Gomez-Corral [1], Corral and Duc [11]. Falin [7] presented a survey of main results and methods concentrating on markovian single channel and multi channel systems with retrial queues. Falin and Templeton [6] analysed single server and multi server retrial queues including steady-state and transient distribution of the number of customers in orbit.

Pegden and Rosenshine [11] have given the probability of exact number of arrivals and departures by a given time for the classical queueing model $M/M/1/\infty$. This measure supplies better insight into the behavior of a queueing system than the probability of the exact number of units in the system at a given time studied in the early literature on queues and therefore is more justified.

Most papers on retrial queues deal with the steady-state distributions of the system state. Falin and Templeton [6] studied a single server retrial queueing system and obtained the steady-state probabilities for the number of units in the orbit when server is free and when server is busy. Garg and Kumar [8] obtained explicit time dependent probabilities of exact number of arrivals and departures from the orbit of a single server retrial queue with impatient customers. Indra and Renu [9] presented the time dependent probabilities of exact arrivals and departures by time t for M/M/1 queueing model with Bernoulli schedule and multiple working vacations.

In this paper, we obtain recursively the time dependent probabilities for the exact number of arrivals and departures from the system by a given time when the server is busy and when the server is idle for a single server retrial queueing system using the concept given in tables of integral transforms by Bateman (1954). The rest of this paper is organized as follows: Section 2 gives a relatively formal description of the queueing model. In Section 3, we defined the two-dimensional state model, derived the difference-differential equations and obtained the time dependent solution for the model. Section 4 presents the performance measures with numerical results and verification of results. In Section 5, more numerical results are obtained, and analysed graphically. In the last Section 6, the busy period distribution of the system and the busy period distribution of the server is presented numerically and graphically.

2. Model Description

We consider a two-state single server retrial queueing system in which calls arrive according to Poisson process with mean arrival rate λ. On finding the server busy, calls go to some virtual place (referred as an orbit) and repeat their request for service from the orbit after some random amount of time. For distribution of arrivals, service times and retrials, we make use of the following assumptions and notations:

i. The arrival of primary calls follow a Poisson distribution with parameter λ.

ii. The repeated calls follow a Poisson distribution with parameter θ.

iii. Service times are exponentially distributed with parameter μ.

iv. The stochastic process involved viz. arrivals of units, departures of units and retrials are statistically independent.

Laplace transformation $\bar{f}(s)$ of $f(t)$ is given by

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)\, dt, \quad \text{Re}(s) > 0$$

The Laplace inverse of

$$\frac{Q(p)}{P(p)} \; is \; \sum_{k=1}^{n} \sum_{l=1}^{m_k} \frac{t^{m_k - l}\, e^{a_k t}}{(m_k - l)!(l-1)!} \times \frac{d^{l-1}}{dp^{l-1}} \frac{Q(p)}{P(p)} \, (p - a_k)^{m_k} \;\; \forall \; p = a_k, \;\; a_i \neq a_k \text{ for } i \neq k.$$

where,

$$P(p) = (p - a_1)^{m_1} \, (p - a_2)^{m_2} \ldots \ldots \ldots (p - a_n)^{m_n}$$

Q(p) is a polynomial of degree < $m_1 + m_2 + m_3 + \ldots\ldots\ldots\ldots m_n - 1$.

If $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$, then

$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(u) G(t - u) du = F * G$, $F * G$ is called the convolution of F and G.

3. The Two-Dimensional State Model

3.1 Definitions

$P_{i, j, 0}(t)$ = Probability that there are exactly i arrivals in the system and j departures from the system by time t when server is idle.

$P_{i, j, 1}(t)$ = Probability that there are exactly i arrivals in the system and j departures from the system by time t when server is busy.

$P_{i, j}(t)$ = Probability that there are exactly i arrivals in the system and j departures from the system by time t.

$$P_{i, j}(t) = P_{i, j, 0}(t) + P_{i, j, 1}(t) \qquad \forall \; i, j \qquad i \geq j$$

also

$$P_{i, j, 1}(t) = 0, \; i \leq j; \; P_{i, j, 0}(t) = 0, \; i < j.$$

Initially

$$P_{0,0,0}(0) = 1; \quad P_{i, j, 0}(0) = 0 \;\&\; P_{i, j, 1}(0) = 0, \; i, j \neq 0.$$

3.2 The differential – difference equations governing the system model are

$\frac{d}{dt} P_{i,j,0}(t) = -(\lambda + (i-j)\theta) P_{i,j,0}(t) + \mu P_{i,j-1,1}(t) \qquad i \geq j \geq 0 \qquad (1)$

$\frac{d}{dt} P_{1,0,1}(t) = -(\lambda + \mu) P_{1,0,1}(t) + \lambda P_{0,0,0}(t) \qquad (2)$

$\frac{d}{dt} P_{i,j,1}(t) = -(\lambda + \mu) P_{i,j,1}(t) + \lambda P_{i-1,j,0}(t) + \lambda(1-\delta_{i-1,j}) P_{i-1,j,1}(t) + (i-j)\theta P_{i,j,0}(t)$

$i>1,\ i > j \geq 0 \qquad (3)$

$$\text{where } \delta_{i-1,j} = \begin{cases} 1, & when\ i-1 = j \\ 0, & otherwise \end{cases}$$

Using the Laplace transformation $\bar{f}(s)$ of $f(t)$ given by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad Re(s) > 0$$

in the equations (1) - (3) along with the initial conditions, we have

$(s + \lambda + (i-j)\theta) \bar{P}_{i,j,0}(s) = \mu \bar{P}_{i,j-1,1}(s) \qquad i \geq j \geq 0 \qquad (4)$

$(s + \lambda + \mu) \bar{P}_{1,0,1}(s) = \lambda \bar{P}_{0,0,0}(s) \qquad (5)$

$(s + \lambda + \mu) \bar{P}_{i,j,1}(s) = \lambda \bar{P}_{i-1,j,0}(s) + \lambda(1-\delta_{i-1,j}) \bar{P}_{i-1,j,1}(s) + (i-j)\theta \bar{P}_{i,j,0}(s) \quad i>1,\ i > j \geq 0 \qquad (6)$

$$\text{where } \delta_{i-1,j} = \begin{cases} 1, & when\ i-1 = j \\ 0, & otherwise \end{cases}$$

3.3 Solution of the Problem

Solving equations (4) to (6) recursively, we have

$\bar{P}_{0,0,0}(s) = \frac{1}{s+\lambda} \qquad (7)$

$\bar{P}_{i,0,1}(s) = \frac{1}{s+\lambda} \left(\frac{\lambda}{s+\lambda+\mu}\right)^i \qquad i \geq 1 \qquad (8)$

$\bar{P}_{i,1,0}(s) = \left[\frac{1}{s+\lambda} \left(\frac{\lambda}{s+\lambda+\mu}\right)^i \frac{\mu}{s+\lambda+(i-1)\theta}\right] \quad i \geq 1 \qquad$ or

$\bar{P}_{i,1,0}(s) = \frac{\mu}{s+\lambda+(i-1)\theta} \bar{P}_{i,0,1}(s) \qquad i \geq 1 \qquad (9)$

$\bar{P}_{i,i,0}(s) = \frac{\mu}{s+\lambda}\left(\frac{\lambda}{s+\lambda+\mu} \bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{s+\lambda+\mu} \bar{P}_{i,i-1,0}(s)\right) \qquad i > 1 \qquad (10)$

$\bar{P}_{i,i-1,1}(s) = \frac{\lambda}{s+\lambda+\mu} \bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{s+\lambda+\mu} \bar{P}_{i,i-1,0}(s) \qquad i > 1 \qquad (11)$

$$\bar{P}_{i,j,0}(s) = \frac{\mu}{s+\lambda+(i-j)\theta} \left(\sum_{k=1}^{i-j+1} \left(\frac{\lambda}{s+\lambda+\mu}\right)^{i-j-k+1} \eta'_k(s) \bar{P}_{j+k-1,j-1,0}(s) + \left(\frac{\lambda}{s+\lambda+\mu}\right)^{i-j} \bar{P}_{j,j-1,1}(s) \right) \quad i>j>1 \qquad (12)$$

where $\eta'_k(s) = \begin{cases} 1 & for\ k = 1 \\ \left(1 + \frac{k\theta}{s+\lambda+\mu}\right) & for\ k = 2\ to\ i-j \\ \frac{k\theta}{s+\lambda+\mu} & for\ k = i-j+1 \end{cases}$

$$\bar{P}_{i,j,1}(s) = \sum_{k=1}^{i-j} \left(\frac{\lambda}{s+\lambda+\mu}\right)^{i-j-k} \eta'_k(s)\, \bar{P}_{j+k,j,0}(s) + \left(\frac{\lambda}{s+\lambda+\mu}\right)^{i-j-1} \bar{P}_{j+1,j,1}(s) \quad i \geq j+2, j \geq 1 \quad (13)$$

where $\eta'_k(s) = \begin{cases} 1 & for\ k = 1 \\ \left(1 + \frac{k\theta}{s+\lambda+\mu}\right) & for\ k = 2\ to\ i-j-1 \\ \frac{k\theta}{s+\lambda+\mu} & for\ k = i-j \end{cases}$

Taking the Inverse Laplace transform of equations (7) to (13), we have

$$P_{0,0,0}(t) = e^{-\lambda t} \quad (14)$$

$$P_{i,0,1}(t) = \lambda^i e^{-\lambda t} \times \left\{\frac{1}{(\mu)^i} - e^{-\mu t} \sum_{r=0}^{i-1} \frac{(t)^r}{r!} \frac{1}{(\mu)^{i-r}}\right\} \quad i \geq 1 \quad (15)$$

$$P_{i,1,0}(t) = \mu e^{-(\lambda+(i-1)\theta)t} * P_{i,0,1}(t) \quad i \geq 1 \quad (16)$$

$$P_{i,i,0}(t) = \left[(\lambda\mu)e^{-\lambda t}\left\{\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right\} * P_{i-1,i-1,0}(t) + (\mu\theta)e^{-\lambda t}\left\{\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right\} * P_{i,i-1,0}(t)\right] \quad i > 1 \quad (17)$$

$$P_{i,i-1,1}(t) = \left(\lambda e^{-(\lambda+\mu)t} * P_{i-1,i-1,0}(t) + \theta e^{-(\lambda+\mu)t} * P_{i,i-1,0}(t)\right) \quad i > 1 \quad (18)$$

$$P_{i,j,0}(t) = \mu\lambda^{i-j} e^{-(\lambda+(i-j)\theta)t} \left\{\frac{1}{(\mu)^{i-j}} - e^{-\mu t} \sum_{r=0}^{i-j-1} \frac{(t)^r}{r!} \frac{1}{(\mu)^{i-j-r}}\right\} * P_{j,j-1,0}(t) +$$
$$e^{-(\lambda+(i-j)\theta)t} \sum_{k=2}^{i-j} \mu\lambda^{i-j-k+1} \left\{\frac{1}{(\mu)^{i-j-k+1}} - e^{-\mu t} \sum_{r=0}^{i-j-k} \frac{(t)^r}{r!} \frac{1}{(\mu)^{i-j-k-r+1}}\right\} *$$
$$P_{j+k-1,j-1,0}(t) +$$
$$e^{-(\lambda+(i-j)\theta)t} \sum_{k=2}^{i-j} (\mu k\theta)\lambda^{i-j-k+1} \left\{\frac{1}{(\mu)^{i-j-k+2}} - e^{-\mu t} \sum_{r=0}^{i-j-k+1} \frac{(t)^r}{r!} \frac{1}{(\mu)^{i-j-k-r+2}}\right\} *$$
$$P_{j+k-1,j-1,0}(t) + e^{-(\lambda+(i-j)\theta)t} \left\{\frac{1}{\mu} - \frac{e^{-\mu t}}{\mu}\right\} ((i-j+1)\mu\theta) * P_{i,j-1,0}(t) +$$
$$\mu\lambda^{i-j} e^{-(\lambda+(i-j)\theta)t} \left\{\frac{1}{(\mu)^{i-j}} - e^{-\mu t} \sum_{r=0}^{i-j-1} \frac{(t)^r}{r!} \frac{1}{(\mu)^{i-j-r}}\right\} * P_{j,j-1,1}(t) \quad i > j > 1 \quad (19)$$

$$P_{i,j,1}(t) = \lambda^{i-j-1} e^{-(\lambda+\mu)t} \frac{(t)^{i-j-2}}{(i-j-2)!} * P_{j+1,j,0}(t) + e^{-(\lambda+\mu)t} \sum_{k=2}^{i-j-1} \lambda^{i-j-k} \frac{(t)^{i-j-k-1}}{(i-j-k-1)!} *$$
$$P_{j+k,j,0}(t) + e^{-(\lambda+\mu)t} \sum_{k=2}^{i-j-1} k\theta\lambda^{i-j-k} \frac{(t)^{i-j-k}}{(i-j-k)!} * P_{j+k,j,0}(t) + (i-j)\theta e^{-(\lambda+\mu)t} *$$

$$P_{i,j,0}(\mathrm{t}) + \lambda^{i-j-1} e^{-(\lambda+\mu)t} \frac{(t)^{i-j-2}}{(i-j-2)!} * P_{j+1,j,1}(\mathrm{t}) \qquad i \geq j+2,\ j \geq 1 \tag{20}$$

4. Measures of Effectiveness

4.1 The Laplace transform of $\bar{P}_{i.}(s)$ of the probability $P_{i.}(t)$ that exactly i units arrive by time t is :

$$\bar{P}_{i.}(s) = \sum_{j=0}^{i} \bar{P}_{i,j}\ (s) = \frac{\lambda^i}{(s+\lambda)^{i+1}}\ ; \qquad i > 0 \tag{21}$$

And its Inverse Laplace transform is

$$P_{i.}(t) = \frac{e^{-\lambda t}\ (\lambda t)^i}{i!}. \tag{22}$$

The very (basic) assumption on primary arrivals is that it forms a Poisson process and above analysis of abstract solution also verifies the same.

4.2 The departure process from M/M/1 queue has the distribution function $P_{.j}\ (t)$, the probability that exactly j customers have been served by time t. $P_{.j}\ (t)$ in terms of $P_{i,j}\ (t)$ is given by:

$$P_{.j}\ (t) = \sum_{i=j}^{\infty} P_{i,j}\ (t)$$

4.3 From the abstract solution of our model, we verify that the sum of all possible probabilities is one i.e. taking summation over i and j on equations (7)-(13) and adding, we get

$$\sum_{i=0}^{\infty} \sum_{j=0}^{i} \{\bar{P}_{i,j,0}\ (s) + \bar{P}_{i,j,1}(s)\} = \frac{1}{s}.$$

After taking the inverse Laplace transformation, we get

$$\sum_{i=0}^{\infty} \sum_{j=0}^{i} \{P_{i,j,0}\ (t) + P_{i,j,1}\ (t)\} = 1.$$

which is a verification of our results.

4.4 When we put λ=1, μ=2, i=1, t=3 in equations (15) and (16), we get

$$\sum_{j=0}^{i} P_{i,j,1}\ (t) = 0.024832$$

and

$$\sum_{j=0}^{i} P_{i,j,0}\ (t) = 0.1245292$$

$$Now \qquad \sum_{j=0}^{i} P_{i,j}\ (t) = \sum_{j=0}^{i} P_{i,j,0}\ (t) + \sum_{j=0}^{i} P_{i,j,1}\ (t) = 0.1493612.$$

Putting λ=2, μ=3, i=1, t=3 in equations (15) and (16), we get

$$\sum_{j=0}^{i} P_{i,j,1}\ (t) = 0.0016522795$$

and

$$\sum_{j=0}^{i} P_{i,j,0}\ (t) = 0.0132202073$$

$$Now \qquad \sum_{j=0}^{i} P_{i,j}\ (t) = \sum_{j=0}^{i} P_{i,j,0}\ (t) + \sum_{j=0}^{i} P_{i,j,1}\ (t) = 0.0148725.$$

Putting λ=3, μ=4, i=1, t=3 in equations (15) and (16), we get

$$\sum_{j=0}^{i} P_{i,j,1}\ (t) = 0.0000925568$$

and

$$\sum_{j=0}^{i} P_{i,j,0}\ (t) = 0.0010181314$$

$$Now \qquad \sum_{j=0}^{i} P_{i,j}\ (t) = \sum_{j=0}^{i} P_{i,j,0}\ (t) + \sum_{j=0}^{i} P_{i,j,1}\ (t) = 0.0011106882.$$

Above results in tabular form are given below by Table-1:

Table-1

λ	μ	i	t	$\frac{e^{-\lambda t}\ (\lambda t)^i}{i!}$	$\sum_{j=0}^{i} P_{i,j,1}\ (t)$	$\sum_{j=0}^{i} P_{i,j,0}\ (t)$	$\sum_{j=0}^{i} P_{i,j}\ (t)$
1	2	1	3	0.149361	0.024832	0.1245292	0.149361
2	3	1	3	0.0148725	0.0016522795	0.0132202073	0.0148725
3	4	1	3	0.0011106	0.0000925568	0.0010181314	0.0011106

Last column of Table-1 shows compete agreement with the Table-1 of **Pegden & Rosenshine [15].**

4.5 Define $Q_{n,k}\ (t)$ = Probability that there are n customers in the system at time t and the server is free or busy according as k=0 or 1.

The probability of exactly n customers in the system at time t in terms of $P_{i,j,0}\ (t)$ and $P_{i,j,1}\ (t)$:

When the server is free, it is defined by probability $Q_{n,0}\ (t)$:

$$Q_{n,0}\ (t) = \sum_{j=0}^{\infty} P_{j+n,j,0}(t)$$

In this case, the number of customers in the orbit is equal to n which is obtained by using:

n = (number of arrivals – number of departures).

When the server is busy, it is defined by probability $Q_{n,1}\ (t)$:

$$Q_{n,1}\ (t) = \sum_{j=0}^{\infty} P_{j+n+1,j,1}(t)$$

In this case, the number of customers in the orbit is equal to n which is obtained by using:

n = (number of arrivals – number of departures – 1).

Using above definitions, from the equations (1) to (3), the set of equations in statistical equilibrium are:

$(\lambda + n\theta)\ Q_{n,0} = \mu\ Q_{n,1} \qquad n \geq 0 \qquad (23)$

$(\lambda + \mu)\ Q_{n,1} = \lambda\ (Q_{n,0} + Q_{n-1,1}) + (n+1)\ \theta\ Q_{n+1,0} \qquad n > 1 \qquad (24)$

To solve these equations, define partial generating functions

$$q_0(z) \equiv \sum_{n=0}^{\infty} Q_{n,0}\, z^n \ \text{ and } \ q_1(z) \equiv \sum_{n=0}^{\infty} Q_{n,1}\, z^n$$

on using equations (23) - (24) and above definitions, we have:

$\lambda q_0(z) + \theta z\, q_0'(z) = \mu\, q_1(z) \qquad (25)$

$(\mu + \lambda - \lambda z)\ q_1(z) = \lambda q_0(z) + \theta q_0'(z) \qquad (26)$

where $q_0'(z)$ represents derivative of $q_0(z)$ with respect to z.

Eliminating $q_1(z)$, we get the following differential equations for $q_0(z)$:

$$q_0'(z) = \frac{\lambda\, \rho}{\theta\,(1-\rho z)}\, q_0(z)$$

with solution

$$q_0(z) \;\; = \frac{c}{(1-\rho z)^{\frac{\lambda}{\theta}}} \tag{27}$$

on putting above results in (26), we have:

$$q_1(z) \;\; = \frac{c\rho}{(1-\rho z)^{\frac{\lambda}{\theta}+1}} \tag{28}$$

where c is a constant and can be found with the help of the normalizing condition

$$\sum_{n=0}^{\infty}(Q_{n,0}+ Q_{n,1}) = q_0(1) + q_1(1) = 1.$$

Putting z = 1 in (27) and in (28), we get

$$c = (1-\rho)^{\frac{\lambda}{\theta}+1}$$

The result coincides with (1.8) – (1.9) of Falin and Templeton [8].

5. Numerical Solution and Graphical Representation

In this section, we discuss some interesting numerical results that qualitatively describe the model under study. Following the thesis K. Neelam [14], the numerical results are generated for the case when $\rho=\left(\frac{\lambda}{\square}\right)=0.3$, $\eta=\left(\frac{\theta}{\square}\right)=0.6$ using MATLAB programming. From the numerical results, it is found that the sum of all the probabilities at any instance approaches to one. In table 2, we show some of the significant probabilities at different instants of time whose sum is found close to one.

Table-2

At time t= 1

t	$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	$P_{3,0,1}$	$P_{3,1,1}$	$P_{3,2,1}$	sum
1	0.7408	0.0818	0.0059	0.021	0.1405	0.0176	0.0077	0.0016	0.001	0.0001	0.99

At time t= 5

t	$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{3,2,0}$	$P_{3,3,0}$	$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	$P_{3,1,1}$	sum
5	0.2231	0.2682	0.0259	0.132	0.0232	0.0353	0.0887	0.0246	0.069	0.027	0.91

At time t= 10

t	$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{3,2,0}$	$P_{3,3,0}$	$P_{4,3,0}$	$P_{4,4,0}$	$P_{2,1,1}$	$P_{4,3,1}$
10	0.0498	0.1344	0.0074	0.172	0.0192	0.1373	0.230	0.0587	0.048	0.0353

t	$P_{5,5,0}$	$P_{6,6,0}$	$P_{1,0,1}$	Sum
10	0.0310	0.0096	0.0149	0.948

At time t= 20

t	$P_{1,1,0}$	$P_{2,2,0}$	$P_{3,3,0}$	$P_{4,4,0}$	$P_{5,5,0}$	$P_{6,6,0}$	$P_{7,7,0}$	$P_{8,8,0}$	$P_{6,5,1}$	$P_{10,10,0}$
20	0.0141	0.0398	0.0737	0.108	0.1097	0.0951	0.0699	0.0438	0.032	0.0317

t	$P_{5,4,0}$	$P_{6,5,0}$	$P_{7,6,0}$	$P_{10,9,0}$	$P_{8,6,1}$	$P_{3,2,1}$	$P_{4,3,1}$	$P_{5,4,1}$	$P_{6,4,1}$	$P_{7,6,1}$
20	0.0110	0.0150	0.0161	0.0133	0.0133	0.0119	0.0221	0.0302	0.0124	0.0284

t	$P_{8,7,1}$	$P_{9,7,1}$	$P_{10,8,1}$	$P_{9,8,1}$	$P_{10,9,1}$	$P_{8,7,0}$	$P_{9,8,0}$	$P_{9,9,0}$	Sum
20	0.0208	0.0103	0.0108	0.0130	0.0120	0.0140	0.0113	0.0299	0.938

At time t= 30

t	$P_{3,3,0}$	$P_{4,4,0}$	$P_{5,5,0}$	$P_{6,6,0}$	$P_{7,7,0}$	$P_{8,8,0}$	$P_{9,9,0}$	$P_{10,9,0}$	$P_{10,10,0}$	$P_{10,9,1}$
30	0.0133	0.0285	0.0485	0.068	0.0816	0.0834	0.0775	0.0298	0.311	0.0401

t	$P_{2,2,0}$	$P_{7,6,0}$	$P_{8,7,0}$	$P_{7,6,1}$	$P_{8,7,1}$	$P_{9,8,1}$	$P_{9,7,1}$	$P_{10,8,1}$	Sum
30	0.0046	0.0073	0.0093	0.0205	0.0245	0.0254	0.0104	0.0177	0.9018

At time t= 40

t	$P_{6,6,0}$	$P_{7,7,0}$	$P_{8,8,0}$	$P_{10,1,0}$	$P_{10,9,0}$	$P_{10,10,0}$	$P_{8,6,1}$	$P_{8,7,1}$	$P_{9,8,1}$	$P_{10,9,1}$	sum
40	0.0249	0.0343	0.049	0.061	0.0178	0.6946	0.0092	0.0103	0.014	0.0313	0.95

The probabilities against time are represented graphically in figures from 1 to 9. Figure 1 shows the plots of probabilities $P_{0,0,0}$ and $P_{1,1,0}$ against time t for the case when $\rho=\left(\frac{\lambda}{?}\right)=0.3$ and $\eta=\left(\frac{\theta}{?}\right)=0.6$. It is clear from the graph that the probability $P_{0,0,0}$ decreases rapidly from the initial value 1 at time t=0. The probability $P_{1,1,0}$ increases rapidly in the starting moments from initial value zero, and then decreases gradually.

Probabilities $P_{0,0,0}$ and $P_{1,1,0}$ against time t

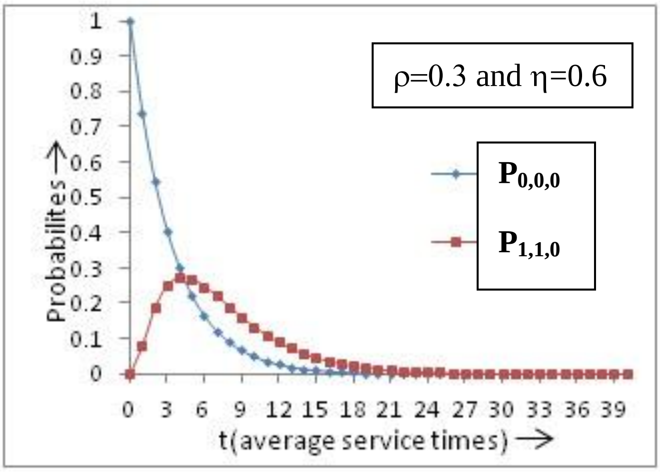

Figure 1

Figure 2 illustrates relationship among four probabilities $P_{4,0,1}$, $P_{4,1,1}$, $P_{4,2,1}$ and $P_{4,3,1}$ for the case when $\rho=0.3$ and $\eta=0.6$. Here we interpret that for higher values of time, the probabilities follow the rule that higher the number of departures, higher is the probability as it is seen here that

$P_{4,3,1} > P_{4,2,1} > P_{4,1,1} > P_{4,0,1}$.

Probabilities $P_{4,0,1}$, $P_{4,1,1}$, $P_{4,2,1}$ and $P_{4,3,1}$ against time t

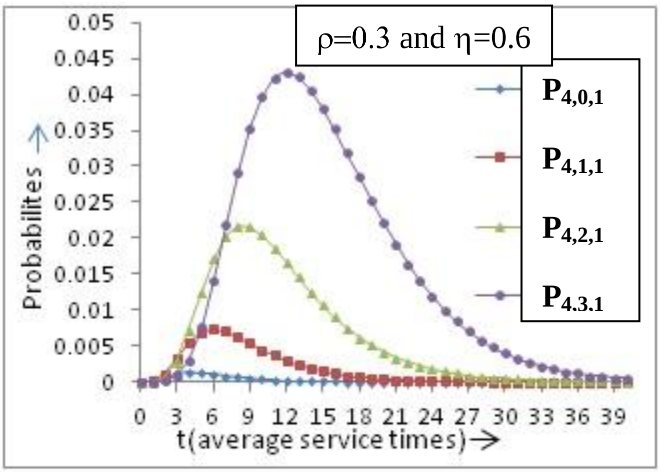

Figure 2

Figure 3 shows the comparison among probabilities $P_{1,0,1}$, $P_{2,0,1}$, $P_{3,0,1}$ and $P_{4,0,1}$ for the case when ρ=0.6 and η=0.6 i.e. In the starting moments, the probabilities increase but afterwards start decreasing due to their peculiar behavior. From the graph, it can also be interpreted for the case under study that probabilities $P_{i,0,1}$ have smaller values when i (the number of arrivals) are more.

Probabilities $P_{1,0,1}$, $P_{2,0,1}$, $P_{3,0,1}$ and $P_{4,0,1}$ against time t

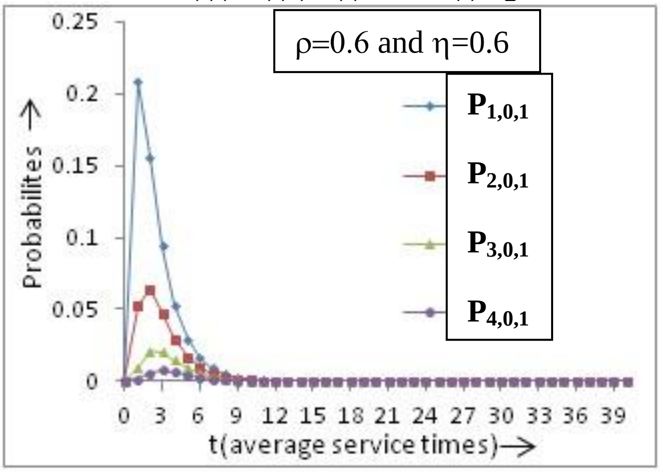

Figure 3

Figure 4 shows relative changes in probabilities $P_{3,1,0}$ and $P_{3,1,1}$ against time t for the case when ρ=0.3 and η=0.6. We see that the Probability $P_{3,1,1}$ increased rapidly in the starting moment, then decreases with a high rate. Probability $P_{3,1,0}$ also increases in the starting moments and then start decreases. It is seen that probability $P_{3,1,1}$ always remained more than $P_{3,1,0}$ i.e. the probability when the server is busy is always more than the probability when the server is free.

Probabilities $P_{3,1,0}$ and $P_{3,1,1}$ against time t

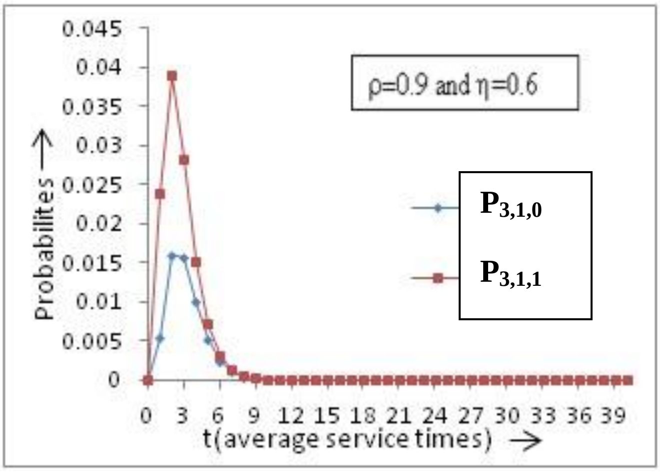

Figure 4

To study the effect of traffic intensity on different probabilities of the model, the data of various probabilities is generated for different values of ρ keeping the other parameter constant. The set of ρ values taken for the comparison is {0.3,0.6,0.9}. Figure 5 shows plot of probability $P_{0,0,0}$ against time t for different value of ρ. From the initial condition, we see that the probability $P_{0,0,0}$ at time t=0 is one and with increasing the time the probability $P_{0,0,0}$ start decreasing. Behavior of the probability $P_{0,0,0}$ is same for all the values of ρ. From the figure it is concluded that as ρ increases $P_{0,0,0}$ decreases. So more the traffic intensity (i.e. more customers are arriving per unit service time) less is the probability of zero units in the system.

Effect of $\rho=[\lambda/\mu]$ on $P_{0,0,0}$ against time

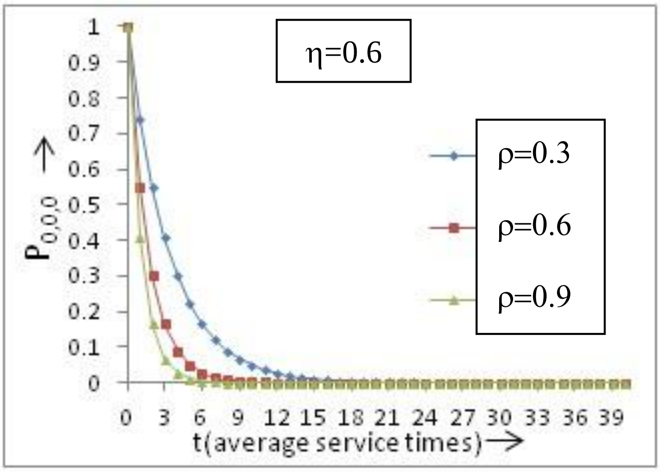

Figure 5

In figure 6, the probability $P_{3,1,0}$ and in figure 7, the probability $P_{3,1,1}$ are plotted against time t for different values of ρ=0.3,0.6 and 0.9. From both the figures, it is concluded that due to their particular behavior, the probabilities $P_{3,1,0}$ and $P_{3,1,1}$ increase in the initial moments and then start decreasing for higher values of time. We also observe that probabilities $P_{3,1,0}$ and $P_{3,1,1}$ remained large value for higher ρ values initially but this trend reverses for higher values of time.

Effect of $\rho=[\lambda/\mu]$ on $P_{3,1,0}$ against time

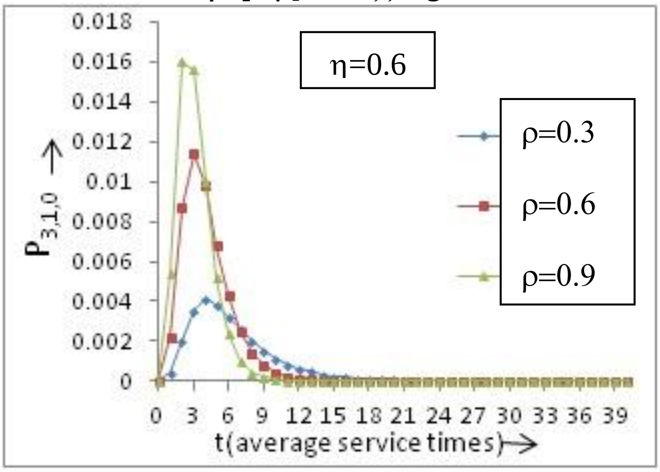

Figure 6

Effect of $\rho=[\lambda/\mu]$ on $P_{3,1,1}$ against time

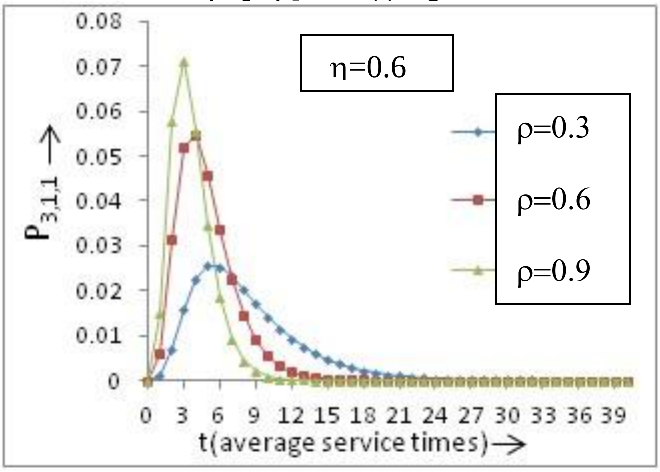

Figure 7

In figure 8, the probability $P_{2,1,0}$ is plotted against time t for different value of η (retrial rate per unit service time). From this figure, it is concluded that as η increases $P_{2,1,0}$ decreases. So we can interpret that more is the retrial rate per unit service time less is the probability of server being free in the system.

Effect of $\eta=[\theta/\mu]$ on $P_{2,1,0}$ against time

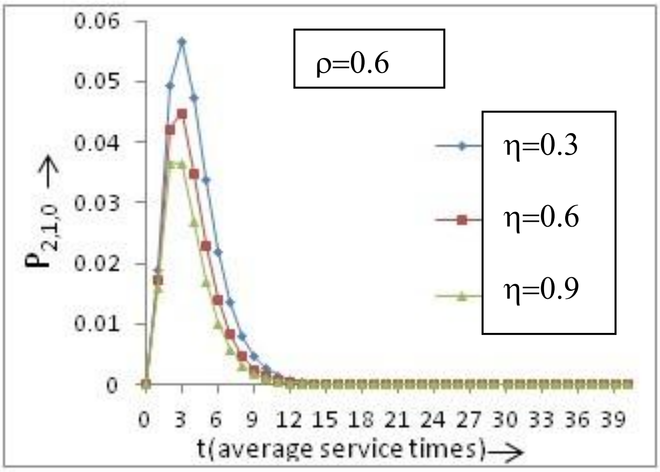

Figure 8

In figure 9, the probability $P_{2,1,1}$ is plotted against time for different value of retrial rate per unit service time i.e. for η=0.3,0.6 and 0.9. From the figure, it is concluded that as η increases, $P_{2,1,1}$ slightly increases . So generally we can say that more is the retrial rate per unit service time more is the probability of server being busy in the system.

Effect of $\eta=[\theta/\mu]$ on $P_{2,1,1}$ against time

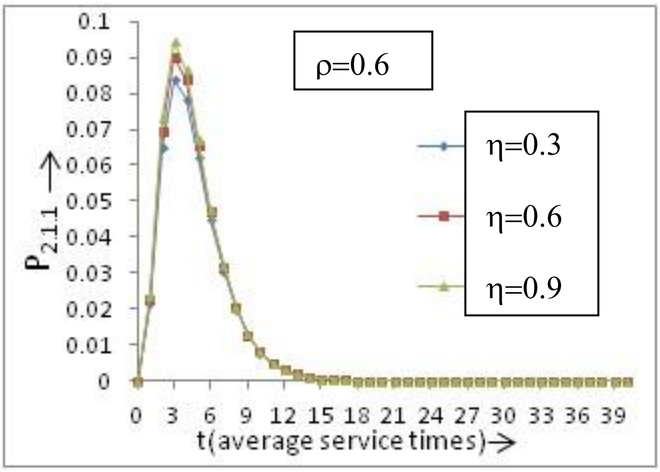

Figure 9

6. Busy Period Distribution

In this section, we discuss some interesting numerical results about busy period distribution of the server and busy period distribution of the system.

The probability when the server is busy is given by

P (Server is busy) = $\sum_{i>j\geq 0} P_{i,j,1}$ (t) (29)

The probability when the system is busy is given by

P (System is busy) = $\sum_{i>j\geq 0}\left(P_{i,j,0}\,(t) + P_{i,j,1}\,(t)\right)$ (30)

Numerical & Graphical representation of busy period

The numerical results are generated using MATLAB programming. The probability when the system is busy and the probability when the server is busy are presented in Table-3 for different values of ρ (traffic intensity) at η=0.6

Table-3

Probability (System busy)				Probability (Server busy)		
t	**ρ=0.3**	**ρ=0.6**	**ρ=0.9**	**ρ=0.3**	**ρ=0.6**	**ρ=0.9**
0	0	0	0	0	0	0
1	0.1753	0.3237	0.4482	0.1688	0.3204	0.4089
2	0.2449	0.4492	0.6084	0.2202	0.3791	0.4959
3	0.2854	0.5268	0.7025	0.2418	0.4142	0.5379
4	0.3137	0.582	0.765	0.2546	0.4385	0.5686
5	0.3348	0.6234	0.8087	0.2638	0.456	0.5928
6	0.3508	0.6554	0.8405	0.2709	0.4732	0.6118
7	0.3633	0.6806	0.8643	0.2764	0.486	0.6253

In figure 10, probability (system busy) and probability (server busy) are plotted for the case ρ=0.9 and η=0.6. It is observed that the two curves are increasing rapidly in starting moments and they start decreasing gradually for higher values of t. Figure 10 also shows that the probability when the system is busy always remains more than the probability when the server is busy, as desired.

Probability (system busy) and Probability (server busy) against time for ρ=0.9, η=0.6

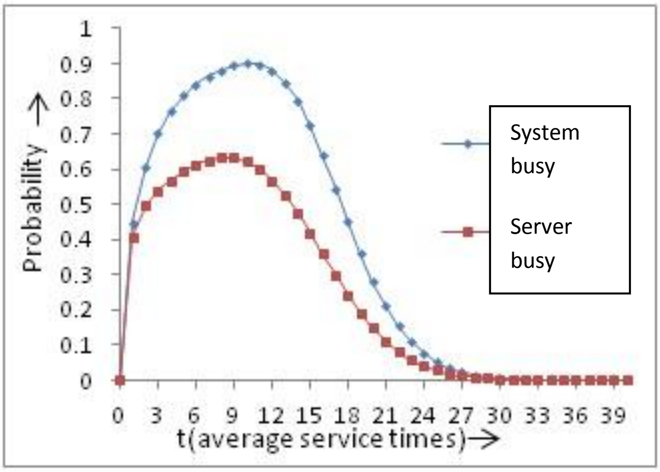

Figure 10

The probability when the system is busy is plotted in figure 11 and the probability when the server is busy is plotted in figure 12 for different values of traffic intensity. From these figures, it is clearly visible that as the value of ρ increases, both the probabilities achieved higher highest values for some t.

Effect of ρ on Probability (system busy) against time

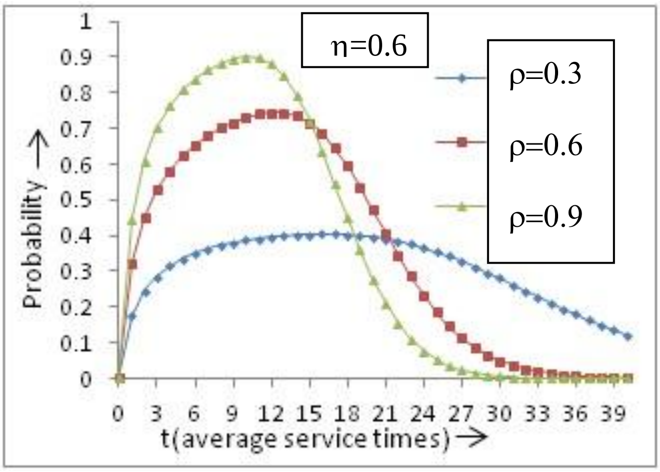

Figure 11

Effect of ρ on Probability (server busy) against time

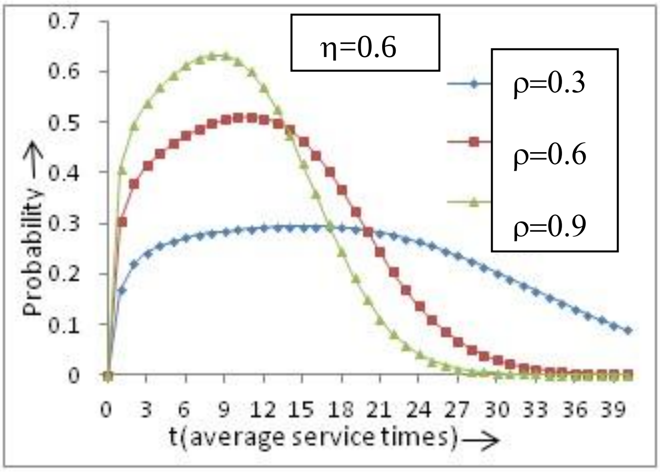

Figure 12

To study the effect of η (retrial rate per unit service time) on the probability of system busy and the probability of server busy are plotted for different values of η in figures 13 and 14 respectively. In figure 13, the probability when the system busy decreases as η increases. It is interpreted that more is the retrial rate per unit service time less is the probability of the system remaining busy. In figure 14, it is seen that the probability (server is busy) increases initially but after certain time the trend is reversed, showing that when retrial rate per unit service time is more, probability when the server busy is more in the initial times.

Effect of η on Probability (system busy) against time

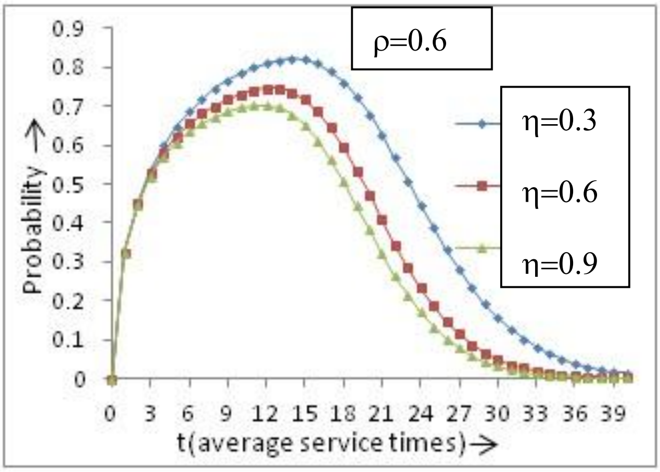

Figure 13

Effect of η on Probability (server busy) against time

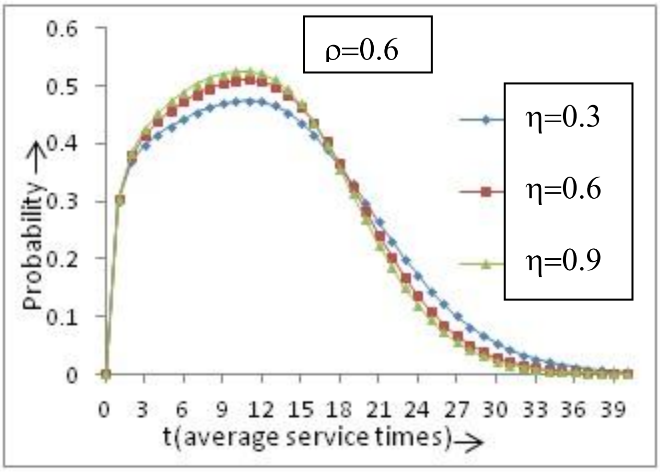

Figure 14

Conclusion

From two-dimensional state queuing model, factors are well understood and quantified. This model mainly applies to telecommunication systems and computer systems that allow us to repeat demand again and again until service. Finally, the numerical analysis clearly demonstrates the meaningful impact of the model.

Acknowledgement: The authors are thankful to the Referee for valuable comments and suggestions.

References

[1] Artalejo, J.R. and Gomez-Corral, A. (2008), Retrial Queueing Systems-A Computational Approach, Springer-Vrelag, Berlin Heidelberg.

[2] Artalejo, J. R. (1999a). A classified bibliography of research on retrial queues: Progress in (1990 – 1999), 7,187-211.

[3] Artalejo, J. R. (1999b). Acessible bibliography on retrial queues, Mathematical and Computer modeling, 30, 1-6.

[4] Bunday, B.D. (1986), Basic Queueing theory, London, Edward Arnold (Publishers) Ltd.

[5] Cohen, J.W. (1957). Basic problems of telephone traffic theory and the influence of repeated calls, Philips telecommunication review, 18(2), 49-100.

[6] Falin, G.I. and Templeton, J.G.C. (1997), Retrial Queues, London, Chapman & Hall.

[7] Falin, G.I. (1990). A survey on retrial queues, Queueing systems, 127-167.

[8] Garg, P.C. and Kumar (2012). A single server Retrial queues with impatient customers, Mathematical journal of interdisciplinary sciences, 1(1), 67-82.

[9] Indra and Renu (2011). Transient analysis of Markovian queueing model with Bernoulli schedule and multiple working vacations, International journal of Computer applications, 20(5), 6975-8887.

[10] Kosten, L. (1947). On the influence of repeated calls in the theory of probabilities of blocking, De Ingenieur, 59, 1-25.

[11] Pegden, C.D. and Rosenshine, M. (1982). Some new results for the M/M/1 queue. Mgt Sci, 28, 821-828.