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BIANCHI TYPE IX PERFECT FLUID COSMOLOGICAL MODELS WITH DARK ENERGY

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Abstract: In the present paper, we investigate Bianchi type IX cosmological models with perfect fluid and dark energy (phantom and quintessence). To get the physical aspects of the model, we assume that the potential $V(\phi)=\frac{\lambda}{n}\phi^n$ where ϕ is Higgs field. The Higgs field is initially large, but decreases with time for phantom and quintessence fields. Physical and kinematical behavior of the models are also discussed.

Keywords: Bianchi type IX, Perfect fluid, Dark Energy.

1. Introduction

The investigations on cosmological observations by Riess et al. [17], Perlmutter et al. [13,14] have confirmed that the present day universe is in a state of accelerated expansion. During the past few years, remarkable progress has been made in cosmology, both at observational and theoretical level. The most surprising results are presented in [17], [2], [3] and [6]. These observations support the fact that only ~ 4% of the total energy density of the universe is in the form of baryonic matter, ~ 24% is non - baryonic matter and almost ~ 72% is of completely unknown component with negative pressure. The presence of an energy component with negative pressure (still undistinguishable from the cosmological constant Λ) and the nature of this new component, commonly termed as dark energy, is a major challenge in cosmology and fundamental physics. Some examples of dark energy models are quintessence [15] and Phantom [4]. Quintessence is a hypothetical form of dark energy and it behaves like a cosmological constant by combining positive energy density and negative pressure. Zlatev et al. [25] has shown that “Tracker Field”, a form of quintessence, may explain the coincidence, adding new motivation for the quintessence scenario. Phantom energy is also a hypothetical form of dark energy satisfying the equation of state with $\omega < -1$in ap = wρ, p being isotropic pressure and ρ the matter density. It possesses negative kinetic energy and predicts expansion of the universe in excess of that predicted by a cosmological constant which leads to a Big Rip [4]. Shear free perfect fluid conjecture in general relativity is discussed by Muzikayise and Petter [12].

The Bianchi type IX universe behaves quite differently than the simple Bianchi type I anisotropic universe because it has both expansion aniosotropy (shear) and 3 - curvature anisotropy [1]. Tyagi et al. [22] have investigated the Bianchi type - IX string cosmological model for perfect fluid distribution under two conditions: $(i)\rho + \lambda = 0$ and $(ii)\rho - \lambda = 0$, λ being string tension density. Bianchi type IX cosmological models include closed FRW models. These models allow not only expansion but also rotation and shear in general relativity. King [9] has shown that every Bianchi type IX model universe can be interpreted as a closed Friedmann universe with superimposed gravitational waves that have the longest wavelength that fit into a closed universe.

Kumar and Singh [10] considered a spatially homogeneous and totally anisotropic Bianchi I space – time with perfect fluid and anisotropic dark energy, which has dynamical energy density. Reddy et al. [16] have investigated spatially homogeneous and anisotropic Bianchi type IX space - time filled with two minimally interacting fields, matter and holographic dark energy components in the scalar - tensor theory of gravitation formulated by Saez and Ballester. Ghate and Sontakke [8] have investigated Binary mixture of anisotropic dark energy and perfect fluid in Bianchi type IX space-time. Frahaman et al. [7] investigated a class of cosmological solutions of massive strings for the Bianchi IX space-time in Lyra geometry. Vaidya and Patel [23] have studied spatially homogeneous Bianchi type IX space-time and they have outlined general scheme for the derivation of exact solutions of Einstein's field equations in presence of perfect fluid and pure radiation fields. Tyagi and Chhajed [21] investigated homogeneous anisotropic Bianchi type IX cosmological model for perfect fluid distribution with electromagnetic field. Shaikh and Katore [19] have studied the solution of plane - symmetric universe in the presence and the absence of magnetic field. Coley and Goliath [5] have investigated closed, spatially homogeneous cosmological models with a perfect fluid and a scalar field with an exponential potential, using dynamical systems method.

Recently some noticeable works have been carried out by several cosmologists in the context of Bianchi Type IX perfect fluid cosmological models with dark energy. Serge and Piechocki [18] have studied the dynamics of the vacuum Bianchi IX model with time like singularity and compare it with the dynamics of the Bianchi IX model with cosmological singularity. Yadav [24] have investigated Bianchi type IX cosmological models for barotropic fluid distribution with vacuum energy density. Singh and

Srivastava [20] have investigated the dynamical evolution of a homogeneous and anisotropic Bianchi V model filled with perfect fluid and scalar field.

In this paper, we have discussed Bianchi type IX cosmological models with perfect fluid and dark energy. In the first section, the introduction and some recent work in the related field has been discussed. In the second section, field equations are constructed and then solved. Further some properties of the model with results and conclusions are presented.

2. The Metric and Field Equations

We consider the Bianchi type IX metric in the form

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 dy^2 + \left(B^2 \sin^2 y + A^2 \cos^2 y\right) dz^2 - 2A^2 \cos y dx dz \tag{1}$$

where A and B are functions of ‘t’ alone.

The energy momentum tensor for perfect fluid is given by

$$_{pf}T_{ab} = \left(\rho_{pf} + p_{pf}\right) u_a u_b + p_{pf}\, g_{ab} \tag{2}$$

where ρ_{pf} is the matter density and p_{pf} is the pressure.

For comoving observer

$$u^1 = 0 = u^2 = u^3 \text{ and } u^4 = 1 \tag{3}$$

The energy momentum tensor for the dark energy is described by scalar field ϕ as

$$_{\phi}T_{ab} = \left(\frac{1}{2}\varepsilon\phi_4^2 + V(\phi)\right) u_a u_b + \left(\frac{1}{2}\varepsilon\phi_4^2 - V(\phi)\right) h_{ab} \tag{4}$$

where $h_{ab} = g_{ab} + u_a u_b$

The phantom field $(\varepsilon = -1)$ may be considered as perfect fluid for comoving observers and the density and pressure are given by [ ]

$$\rho_\phi = -\frac{1}{2}\phi_4^2 + V(\phi) \tag{5}$$

$$p_\phi = -\frac{1}{2}\phi_4^2 - V(\phi) \tag{6}$$

where as for quintessence field $(\varepsilon = 1)$ the density and pressure are

$$\rho_\phi = \frac{1}{2}\phi_4^2 + V(\phi) \tag{7}$$

$$p_{\phi} = \frac{1}{2}\phi_4^2 - V\left(\phi\right) \tag{8}$$

For phantom field and for quintessence field equations (5), (6) and (7), (8) give

$$\rho_{\phi} - p_{\phi} = 2V\left(\phi\right) \tag{9}$$

The Einstein's field equations are given by

$$R_i^j - \frac{1}{2}Rg_i^j = -8\pi G\left({}_{pf}T_i^j + {}_{\phi}T_i^j \right) + \Lambda g_i^j \tag{10}$$

The Klein-Gordon equation for dark energy (quintessence and phantom) with potential $V(\phi)$ is given by

$$\phi_{44} + \left(\frac{A_4}{A} + \frac{2B_4}{B} \right)\phi_4 = -\varepsilon \frac{dV(\phi)}{d\phi} \tag{11}$$

Where $\varepsilon = 1$ (quintessence) and $\varepsilon = -1$ (phantom)

For the line element (1) Einsterin's field equations reduce to the following system of equations

$$\frac{2B_{44}}{B} + \frac{B_4^2}{B^2} + \frac{1}{B^2} - \frac{3A^2}{4B^4} = -8\pi G\left(p_{pf} + \varepsilon\frac{1}{2}\phi_4^2 - V(\phi) \right) + \Lambda \tag{12}$$

$$\frac{A_{44}}{A} + \frac{A_4B_4}{AB} + \frac{B_{44}}{B} + \frac{A^2}{4B^4} = -8\pi G\left(p_{pf} + \varepsilon\frac{1}{2}\phi_4^2 - V(\phi) \right) + \Lambda \tag{13}$$

$$\frac{2A_4B_4}{AB} + \frac{B_4^2}{B^2} - \frac{A^2}{4B^4} + \frac{1}{B^2} = -8\pi G\left(-\rho_{pf} - \varepsilon\frac{1}{2}\phi_4^2 - V(\phi) \right) + \Lambda \tag{14}$$

The Klein – Gordon equation for dark energy (quintessence and phantom) with potential $V(\phi)$ is given by

$$\phi_{44} + 3H\phi_4 = -\varepsilon \frac{dV(\phi)}{d\phi} \tag{15}$$

where the sub – indices $_4$ denotes ordinary differentiation with respect to t.

3. Solution of the field equations

To get the deterministic solution of Einstein's field equation (12) – (15) in terms of cosmic time t, We assume that

$$A = B^n \tag{16}$$

From equation (12) and (13) we get

$$\frac{A_{44}}{A} - \frac{B_{44}}{B} + \frac{B_4}{B}\left(\frac{A_4}{A} - \frac{B_4}{B}\right) + \frac{A^2}{B^4} - \frac{1}{B^2} = 0 \tag{17}$$

Using equation (16) in equation (17),we get

$$BB_{44} + \alpha B_4^2 = \frac{B^{2n-2}}{(1-n)} - \frac{1}{1-n}, \tag{18}$$

where $\alpha = n+1$.

Let $B_4 = f(B)$, (19)

then $B_{44} = ff'$, (20)

where $f' = \frac{df}{dB}$.

Using equation (19) and (20) in equation (18), we get

$$\frac{d\left(f^2\right)}{dB} + \frac{2\alpha}{B} f^2 = \frac{2B^{2n-3}}{(1-n)} - \frac{2}{(1-n)B}, \tag{21}$$

which is a linear differential equation in f^2. Its solution is

$$f^2 = \left(\frac{dB}{dt}\right)^2 = \frac{B^{2n-2}}{(1-n)(n+\alpha-1)} + \frac{\beta}{B^{2\alpha}} - c_1 \tag{22}$$

Where β is the constant of integration and $c_1 = \frac{1}{\alpha(1-n)}$

Therefore the metric (1) reduces to

$$ds^2 = -\left(\frac{dt}{dB}\right)^2 dB^2 + B^{2n}dx^2 + B^2dy^2 + \left(B^2\sin^2 y + B^{2n}\cos^2 y\right)dz^2 - 2B^{2n}\cos y\,dx\,dz \tag{23}$$

After taking suitable transformations of coordinates as follows

$$B = T, x = X, y = Y, z = Z$$

the metric (23) reduces to

$$ds^2 = -\frac{dT^2}{\frac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \frac{\beta}{T^{2\alpha}} - \frac{1}{\alpha(1-n)}} + T^{2n}dX^2 + T^2 dY^2 + \left(T^2 \sin^2 Y + T^{2n} \cos^2 Y\right)dZ^2 - 2T^{2n} \cos Y dX dZ \tag{24}$$

4. Some Physical Parameters

Scalar expansion (θ) for the model (24) is given as

$$\theta = \frac{(n+2)}{T}\sqrt{\frac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \frac{\beta}{T^{2\alpha}} - c_1} \tag{25}$$

Shear (σ) for the model (24) is given as

$$\sigma^2 = \frac{\kappa}{T^{n+2}} \tag{26}$$

The scale factor is defined as

$$R^3 = AB^2 = T^{n+2} \tag{27}$$

Case I

For phantom field $(\varepsilon = -1)$

The equation (11) leads to

$$\phi_{44} + \left(\frac{A_4}{A} + \frac{2B_4}{B}\right)\phi_4 = \frac{dV(\phi)}{d\phi} \tag{28}$$

Which leads to

$$\phi_{44} + (n+2)\frac{B_4}{B}\phi_4 = \lambda\phi^{n-1} \tag{29}$$

Taking $\phi_{44} << \frac{dV(\phi)}{d\phi}$ equation (29) reduces to

$$\frac{d\phi}{\phi^{n-1}} = \frac{\lambda}{(n+2)}\frac{T}{f^2}dT \tag{30}$$

where $f = \frac{dB}{dt}$

Equation (30) on integration leads to

$$\phi^{-n+2}=\lambda\frac{2-n}{2+n}\int\frac{TdT}{\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}+\frac{\beta}{T^{2\alpha}}-c_1}+c_2 \tag{31}$$

Where c_2 is the constant of integration.

In particular if we take $\beta=0$ and $c_1=0$, equation (31) leads to

$$\phi^{-n+2}=\frac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}}+c_2 \tag{32}$$

The matter density (ρ_{pf}) and pressure (p_{pf}) for the model (25) are given by

$$8\pi G\rho_{pf}=\frac{(2n+1)}{T^2}\left[\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}+\frac{\beta}{T^{2\alpha}}-c_1\right]-\frac{1}{4}T^{2n-4}+\frac{1}{T^2}+$$

$$\frac{\frac{4\pi G\lambda^2}{(n+2)^2}T^2\left[\frac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}}+c_2\right]^{\frac{2n-2}{2-n}}}{\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}+\frac{\beta}{T^{2\alpha}}-c_1}- \tag{33}$$

$$\frac{8\pi G\lambda}{n}\left[\frac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}}+c_2\right]^{\frac{n}{2-n}}-\Lambda$$

$$8\pi Gp_{pf}=\frac{(2\alpha-1)}{T^2}\left[\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}+\frac{\beta}{T^{2\alpha}}-c_1\right]-\left(\frac{2}{1-n}-\frac{3}{4}\right)T^{2n-4}+\frac{2}{T^2(1-n)}-\frac{1}{T^2}$$

$$\frac{\frac{4\pi G\lambda^2}{(n+2)^2}T^2\left[\frac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}}+c_2\right]^{\frac{2n-2}{2-n}}}{\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}+\frac{\beta}{T^{2\alpha}}-c_1} \tag{34}$$

$$+\frac{8\pi G\lambda}{n}\left[\frac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}}+c_2\right]^{\frac{n}{2-n}}+\Lambda$$

Case II

For quintessence field $(\varepsilon=1)$

The equation (11) leads to

$$\phi_{44}+\left(\frac{A_4}{A}+\frac{2B_4}{B}\right)\phi_4=-\frac{dV(\phi)}{d\phi} \tag{35}$$

Which leads to

$$\phi_{44}+(n+2)\frac{B_4}{B}=-\lambda\phi^{n-1} \tag{36}$$

Taking $\phi_{44}<<\frac{dV(\phi)}{d\phi}$ equation (29) reduces to

$$\frac{d\phi}{\phi^{n-1}}=\frac{-\lambda}{(n+2)}\frac{T}{f^2}dT \tag{37}$$

where $f=\frac{dB}{dt}$

Equation (37) on integration leads to

$$\phi^{-n+2}=-\lambda\frac{2-n}{n+2}\int\frac{TdT}{\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}+\frac{\beta}{T^{2\alpha}}-c_1}+c_3 \tag{38}$$

Where c_3 is the constant of integration and $c_1=\frac{1}{\alpha(1-n)}$

In particular if we take $\beta=0$ and $c_1=0$, equation (38) leads to

$$\phi^{-n+2}=\frac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}}+c_3 \tag{39}$$

The matter density (ρ_{pf}) and pressure (p_{pf}) for the model (25) are

$$\begin{aligned}8\pi G\rho_{pf}=&\frac{(2n+1)}{T^2}\left[\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}+\frac{\beta}{T^{2\alpha}}-c_1\right]-\frac{1}{4}T^{2n-4}+\frac{1}{T^2}+\\&\frac{\frac{4\pi G\lambda^2}{(n+2)^2}T^2\left[\frac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}}+c_3\right]^{\frac{2n-2}{2-n}}}{\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}+\frac{\beta}{T^{2\alpha}}-c_1}-\\&\frac{8\pi G\lambda}{n}\left[\frac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}}+c_3\right]^{\frac{n}{2-n}}-\Lambda\end{aligned} \tag{40}$$

$$
\begin{aligned}
8\pi G p_{pf} &= \frac{(2\alpha-1)}{T^{2}}\left[\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}+\frac{\beta}{T^{2\alpha}}-c_{1}\right]-\left(\frac{2}{1-n}-\frac{3}{4}\right)T^{2n-4}+\frac{2}{T^{2}(1-n)} \\
&-\frac{1}{T^{2}}-\frac{\frac{4\pi G\lambda^{2}}{(n+2)^{2}}T^{2}\left[\frac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}}+c_{3}\right]^{\frac{2n-2}{2-n}}}{\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}+\frac{\beta}{T^{2\alpha}}-c_{1}} \\
&+\frac{8\pi G\lambda}{n}\left[\frac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}}+c_{3}\right]^{\frac{n}{2-n}}+\Lambda
\end{aligned}
\tag{41}
$$

5. Conclusion

We find that the spatial volume increases with time. Hence the model (24) represents inflationary scenario. Since $\frac{\sigma}{\theta} \neq 0$, hence anisotropy is maintained throughout. However for $\beta = 0 \, and \, c_1 = 0$ the model (24) isotropizes. The model has POINT TYPE singularity at T = 0 if $n > 0$. For $n < 0$, the model (24) has CIGAR TYPE singularity at T = 0 (MacCallum[11]). The model (24) starts with a big bang at T = 0 and the scalar expansion (θ) decreases with time. The Higgs field (ϕ) for phantom field is initially large but decreases due to lapse of time and for quintessence field the Higgs field (ϕ) also decreases with time. In phantom field $(\varepsilon = -1)$ as well as in quintessence field $(\varepsilon = 1)$ as $T \to \infty$ then $\phi^{(n-2)} \to 0$, and for n = 4, $\phi^2 \to 0$. The Scalar expansion (θ) are initially large, but decreases due to the lapse of time provided $n > -2$.

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Appendix

The calculation after equation no.(29) are as follows.

$$\phi_{44} + (n+2)\frac{B_4}{B}\phi_4 = \lambda\phi^{n-1} \tag{29}$$

Taking $\phi_{44} << \dfrac{dV(\phi)}{d\phi}$ in equation (29) we get

$$\frac{d\phi}{\phi^{n-1}} = \frac{\lambda}{n+2}\frac{T}{f^2}dT \tag{30}$$

Where $f = \dfrac{dB}{dt}$

Equation (30) on integrating leads to

$$\phi^{-n+2} = \frac{\lambda(2-n)}{(2+n)}\int \frac{TdT}{\dfrac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \dfrac{\beta}{T^{2\alpha}} - c_1} + c_2 \tag{31}$$

Where $c_1 = \dfrac{1}{\alpha(1-n)}$ and c_2 is constant of integration

In particular we take $\beta = 0$ and $c_1 = 0$ equation (31) leads to

$$\phi^{-n+2} = \frac{\lambda(2-n)}{(2+n)}\int \frac{TdT}{\dfrac{T^{2n-2}}{(1-n)(n+\alpha-1)}} + c_2$$

Then $\phi^{-n+2} = \dfrac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}} + c_2$ (32)

Then $\phi^{n} = \left[\dfrac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}} + c_2\right]^{\frac{n}{2-n}}$

And $\phi^{n-1} = \left[\dfrac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}} + c_2\right]^{\frac{n-1}{2-n}}$

And $\phi^{2n-2} = \left[\dfrac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}} + c_2 \right]^{\frac{2n-2}{2-n}}$

In equation (29) taking $\phi_{44} << \dfrac{dV(\phi)}{d\phi}$ then

$(n+2)\dfrac{B_4}{B}\phi_4 = \lambda\phi^{n-1}$ then

$\phi_4 = \dfrac{\lambda}{(n+2)}\phi^{n-1}\dfrac{B}{B_4}$ Replace $B = T$ and $f = \dfrac{dB}{dt}$ then

$\phi_4 = \dfrac{\lambda}{(n+2)}\phi^{n-1}\dfrac{T}{f}$

$\phi_4^2 = \dfrac{\lambda^2}{(n+2)^2}\phi^{2n-2}\dfrac{T^2}{f^2}$

Putting value of ϕ^{2n-2} and f^2 then

$$\phi_4^2 = \frac{\lambda^2}{(n+2)^2}\left[\frac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}} + c_2 \right]^{\frac{2n-2}{2-n}} \frac{T^2}{\dfrac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \dfrac{\beta}{T^{2\alpha}} - c_1}$$

Now from equation no.(14)

$$8\pi G\rho_{pf} + 4\pi G\varepsilon\phi_4^2 + 8\pi GV(\phi) + \Lambda = \frac{2A_4B_4}{AB} + \frac{B_4^2}{B^2} - \frac{A^2}{4B^4} + \frac{1}{B^2}$$

$$8\pi G\rho_{pf} = \frac{2A_4B_4}{AB} + \frac{B_4^2}{B^2} - \frac{A^2}{4B^4} + \frac{1}{B^2} - 4\pi G\varepsilon\phi_4^2 - 8\pi GV(\phi) - \Lambda$$

Since $A = B^n$ then

$$8\pi G\rho_{pf} = (2n+1)\frac{B_4^2}{B^2} - \frac{B^{2n}}{4B^4} + \frac{1}{B^2} - 4\pi G\varepsilon\phi_4^2 - 8\pi GV(\phi) - \Lambda \qquad \text{(A)}$$

$8\pi G\rho_{pf} = \dfrac{(2n+1)}{T^2}f^2 - \dfrac{T^{2n}}{4T^4} + \dfrac{1}{T^2} - 4\pi G(-1)\phi_4^2 - 8\pi G\dfrac{\lambda}{n}\phi^n - \Lambda$ where $\varepsilon = -1$ and $V(\phi) = \dfrac{\lambda}{n}\phi^n$

$$8\pi G\rho_{pf} = \frac{(2n+1)}{T^2}\left[\frac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \frac{\beta}{T^{2\alpha}} - c_1\right] - \frac{1}{4}T^{2n-4} + \frac{1}{T^2} +$$

$$\frac{4\pi G\lambda^2}{(n+2)^2}\left[\frac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}} + c_2\right]^{\frac{2n-2}{2-n}} \frac{T^2}{\dfrac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \dfrac{\beta}{T^{2\alpha}} - c_1} -$$

$$8\pi G\frac{\lambda}{n}\left[\frac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}} + c_2\right]^{\frac{n}{2-n}} - \Lambda$$

$$8\pi G\rho_{pf} = \frac{(2n+1)}{T^2}\left[\frac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \frac{\beta}{T^{2\alpha}} - c_1\right] - \frac{1}{4}T^{2n-4} + \frac{1}{T^2} +$$

$$\frac{\dfrac{4\pi G\lambda^2}{(n+2)^2}T^2\left[\dfrac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}} + c_2\right]^{\frac{2n-2}{2-n}}}{\dfrac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \dfrac{\beta}{T^{2\alpha}} - c_1} -$$

$$\frac{8\pi G\lambda}{n}\left[\frac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}} + c_2\right]^{\frac{n}{2-n}} - \Lambda \tag{33}$$

Now from equation (12)

$$\frac{2B_{44}}{B} + \frac{B_4^2}{B^2} + \frac{1}{B^2} - \frac{3A^2}{4B^4} = -8\pi GP_{pf} - 4\pi G\varepsilon\phi_4^2 + 8\pi GV(\phi) + \Lambda$$

$$8\pi GP_{pf} = -\frac{2B_{44}}{B} - \frac{B_4^2}{B^2} - \frac{1}{B^2} + \frac{3A^2}{4B^4} - 4\pi G\varepsilon\phi_4^2 + 8\pi GV(\phi) + \Lambda$$

$$8\pi GP_{pf} = -\frac{2B_{44}}{B} - \frac{B_4^2}{B^2} - \frac{1}{B^2} + \frac{3B^{2n}}{4B^4} - 4\pi G(-1)\phi_4^2 + 8\pi G\frac{\lambda}{n}\phi^n + \Lambda$$ where $\varepsilon = -1$ and

$$V(\phi) = \frac{\lambda}{n}\phi^n$$

since $BB_{44} + \alpha B_4^2 = \dfrac{B^{2n-2}}{1-n} - \dfrac{1}{1-n}$ then

$$B_{44} = -\frac{\alpha B_4^2}{B} + \frac{B^{2n-3}}{(1-n)} - \frac{1}{B(1-n)}$$

Putting value of B_{44} we get

$$8\pi G P_{pf} = -\frac{2}{B}\left[\frac{-\alpha B_4^2}{B} + \frac{B^{2n-3}}{(1-n)} - \frac{1}{B(1-n)}\right] - \frac{B_4^2}{B^2} - \frac{1}{B^2} + \frac{3B^{2n}}{4B^4} + 4\pi G\phi_4^2 + 8\pi G\frac{\lambda}{n}\phi^n + \Lambda$$

$$= (2\alpha-1)\frac{B_4^2}{B^2} - \left(\frac{2}{1-n} - \frac{3}{4}\right)B^{2n-4} + \frac{2}{B^2(1-n)} - \frac{1}{B^2} + 4\pi G\phi_4^2 + 8\pi G\frac{\lambda}{n}\phi^n + \Lambda$$

$$= \frac{(2\alpha-1)}{T^2} f^2 - \left(\frac{2}{1-n} - \frac{3}{4}\right)T^{2n-4} + \frac{2}{T^2(1-n)} - \frac{1}{T^2} + 4\pi G\phi_4^2 + 8\pi G\frac{\lambda}{n}\phi^n + \Lambda$$

$$8\pi G P_{pf} = \frac{(2\alpha-1)}{T^2}\left[\frac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \frac{\beta}{T^{2\alpha}} - c_1\right] - \left(\frac{2}{1-n} - \frac{3}{4}\right)T^{2n-4} + \frac{2}{T^2(1-n)} - \frac{1}{T^2} +$$

$$\frac{4\pi G\lambda^2}{(n+2)^2}\left[\frac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}} + c_2\right]^{\frac{2n-2}{2-n}} \frac{T^2}{\frac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \frac{\beta}{T^{2\alpha}} - c_1} + 8\pi G\frac{\lambda}{n}$$

$$\left[\frac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}} + c_2\right]^{\frac{n}{2-n}} + \Lambda$$

$$8\pi G p_{pf} = \frac{(2\alpha-1)}{T^2}\left[\frac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \frac{\beta}{T^{2\alpha}} - c_1\right] - \left(\frac{2}{1-n} - \frac{3}{4}\right)T^{2n-4} + \frac{2}{T^2(1-n)} - \frac{1}{T^2}$$

$$\frac{\frac{4\pi G\lambda^2}{(n+2)^2}T^2\left[\frac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}} + c_2\right]^{\frac{2n-2}{2-n}}}{\frac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \frac{\beta}{T^{2\alpha}} - c_1}$$

$$+\frac{8\pi G\lambda}{n}\left[\frac{\lambda(1-n)(n+\alpha-1)}{2(2+n)T^{2n-4}} + c_2\right]^{\frac{n}{2-n}} + \Lambda \tag{34}$$

Case: 2 For quintessence field $\varepsilon = 1$

The equation (11) leads to

$$\phi_{44} + \left(\frac{A_4}{A} + \frac{2B_4}{B}\right)\phi_4 = -\frac{dV(\phi)}{d\phi} \tag{35}$$

Taking $\phi_{44} << \frac{dV(\phi)}{d\phi}$

$$\left(\frac{A_4}{A}+\frac{2B_4}{B}\right)\phi_4=-\frac{dV(\phi)}{d\phi}$$

$$\left(\frac{nB_n}{B}+\frac{2B_4}{B}\right)\phi_4=-\frac{d}{d\phi}\left(\frac{\lambda}{n}\phi^n\right)$$

$$(n+2)\frac{B_4}{B}\phi_4=-\lambda\phi^{n-1} \tag{36}$$

Then

$$\frac{d\phi}{\phi^{n-1}}=\frac{-\lambda}{(n+2)}\frac{T}{f}dT \tag{37}$$

$$\phi^{-n+2}=\frac{-\lambda(2-n)}{(n+2)}\int\frac{TdT}{\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}+\frac{\beta}{T^{2\alpha}}-c_1}+c_3 \tag{38}$$

Where $c_1=\frac{1}{\alpha(1-n)}$ and c_3 is constant of integration.

if $\beta=0$ and $c_1=0$ then

$$\phi^{-n+2}=\frac{-\lambda(2-n)}{(n+2)}\int\frac{TdT}{\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}}+c_3$$

$$\phi^{-n+2}=\frac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}}+c_3 \tag{39}$$

$$\phi^n=\left[\frac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}}+c_3\right]^{\frac{n}{2-n}}$$

From equation (36)

$\phi_4=\frac{-\lambda}{(n+2)}\phi^{n-1}\frac{B}{B_4}$ replace $B=T$ and $f=\frac{dB}{dt}$

$\phi_4 = \frac{-\lambda}{(n+2)}\phi^{n-1}\frac{T}{f}$ then

$$\phi_4^2 = \frac{\lambda^2}{(n+2)^2}\phi^{2n-2}\frac{T^2}{f^2}$$

$$\phi_4^2 = \frac{\lambda^2}{(n+2)^2}\left[\frac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}}+c_3\right]^{\frac{2n-2}{2-n}}\frac{T^2}{\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}+\frac{\beta}{T^{2\alpha}}-c_1}$$

From equation no.(A)

$$8\pi G\rho_{pf} = (2n+1)\frac{B_4^2}{B^2}-\frac{B^{2n}}{4B^4}+\frac{1}{B^2}-4\pi G\varepsilon\phi_4^2-8\pi GV(\phi)-\Lambda$$

$8\pi G\rho_{pf} = \frac{(2n+1)}{T^2}f^2-\frac{T^{2n}}{4T^4}+\frac{1}{T^2}-4\pi G(1)\phi_4^2-8\pi G\frac{\lambda}{n}\phi^n-\Lambda$ where $\varepsilon = 1$ and $V(\phi) = \frac{\lambda}{n}\phi^n$

$$8\pi G\rho_{pf} = \frac{(2n+1)}{T^2}\left[\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}+\frac{\beta}{T^{2\alpha}}-c_1\right]-\frac{1}{4}T^{2n-4}+\frac{1}{T^2}+$$

$$\frac{4\pi G\lambda^2}{(n+2)^2}\left[\frac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}}+c_3\right]^{\frac{2n-2}{2-n}}\frac{T^2}{\frac{T^{2n-2}}{(1-n)(n+\alpha-1)}+\frac{\beta}{T^{2\alpha}}-c_1}-$$

$$8\pi G\frac{\lambda}{n}\left[\frac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}}+c_3\right]^{\frac{n}{2-n}}-\Lambda$$

$$8\pi G\rho_{pf} = \frac{(2n+1)}{T^2}\left[\frac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \frac{\beta}{T^{2\alpha}} - c_1\right] - \frac{1}{4}T^{2n-4} + \frac{1}{T^2} +$$

$$\frac{\dfrac{4\pi G\lambda^2}{(n+2)^2}T^2\left[\dfrac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}} + c_3\right]^{\frac{2n-2}{2-n}}}{\dfrac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \dfrac{\beta}{T^{2\alpha}} - c_1} - \tag{40}$$

$$\frac{8\pi G\lambda}{n}\left[\frac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}} + c_3\right]^{\frac{n}{2-n}} - \Lambda$$

and

$$8\pi GP_{pf} = -\frac{2B_{44}}{B} - \frac{B_4^2}{B^2} - \frac{1}{B^2} + \frac{3A^2}{4B^4} - 4\pi G\varepsilon\phi_4^2 + 8\pi GV(\phi) + \Lambda$$

$$8\pi GP_{pf} = -\frac{2B_{44}}{B} - \frac{B_4^2}{B^2} - \frac{1}{B^2} + \frac{3B^{2n}}{4B^4} - 4\pi G(1)\phi_4^2 + 8\pi G\frac{\lambda}{n}\phi^n + \Lambda \text{ where } \varepsilon = -1 \text{ and}$$

$$V(\phi) = \frac{\lambda}{n}\phi^n$$

$$8\pi GP_{pf} = \frac{(2\alpha-1)}{T^2}f^2 - \left(\frac{2}{1-n} - \frac{3}{4}\right)T^{2n-4} + \frac{2}{T^2(1-n)} - \frac{1}{T^2} - 4\pi G\phi_4^2 + 8\pi G\frac{\lambda}{n}\phi^n + \Lambda$$

$$8\pi GP_{pf} = \frac{(2\alpha-1)}{T^2}\left[\frac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \frac{\beta}{T^{2\alpha}} - c_1\right] - \left(\frac{2}{1-n} - \frac{3}{4}\right)T^{2n-4} + \frac{2}{T^2(1-n)} - \frac{1}{T^2} -$$

$$\frac{4\pi G\lambda^2}{(n+2)^2}\left[\frac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}} + c_3\right]^{\frac{2n-2}{2-n}} \frac{T^2}{\dfrac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \dfrac{\beta}{T^{2\alpha}} - c_1} + 8\pi G\frac{\lambda}{n}$$

$$\left[\frac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}} + c_3\right]^{\frac{n}{2-n}} + \Lambda$$

$$
\begin{aligned}
8\pi G p_{pf} = {} & \frac{(2\alpha - 1)}{T^{2}}\left[\frac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \frac{\beta}{T^{2\alpha}} - c_{1}\right] - \left(\frac{2}{1-n} - \frac{3}{4}\right)T^{2n-4} + \frac{2}{T^{2}(1-n)} \\
& - \frac{1}{T^{2}} - \frac{\dfrac{4\pi G\lambda^{2}}{(n+2)^{2}}\, T^{2}\left[\dfrac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}} + c_{3}\right]^{\frac{2n-2}{2-n}}}{\dfrac{T^{2n-2}}{(1-n)(n+\alpha-1)} + \dfrac{\beta}{T^{2\alpha}} - c_{1}} \\
& + \frac{8\pi G\lambda}{n}\left[\frac{-\lambda(1-n)(n+\alpha-1)}{2(n+2)T^{2n-4}} + c_{3}\right]^{\frac{n}{2-n}} + \Lambda
\end{aligned}
\tag{41}
$$