

ON THE SOLUTIONS OF GENERALIZED FRACTIONAL KINETIC EQUATIONS INVOLVING GENERALIZED K-BESSEL FUNCTION

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Abstract: In this paper, we derive the solution of generalized fractional kinetic equations involving generalized k-Bessel function [4] by using Sumudu transform [9]. Here, the results obtained in terms of generalized Mittag-Leffler function of two parameters [10] are rather general in nature and can easily construct various known and (presumably) new results.

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1. Introduction and Preliminaries

Recently, a remarkable interest has been developed in the study of the solution of fractional kinetic equations due to their importance in mathematical physics and astrophysics. During the last several decades, fractional kinetic equations of different forms have been widely used in describing several important problems of physics and astrophysics. Saxena et al.[5] introduced the solution of the generalized fractional kinetic equations associated with the generalized Mittag-Leffler function. Subsequently, Saxena et al.[7] developed an alternative derivation for the solution of the generalized fractional kinetic equations in terms of special functions by applying Sumudu transform. In view of the effectiveness and a great importance of the kinetic equations, the authors develop a further generalized form of the fractional kinetic equation involving generalized k-Bessel function.

The fractional differential equation between the rate of change of reaction was established by Haubold and Mathai [3], and the destruction rate and the production rate are given as follows:

$$\frac{dN}{dt} = -d(N_t) + P(N_t), \quad (1)$$

where $N = N(t)$ is the rate of reaction, $d = d(N)$ is the rate of destruction, $p = p(N)$ is the rate of production and N_t denotes the function defined by $N_t(t^*) = N(t - t^*)$, $t^* > 0$.

The special case of (1), for spatial fluctuations and inhomogeneities in $N(t)$ the quantities are neglected, that is the equation

$$\frac{dN}{dt} = -c_i N_i(t), \quad (2)$$

with the initial condition that $N_i(t=0) = N_0$ is the number density of species i at time $t=0$ and constant $c_i > 0$. If we remove the index i and integrate the standard kinetic equation (2), we have

$$N(t) - N_0 = -c {}_0D_t^{-1} N(t), \quad (3)$$

where ${}_0D_t^{-1}$ is the special case of the Riemann-Liouville fractional integral operator ${}_0D_t^{-\nu}$ defined as

$${}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s) ds, \quad (t > 0, \operatorname{Re}(\nu) > 0). \quad (4)$$

The fractional generalization of the standard kinetic equation (3) is given by Haubold and Mathai [3] as follows:

$$N(t) - N_0 = -c^\nu {}_0D_t^{-\nu} N(t), \quad (5)$$

and obtained the solution of (5) as

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu n + 1)} (ct)^{\nu n}. \quad (6)$$

Further, Saxena and Kalla [6] considered the following fractional kinetic equation

$$N(t) - N_0 f(t) = -c^\nu {}_0D_t^{-\nu} N(t) \quad (\operatorname{Re}(\nu) > 0), \quad (7)$$

where $N(t)$ denotes the number density of a given species at time t , $N_0 = N(0)$ is the number density of that species at time $t=0$ and c is a constant.

Recently, the generalized k-Bessel function was introduced by Mondal [4] and given as follows:

$$W_{p,c}^k(z) = \sum_{n=0}^{\infty} \frac{(-c)^n \left(\frac{z}{2}\right)^{2n+\frac{p}{k}}}{\Gamma_k(nk + p + k) n!} \quad (8)$$

where $k > 0, \text{Re}(p) > -1$ and $c \in R$.

If we take $k=1$ and $c=1$ in (8), then generalized k-Bessel function reduce to the classical Bessel function $J_p(z)$ of order p as (See [1], [2], [8])

$$J_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+p}}{\Gamma(n+p+1) n!}, \quad (z, p \in C, \text{Re}(p) > -1) \tag{9}$$

Further, if we set $k=1$ and $c=-1$ in (1.8), then generalized k-Bessel function $W_{p,c}^k$ reduce to the modified Bessel function I_p ([8], p – 77) as

$$I_p(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n+p}}{\Gamma(n+p+1) n!}, \quad (z, p \in C, \text{Re}(p) > -1) \tag{10}$$

The generalized Mittag-Leffler function of two parameters was given and studied by Wiman [10] as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (\alpha, \beta \in C, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0) \tag{11}$$

2. Solution of generalized fractional kinetic equations by using Sumudu transform

In this section, we derive the solution of the generalized kinetic equations involving generalized k-Bessel function by applying Sumudu transform.

Sumudu transform was defined and studied by Watugala [9] to facilitate the process of solving differential and integral equations in the time domain, and for the use in various applications of system engineering and applied physics. Sumudu transform has very special and useful properties and it is useful in solving the problems of science and engineering governing kinetic equations. The Sumudu transform has been shown to be the theoretical dual of the Laplace transform.

The Sumudu transform is derived from the classical Fourier integral and defined over the set of the functions

$$A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{|t|/\tau_j} \text{ if } t \in (-1)^j \times [0, \infty) \right\}, \tag{12}$$

by the following formula

$$G(u) = S\{f(t):u\} = \int_0^{\infty} e^{-t} f(ut) dt, \quad (-\tau_1 < u < \tau_2), \tag{13}$$

where M is a real finite number and τ_1 and τ_2 can be finite or infinite (see [9]).

Hence, $G(u)$ is called as the Sumudu transform of $f(t)$. It is obvious that this is a linear operator.

The Sumudu and Laplace transforms exhibit a duality relation that may be expressed either as

$$G\left(\frac{1}{u}\right) = u F(u) \text{ or } G(u) = \frac{1}{u} F\left(\frac{1}{u}\right), \quad (14)$$

$$F\left(\frac{1}{p}\right) = p G(p) \text{ or } F(p) = \frac{1}{p} G\left(\frac{1}{p}\right). \quad (15)$$

The convolution theorem for Sumudu transform is given by

$$S\{f * g : u\} = u S\{f : u\} S\{g : u\}, \quad (16)$$

Theorem 1 Let $a, t, v \in R^+$ and $k > 0, c \in R$. Also let $p \in C$ with $Re(p) > -1$ and $Re(u) > 0$ with $|u| < a^{-1}$. Then the solution of following generalized fractional kinetic equation:

$$N(t) - N_0 W_{p,c}^k(t) = -a^v {}_0D_t^{-v} N(t) \quad (17)$$

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-c)^n \Gamma(2n + \frac{p}{k} + 1) \left(\frac{t}{2}\right)^{2n + \frac{p}{k}}}{\Gamma_k(nk + p + k) n!} E_{v, 2n + \frac{p}{k} + 1}(-a^v t^v) \quad (18)$$

where $E_{v, 2n + \frac{p}{k} + 1}(\cdot)$ is the generalized Mittag-Leffler function in (11).

Proof : By taking Sumudu transform on both sides of (17) and using (8) and (13), we obtain

$$S\{N(t) : u\} - N_0 S\{W_{p,c}^k(t) : u\} = -a^v S\{{}_0D_t^{-v} N(t) : u\}$$

$$N(u) - N_0 S\{W_{p,c}^k(t) : u\} = -a^v u^v N(u)$$

where $S\{N(t) : u\} = N(u)$ and $S\{{}_0D_t^{-v} N(t) : u\} = u^v N(u)$.

$$N(u)[1 + (au)^v] = N_0 S\left\{\sum_{n=0}^{\infty} \frac{(-c)^n \left(\frac{t}{2}\right)^{2n + \frac{p}{k}}}{\Gamma_k(nk + p + k) n!} : u\right\}$$

$$= N_0 \sum_{n=0}^{\infty} \frac{(-c)^n \left(\frac{1}{2}\right)^{2n + \frac{p}{k}}}{\Gamma_k(nk + p + k) n!} S\left\{t^{2n + \frac{p}{k}} : u\right\}$$

$$= N_0 \sum_{n=0}^{\infty} \frac{(-c)^n \left(\frac{1}{2}\right)^{2n + \frac{p}{k}}}{\Gamma_k(nk + p + k) n!} \int_0^{\infty} e^{-t} (ut)^{2n + \frac{p}{k}} dt$$

$$N(u)[1 + (au)^v] = N_0 \sum_{n=0}^{\infty} \frac{(-c)^n \left(\frac{u}{2}\right)^{2n + \frac{p}{k}}}{\Gamma_k(nk + p + k) n!} \Gamma\left(2n + \frac{p}{k} + 1\right)$$

Expanding $[1 + (au)^v]^{-1}$ as a geometric series for $|u| < a^{-1}$, we have

$$N(u) = N_0 \sum_{n=0}^{\infty} \frac{(-c)^n \left(\frac{u}{2}\right)^{2n+\frac{p}{k}} \Gamma(2n + \frac{p}{k} + 1)}{\Gamma_k(nk + p + k) n!} \sum_{r=0}^{\infty} (-1)^r (au)^{vr}$$

Finally, by taking inverse Sumudu transform and make use of the formulas $S^{-1}\{u^{v-1}:t\} = \frac{t^{v-1}}{\Gamma(v)}$ ($\min\{Re(v), Re(u)\} > 0$) and $S^{-1}\{N(u):t\} = N(t)$, then we get

$$\begin{aligned} N(t) &= N_0 \sum_{n=0}^{\infty} \frac{(-c)^n \left(\frac{t}{2}\right)^{2n+\frac{p}{k}} \Gamma(2n + \frac{p}{k} + 1)}{\Gamma_k(nk + p + k) n!} \sum_{r=0}^{\infty} (-1)^r a^{vr} S^{-1}\{u^{2n+\frac{p}{k}+vr}:t\} \\ &= N_0 \sum_{n=0}^{\infty} \frac{(-c)^n \left(\frac{t}{2}\right)^{2n+\frac{p}{k}} \Gamma(2n+\frac{p}{k}+1)}{\Gamma_k(nk+p+k) n!} \sum_{r=0}^{\infty} (-1)^r a^{vr} \frac{t^{2n+\frac{p}{k}+vr}}{\Gamma(2n+\frac{p}{k}+vr+1)} \\ &= N_0 \sum_{n=0}^{\infty} \frac{(-c)^n \left(\frac{t}{2}\right)^{2n+\frac{p}{k}} \Gamma(2n+\frac{p}{k}+1)}{\Gamma_k(nk+p+k) n!} \sum_{r=0}^{\infty} \frac{(-1)^r (a^v t^v)^r}{\Gamma(2n+\frac{p}{k}+1+vr)} \\ &= N_0 \sum_{n=0}^{\infty} \frac{(-c)^n \left(\frac{t}{2}\right)^{2n+\frac{p}{k}} \Gamma(2n+\frac{p}{k}+1)}{\Gamma_k(nk+p+k) n!} E_{v,2n+\frac{p}{k}+1}(-a^v t^v). \end{aligned}$$

This completes the proof of Theorem 1.

If we set $k = 1$ and $c = 1$ in Theorem 1, then we arrive at the following corollary.

Corollary 1.1 Let $a, t, v \in R^+$. Also let $p \in C$ with $Re(p) > -1$ and $Re(u) > 0$ with $|u| < a^{-1}$. Then the solution of the following generalized fractional kinetic equation involving classical Bessel function (9) :

$$N(t) - N_0 J_p(t) = -a^v {}_0D_t^{-v} N(t) \tag{19}$$

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(2n + p + 1) \left(\frac{t}{2}\right)^{2n+p}}{\Gamma(n + p + 1) n!} E_{v,2n+p+1}(-a^v t^v) \tag{20}$$

where $E_{v,2n+p+1}(\cdot)$ is the generalized Mittag-Leffler function in (11).

Further, if we put $k = 1$ and $c = -1$ in Theorem 1, then we obtain the following result :

Corollary 1.2 Let $a, t, v \in R^+$. Also let $p \in C$ with $Re(p) > -1$ and $Re(u) > 0$ with $|u| < a^{-1}$. Then the solution of the following generalized fractional kinetic equation involving Modified Bessel function (10) :

$$N(t) - N_0 I_p(t) = -a^v {}_0D_t^{-v} N(t) \tag{21}$$

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{\Gamma(2n + p + 1) \left(\frac{t}{2}\right)^{2n+p}}{\Gamma(n + p + 1) n!} E_{v,2n+p+1}(-a^v t^v) \tag{22}$$

where $E_{\nu, 2n+p+1}(\cdot)$ is the generalized Mittag-Leffler function in (11).

Theorem 2 Let $a, t, \nu \in R^+$ and $k > 0, c \in R$. Also let $p \in C$ with $Re(p) > -1$ and $Re(u) > 0$ with $|u| < a^{-1}$. Then the solution of following generalized fractional kinetic equation:

$$N(t) - N_0 W_{p,c}^k(a^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \quad (23)$$

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-c)^n \Gamma(2n\nu + \frac{p\nu}{k} + 1) \left(\frac{a^\nu t^\nu}{2}\right)^{2n + \frac{p}{k}}}{\Gamma_k(nk + p + k) n!} E_{\nu, \nu(2n + \frac{p}{k}) + 1}(-a^\nu t^\nu) \quad (24)$$

where $E_{\nu, \nu(2n + \frac{p}{k}) + 1}$ is the generalized Mittag-Leffler function given by (11).

Proof : Above Theorem 2 can be proved in parallel with the proof of Theorem 1. So, we omit the details of their proof.

If we take $k = 1$ and $c = 1$ in Theorem 2, then we arrive at the following corollary.

Corollary 2.1 Let $a, t, \nu \in R^+$. Also let $p \in C$ with $Re(p) > -1$ and $Re(u) > 0$ with $|u| < c^{-1}$. Then the solution of the following generalized fractional kinetic equation involving classical Bessel function (9) :

$$N(t) - N_0 J_p(a^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \quad (25)$$

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(2n\nu + p\nu + 1) \left(\frac{a^\nu t^\nu}{2}\right)^{2n+p}}{\Gamma(n + p + 1) n!} E_{\nu, \nu(2n+p)+1}(-a^\nu t^\nu) \quad (26)$$

where $E_{\nu, \nu(2n+p)+1}(\cdot)$ is the generalized Mittag-Leffler function given by (11).

Further, if we set $k = 1$ and $c = -1$ in Theorem 2, then we obtain the following result.

Corollary 2.2 Let $a, t, \nu \in R^+$. Also let $p \in C$ with $Re(p) > -1$ and $Re(u) > 0$ with $|u| < c^{-1}$. Then the solution of the following generalized fractional kinetic equation involving modified Bessel function (10) :

$$N(t) - N_0 I_p(a^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \quad (27)$$

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{\Gamma(2n\nu + p\nu + 1) \left(\frac{a^\nu t^\nu}{2}\right)^{2n+p}}{\Gamma(n + p + 1) n!} E_{\nu, \nu(2n+p)+1}(-a^\nu t^\nu) \quad (28)$$

where $E_{\nu, \nu(2n+p)+1}(\cdot)$ is the generalized Mittag-Leffler function given by (11).

3. Conclusion

In this paper, we give a new fractional generalization of the standard kinetic equation and derived solutions for the same. From the close relationship of the generalized k-Bessel function $W_{p,c}^k(z)$ with many special functions, we can easily construct various and new fractional kinetic equations.

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