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AN INTERIOR GRIFFITH-CRACK OPENED BY THERMAL STRESS IN AN ORTHOTROPIC INFINITE STRIP

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Abstract : The closed form expressions of stress-intensity factors and of crackshape are obtained by using Fourier transform method for a ring shaped crack in an isotropic solid.
Keywords : (1) Stress Intensity Factor (S.I.F.), (2) Crack Opening Displacement (C.O.D.), (3) Fourier Transform (F.T.), (4) Modified Bessels Function (MBF).

1. Introduction

Civil Engineers use circular pillars in the constructions of bridges. These pillars are made by iron frame and concrete with cement. The continuous use of bridges, the iron frame which is circular is shape leave the matrix (made of concrete and cement).

This causes discontinuity in the medium. The shape of discontinuity is in ring shape.

The height and radios of the pillar are large in comparison to radios or width of ring shaped discontinuity. Therefore it is considered as infinite three dimensional isotropic solid with ring shaped discontinuity whose axis coincides with z-axis.

Linear fracture mechanics has established itself highlyas satisfactory working tool in studying the phenomenon of brittle fracture and crack propagation in solid structures. The technique appear to be more effective when plane-strain conditions prevail.

The crack problems in shell's type solid structure or crack discontinuity in shell shape in solids poses limitation. Two major limitations arise from geometry and material behavior. The geometrical factors include the relative size of the crack with respect to radii of curvature of shells and shape and orientation of crack. So far as material properties are concerned we take up isotropic homogeneous solid having shell-type discontinuity.

Erdogan and Ratwani [2] calculated the stresses causing fatigue and fracture of isotropic cylindrical shell containing circumferential crack by using numerical method. Erdogan [3] extended above method to orthotropic cylindrical shell having axial crack. Ma et. al [9] obtained stress-intensity factors for axial cracks in hollow isotropic cylindrical shell by using finite-element technique.

Liu et. al [8] analyzed the crack closure effect on stress-intensity factors for circumferentially cracked cylindrical shell. Lal and Pandey [6] and Lal [7] has discussed thermo-elastic problem with penny-shaped crack reducing the problem to able integral equation.

Jaunky et.al [5] disused the mechanical response of laminated composite cylindrical panel in axial compression by using shell theories.

The problem in present research endeavor is of ring shaped crack having axis parallel to z-axis. The infinite 3-D isotropic is now cylinder of infinite radius and axis as z-axis. The ring shaped crack occupies the space $r = d, 0 \le |z| < c,\ \ 0 \le \theta \le 2\pi$ see figure-1.

The crack is formed by hydro-static force acting in medium and it is such that the cross-sections obtained by any $\theta = \alpha$ are same. It reduces the 3-dimensional problem to 2-dimension i.e. *r* and *z*. We take cross-section by $\theta = 0$ and $\theta = \pi$, see figure 2a. It is being assumed that $\sigma_{\theta\theta} = 0$ and the operator $\frac{\partial}{\partial\theta}$ is null operator. Thus the co-ordinates of any point will be *r* and *z* when cylindrical co-ordinate system is taken.

Thus the physical problem is reduced to the following mixed boundary value problem.

$$\sigma_{rr}(d,z) = -p(z), 0 \le |z| < c,\ u_r(d,z) = 0, c \le |z| < \infty, \qquad \ldots(1)\text{ - }(2)$$

$$\sigma_{rz}(d,z) = 0, 0 \le |z| < \infty, \qquad \ldots(3)$$

and all physical quantities, i.e., the components of stress and of displacement are zero as $r, z \Rightarrow \infty$.

We checked throughout that

$$u_r(d,z) > 0, 0 \le |z| < c \qquad \ldots(4)$$

which means that crack really opens out and the faces of crack do not meet each other than at crack tips, see Burniston [1].

The symmetry of geometry and of loading reduce the boundary and mixed-boundary conditions (1)-(3) reduce to, see figure 2b

$$\sigma_{rr}(d,z) = -p(z), 0 \le z < c\ ,\ u_r(d,z) = 0, c \le z < \infty \qquad \ldots(5)\text{ - }(6)$$

$$\sigma_{rz}(d,z) = 0, 0 \le z < \infty \qquad \ldots(7)$$

The plan of the paper is as to follows : Section 1 introduction the problem and reduce to mixed-boundary value problem. Section-2 formulate the mixed-boundary value problem and reduces to dual integral equation. Section-3 solves the dual integral equation and reduces to Fredholm integral equation of second kind. Section-4 solves the Fredholm integral equation. Physical quantities are given in Section-5. This section takes one special case of loading, too.

2. Formulation and Reduction to Dual Integral Equation

The equations of equilibrium, after using stress-strain equations, are reduced to fourth order partial differential equation in u_r as :

$$\Delta^2(\Delta^2 u_r(r,z)) = 0\,,\ \Delta^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\,\mathrm{d} \qquad \ldots(8)$$

with,

The other displacement component u_z is related with u_r in the following manner

$$u_z(r,z) = \frac{1}{p}\left[(\lambda+2\mu)\int \left\langle \frac{u_r}{r} + \frac{\partial u_r}{\partial r} \right\rangle dz + \int \frac{\partial u_r}{dz}\,dr\right]_{\mathrm{f}} \qquad \ldots(9)$$

where λ and μ are Lame's constants. We assume the solution of (8) as

$$u_r(r,z) = \int_0^\infty \cos(sz)[A(s)I_1(sr) + rB(s)I_0(sr)]ds \qquad \ldots(10)$$

Then

$$u_z(r,z) = -\frac{1}{P}\int_0^\infty \sin(sz)\left[Q\,A(s)I_0(sr) + B(s)\{(2+rs)I_1(sr) + \frac{(2-rs)}{rs}I_0(sr)\right]ds \qquad \ldots(11)$$

Where $P = \lambda + \mu,\ Q = 1 + \mu + P$ and $I_0(sr),\ I_1(sr)$ are modified Bessel's functions of kind first with order zero and one. The use of stress-strain relations and (10)-(11) we get

$$\sigma_{rr}(r,z) = \int_0^\infty s\cos(sz)\left[-A(s)\left\{I_0(rs)\alpha_0 + \frac{I_1(sr)}{sr}\right\} + B(s)\left\{\alpha_1 I_0(sr) + (sr(1+Q)+1)I_1(sr)\right\}\right]ds \qquad \ldots(12)$$

$$\sigma_{rz}(r,z) = -\frac{\mu}{P}\int_0^\infty s\sin(sz)\left[(P+Q)A(s)I_1(sr) + \frac{B(s)}{s}\left\{I_0(rs)(2r^2sp + r^2s^2) + (rs+2)I_1(sr)\right\}\right]ds \qquad \ldots(13)$$

Where $A(s)$ and $B(s)$ are two arbitrary constants to be determined. The quantities in (10)-(13) vanish as $|z|$ or $|r| \to \infty$. The boundary condition (7), with (13), gives

$$b_1 A(s) = -\frac{b_2 B(s)}{2} \qquad \ldots(14)$$

with $b_1 = I_0(ds)[2d^2 + \frac{d}{s}] + \frac{s+2}{s}I_1(ds)$, $b_2 = (p+Q)I_1(ds)$ $\qquad \ldots(15)$

Now, the substitution of u_r and σ_{rr} from (10) and (12), respectively, and using (14) – (15), gives

$$\int_0^\infty \psi(s)\cos(sz)dx = 0, c \le z < \infty \qquad \ldots(16)$$

$$\int_0^\infty s(\psi)\cos(sz)ds = -P_1(z), 0 \le z < c \qquad \ldots(17)$$

$$b_1\psi(s) = B(s)[db, I_0(ds) - b_2 I_1(sd)] \qquad \ldots(18)$$

$$P_1(z) = p(z) + \int_0^\infty s\cos(sz)\psi(s)M(ds)ds \qquad \ldots(19)$$

$$M(sd) = (b_2 b_4 - b_1 b_3 - b_5)/b_3,\ b_3(3-Q)I_0(sd) + I_1(sd)[1+s(1+Q)] \qquad \ldots(20)$$

$$b_4 = I_0(sd)\left[\frac{Q-P}{P} + \frac{I_1(sd)}{s}\right], b_5 = b_1 d\, I_0(sd) - b_2 I_1(sd) \qquad \ldots(21)$$

Thus the problem is reduced to dual integral equation (16) – (17).

3. Solution of Integral Equation and Expansion of Some Function

Solution of Integral Equation

The solution of dual integral equation (16) - (17) is obtained through the method of Srivastava and Lowengrub[10].

The solution is assumed as,

$$\pi\psi(s) = 2\int_0^c g(t)\frac{\sin st}{t}dt, \qquad \ldots(22)$$

with no loss of generality as $g(0) = 0$ .The use of integral

$$\int_0^\infty \frac{\sin st \cos xt}{t}dt = \begin{cases} \pi/2, s > x \\ \pi/4, s = x \\ 0, \quad s < x \end{cases}$$

will satisfy (16) through (22).

Then using (22) in (17) and the value of integral

$$\int_0^\infty \frac{\sin st \sin xt}{s}ds = \frac{1}{2}\log\left|\frac{t+x}{t-x}\right|$$

will give,

$$g(t) = -\frac{2t}{\pi^2\sqrt{c^2-t^2}}\left[\Delta_0(t) + \int_0^c g(\alpha)M_1(\alpha,t)dx\right], 0 \le t < c \qquad \ldots(23)$$

$$M_1(\alpha,t) = \int_0^c \frac{\sqrt{c^2-z^2}}{z^2-t^2}K_1(\alpha,z)dz,\ \Delta_0(t) = \int_0^c \frac{\sqrt{c^2-z^2}}{z^2-t^2}p(z)dz \qquad \ldots(24)\text{-}(25)$$

$$K_1(\alpha, z) = \int_0^\infty M(sd)\cos(st)\sin(s\alpha)ds \qquad \ldots(26)$$

M(sd) is defined in (20).

Expansion of Some Functions

We make use of expansion of modified function $I_\nu(z)$, of order ν. In this case $\nu = 0$ and $\nu = 1$.

$$I_\nu(z) = e^{-z}\sum_{m=0}^{d-i}(\nu, m)(-1)^m\left(\frac{z}{2}\right)^m, \ (\nu, m) = \left\lceil \nu + m + \frac{1}{2} \middle/ m! \left\lceil \nu + \frac{m}{2} \right. \right. \qquad \ldots(27)$$

see [4]. It is real part of $I_\nu(z)$.

To get the approximate expansion of *M*(*sd*) it is needed

$$b_2b_4 - b_1b_3 = e^{-sd}\sum_{m=0}^{n-1}\sum_{r=0}^{n-1}(-1)^{m+r}\left(\frac{sd}{2}\right)^{m+r}$$

with

$$e_1(m, r, d)$$

$$e_1(m, r) = \left\lceil m + \frac{1}{2} \right. \left\lceil r + \frac{1}{2} e_{11}(m, r, s) \right.$$

$$e_{11}(m, r) = \left(r + \frac{1}{2}\right)d_4 + d_5\left(r + \frac{1}{2}\right)\left(m + \frac{1}{2}\right)d_6$$

$$b_5 = \sum_{r=0}^{n-1}\sum_{m=0}^{n-1}(-1)^{m+r}\left(\frac{sd}{2}\right)^{m+r}\left[d_7\left\lceil m + \frac{1}{2} \right. \left\lceil r + \frac{3}{2} \right. + d_6\left\lceil m + \frac{3}{2} \right. \left\lceil r + \frac{3}{2} \right. \right]$$

$$b_5^{-1} = \frac{\pi}{4}\sum_{e=0}^{\infty}(2d_7 - d_8)^{-1}\left[\frac{sd}{4}\left(\frac{2d_7 - 3d_8}{2d_7 - d_8}\right)^e\right]$$

$$M(sd) = \pi\sum_{R=0}^{\infty}\sum_{l=0}^{\infty}\sum_{p=0}^{l}\sum_{m=0}^{n-1}\sum_{r=0}^{n-1}(-1)^{m+r+p+l}e_2(m, r, d)$$

$$\left(\frac{\alpha d}{2}\right)^{m+r+l-2p-2k} {}^lC_p\binom{p}{k}(d_1 s - 2)^k \qquad \ldots(28)$$

where $d_1 \sim d_8$ alongwith other are given in appendix-I.

4. Solution of Fredholm Integral Equation

To solve Fredholm integral equation given in (23), we expand the function $g(t)$ is terms of 'd' i.e. distance of ring shaped crack from axis.

$$g(t) = \sum_{r=0}^{\infty} g_r(t) d^{-r} \quad \ldots(29)$$

And then substitute (29) and compare the coefficients of $\{d^{-m}\}$ from both sides. Before we proceed for above analysis we take appropriate values of k, l, p, m, r so that in the expansion of $M(sd)$ we retain upto d^{-5} only. Then from (28).

$$M(sd) = \frac{2}{3P}\left[\frac{t_6}{d^2 s} + \frac{1}{d^4 s^2}\left\langle t_1 + \frac{2\sqrt{\pi}}{3}\right\rangle + \frac{1}{d^6 s^3}\left\langle \frac{2\sqrt{p}}{3} t_7 + \frac{4\pi}{9} t_6 - t_2\right\rangle\right]\left[1 + \frac{P+Q}{2Pd} + \frac{\sqrt{\pi}}{Pd^2}\right]$$

This $M(sd)$ given $K_1(\alpha, z)$ from (26), after evaluating integrals, as

$$K_1(\alpha,t) = \frac{1}{d^2}\left[t_8 + \frac{t_9}{d} + \frac{t_{10}}{d^3}\right] T(\alpha,t) \quad \ldots(30)$$

$$T(\alpha,t) = \frac{1}{2}\left[\alpha \log\left|\frac{\alpha+t}{\alpha-t}\right| + t^2 \log\left|\alpha^2 - t^2\right| - \left|\alpha - t\right|\right], 0 \le t < c \quad \ldots(31)$$

where $t_i, i = 1, 2, \ldots 10$ are given in Appendix-II. Evaluate $M_1(\alpha,t)$ from (24) after using (30) and evaluating integrals which is given as,

$$M_1(\alpha,t) = \left(\frac{t_8}{d^2} + \frac{t_9}{d^3} + \frac{t_{10}}{d^5}\right) T(\alpha,t) \quad \ldots(32)$$

$$T(\alpha,t) = \frac{\pi}{2}\left(\alpha^2 - 3t^2\right)\alpha^2 + \frac{\pi\alpha}{2} - I_2(t)$$

$$I_2(t) = -c + \frac{\sqrt{c^2 - t^2}}{2} I_{21}(t), I_{21}(t) = \log\left|\frac{c + \sqrt{c^2 + t^2}}{c - \sqrt{c^2 - t^2}}\right| \quad \ldots(33)$$

Now we use (29) in (23) and relevant function there in and compare coefficients of $\{d^{-m}\}$ $m = 0,1,2,3,4,5$ only. There

$$g_0(t) = -\frac{2}{\pi^2}\psi_0(t) H_0(t), \; g_1(t) = 0, \psi_0(t) = \frac{t}{\sqrt{c^2 - t^2}}, \quad 0 \le t < c$$

$$g_2(t)=\frac{4t_8}{\pi}\pi_0(t)\left[-d_0I_2(t)+\frac{\pi}{12}\left\langle T_5(t)-3t^2I_3(t)-\frac{\pi^2}{4}\right\rangle\right]$$

$$d_0=\int_0^\infty H_0(t)I_{21}(t)dt$$

$$T_{2n+1}(t)=\int_0^c\frac{\alpha^{2n+1}}{\sqrt{c^2-\alpha^2}(\alpha^2-t^2)}=-P_{2n-1}+t^2T_{2n-1}(t)$$

$$P_{2n+1}=\frac{(n!)^2c^{2n+1}}{2^{2n+2}(4n+1)!},\ T_1(t)=\frac{2}{\sqrt{c^2+t^2}}I_{21}(t)$$

$$g_3(t)=0$$

$$g_4(t)=-\frac{2}{3\pi^2}\psi_0(t)t_4^2[q_{14}-t^2q_{15}-q_{16}I_2(t)]$$

$$g_5(t)=-\frac{34t_8^2}{\pi^2}\psi_0(t)[q_{17}+t^2q_{18}-I_{21}(t)I_{22}(t)]$$

$$I_{22}(t)=d_0t_{10}\sqrt{c^2-t^2}+\frac{5\pi t^4}{6\sqrt{c^2-t^2}}+t_9q_1b$$

where $q_i, i=0,1,2,...25$ are given

Appendix III. These are constants depending upon elastic properties and geometrical parameters d and c. Thus substituting $g_i(t)$, i= 0, 1, 2, …5 in (29) from above

$$g(t)=-\frac{2}{\pi^2}\psi_0(t)\left[H_0(t)+\frac{H_1(t)}{d^2}\left\langle q_{23}+\frac{t^4q_{24}}{d^2}+\frac{1}{a^3}\{I_{21}(t)+I_{29}(t)\}\right\rangle\right], \quad \text{…(34)}$$

$$I_{23}(t)=6q_{22}c^2-6t^2q_3+t^4q_{25}+\frac{t_8}{d}I_2(t) \quad \text{…(35)}$$

5. Physical Quantities in General and a Special Loading

The crack opening displacement and normal stress-components are quantities which are important in fracture design parameters.

Crack Opening Displacement

The crack opening displacement in the value of integral in (16) for z in [0, c). Now using (22) in (16) and evaluating the integral we get.

$$u_r(d,z)=\int_z^c g(t)dt, 0\le z<c, \quad \text{…(36)}$$

where $g(t)$ is to be taken from (34).

Stress Components

The component of shear stress at $r = d$ is assumed to be zero for all z.

Normal Stress

The normal stress component is obtained from (17) for z in (c,∞) after taking second term on left hand side and is given as

$$\sigma_{rr}(d,z) = \frac{1}{\pi}\psi_0(z)\left[H_0(z) + \frac{1}{d^2}\left\langle q_{23} + \frac{z^4 q_{24}}{d^2} + \frac{1}{d^3}\{I_{21}(z) + I_{23}(z)\}\right\rangle\right]$$

$$-\int_0^c g(p)m_2(p,z)dp, c < z < \infty \quad \ldots(37)$$

$$m_2(p,z) = \int_0^\infty M(ds)\cos(sz)\sin(ds)ds$$

It possesses Cauchy type singularly at crack tip (d,c).

Stress intensity factor

The stress-intensity factors at crack tips are defined as

$$(K_e, N_c) = \lim_{c \to e^-} \sqrt{z-c}\left(\sigma_{rr}(d,z), \sigma_{rz}(d_0 z)\right) \quad \ldots(38)$$

But $N_c = 0$. using (37) in (38) and evaluating the limit K_c is given as :

$$K_c = \frac{1}{\pi}\sqrt{\frac{c}{2}}\left[H_0(c) + \frac{1}{d^2}H_1(c)\right], \quad \ldots(39)$$

$$H_1(c) = q_{23} + \frac{c^4 q_{24}}{d^2} + \frac{1}{d^3}\{I_{21}(c) + I_{23}(c)\}$$

Special Case of Loading

We consider that crack was opened by constant and uniform force at crack faces, therefore,

$$p(z) = p_0 = \text{constant} \quad \ldots(40)$$

Thus, $H_0(t) = -p_0 \dfrac{\pi}{2}$ = constant $\quad \ldots(41)$

Substituting (41) in (34) and evaluating integrals

$$U_r(d,z) = \frac{p_0}{\pi}\sqrt{c^2 - z^2}\left[1 + \frac{1}{d^2}\left\{q_{23} + \frac{q_{24}c^4}{d^2} + \frac{1}{d^3}\left\langle \frac{\sqrt{c^2 - z^2}}{c}\log\left|\frac{c + \sqrt{c^2 - z^2}}{c - \sqrt{c^2 - z^2}}\right|\right.\right.\right.$$

$$\left.\left.\left.-2\left(c - \sqrt{c^2 - z^2} + \left(3c^2 q_{22} + 3c^2 q_{23}\right) + \left(1 - \frac{1c^2 - z^2}{3c^2}\right)\right)\right\rangle\right\}\right], \ 0 \le z < c \quad \ldots(42)$$

Thus, he closed form expressions for crack opening displacement and of stress-intensity factor k_c are obtained and given by (42) and (39), respectively.

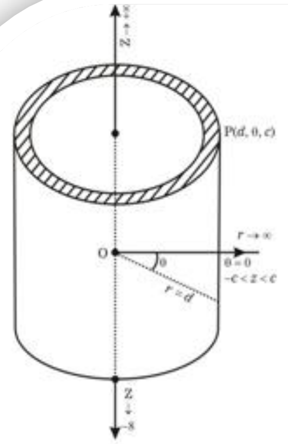

FIGURE-1. DISCONTINUITY IN THE SHAPE OF CYLINDRICAL SHELL WITH AXIS AS Z-AXIS

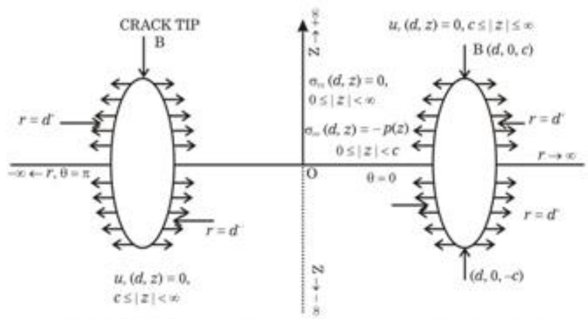

FIGURE-2a. THE GEOMETRY OF CROSS SECTION OF CYLINDERICAL SHELL WITH θ = 0 AND H θ = π REDUCED TO TWO GRIFFITH CRACKS

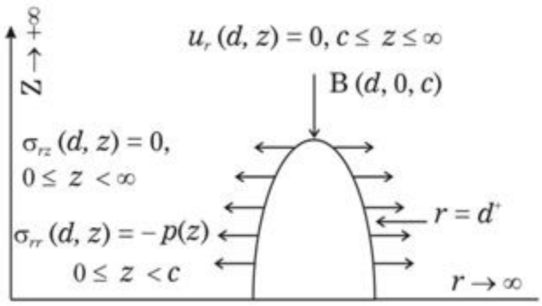

FIGURE-2b. SYMMETRY OF GEOMETRY REDUCED THE PROBLEM TO Ist QUADRANT ONLY.

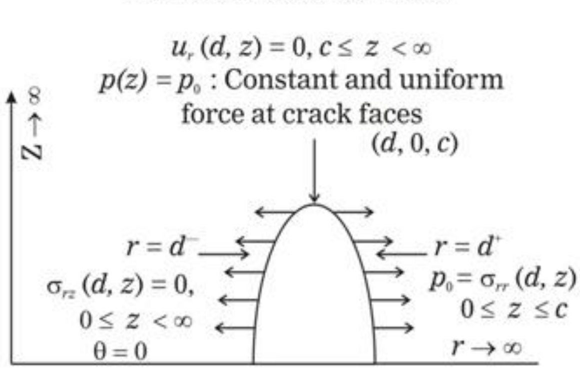

FIGURE- 3. THE CRACKS OPENING DUE TO CONSTANT AND UNIFORM FORCE AT CRACK FACES

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Appendix-I

$$d_1 = d^2(2p+s), d_2 = \frac{s+2}{s}, d_3 = 14s(1+Q),\ d_4 = \frac{d_1}{s} + \frac{Q-P}{P} d_2 - (P+Q)(3-Q)$$

$$d_5 = \frac{d_1(Q-P)}{P}, d_6 = \frac{d_2}{s} - (P+Q)d_3,\ d_7 = P+Q+d_1, d_8 = d_2$$

$$e_2(m,r,s) = e_{21}(m,r,s)\left|m+\frac{1}{2}\right|r+\frac{1}{2}$$

$$e_{21}(m,r,s) = e_{11}(m,r,s) - \left|r+\frac{1}{2}\right.(d_7) + d_8\left(m+\frac{1}{2}\right)\left(r+\frac{1}{2}\right)$$

Appendix-II

$$t_1 = \left[2(1+Q)\sqrt{\pi}-3\right]/4, 2t_2 = \sqrt{2} = 6Q, t_3 = t_2,\ t_4 = 2(P+\sqrt{\pi}-2/p), t_5 = (4Q-7P)/4P$$

$$t_6 = t_4 t_5 + 3(1+Q)P,\ t_7 = \sqrt{2}(t_4+8t_5)/2 - t_1, t_8 = 2[t_7 + 2\sqrt{\pi}/3]/3P,\ t_9 = (P+Q)t_{10}/\sqrt{\pi}$$

$$t_{10} = \sqrt{\pi}\ t_8/p$$

Appendix-III

$$(q_0,q_1) = \int_0^c \psi_0(\alpha)\alpha^2 I_2(\alpha)(\alpha^2,1)d\alpha,\ (q_2,q_3) = \int_0^c \psi_0(\alpha)\alpha^2 T_5(\alpha)(\alpha^2,1)d\alpha$$

$$(q_4,q_5) = \int_0^c \psi_0(\alpha)T_3(\alpha)\alpha^4(\alpha^2,1)d\alpha,\ q_6 = \pi^4(P_5 - 3e^2P_1)/576, q_{10} = \frac{\pi c^2}{4}$$

$$(q_7,q_8,q_9) = \int_0^c \alpha\psi_0(\alpha)(I_0(\alpha), T_5(\alpha), \alpha^2 T_3(\alpha))d\alpha$$

$$(q_{11},q_{12},q_{13}) = \int_0^c \psi_0(\alpha)(I_0(\alpha), T_5(\alpha), \alpha^2 T_3(\alpha))d\alpha$$

$$q_{14} = \frac{\pi q_0 d_0}{12} + \frac{\pi^2}{144}(q_2 - 3q_4) - \frac{\pi^4}{576}P_5 + \frac{\pi}{2}\left[d_0 q_7 + \frac{\pi}{12}q_8 - \frac{\pi}{4}q_9 - \frac{\pi}{96}P_3\right]$$

$$q_{15} = \frac{\pi}{4}\left[d_0 q_1 - \frac{\pi}{12}q_3 - \frac{\pi}{4}q_5 - \frac{\pi}{48}P_3\right],\ q_{16} = d_0 q_{11} + \frac{\pi}{2}\left[q_{12} - \frac{\pi}{6}q_{13} - \frac{\pi^2}{24}q_{10}\right]$$

$$q_{17} = t_9(q_{14} + cq_{16}) - \left(cd_0 + \frac{\pi^2}{4} - \frac{\pi}{12}P_3\right)t_{10},\ q_{18} = \frac{\pi}{12}\left(1 - \frac{\pi}{2}\right)P_1 - t_9 t_{15}$$

$$q_{19} = -\left[2d_0 c t_8 + \frac{\pi^3}{48}t_8 - \frac{\pi}{12}P_3\right],\ q_{20} = \frac{\pi}{12}t_8\left[q_0 d_0 + \frac{\pi}{2}(q_2 - 3q_3)\right]$$

$$q_{21} = -\frac{\pi}{3}P_1, q_{22} = -2d_0 t_8, q_{23} = q_{19} + \frac{q_{20}}{d^2} - \frac{\pi t_8 q_1}{d^3},\ I_0(x) = (c^2 - 2\alpha^2)\frac{\pi}{2}$$

$$q_{24} = q_{21} - \frac{\pi q_1 d_0 d_8}{12d^2} - \frac{8t_8 q_1}{d^3}, q_{25} = (6+\pi)t_8$$