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A NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTION DEFINED BY FRACTIONAL CALCULUS OPERATOR WITH FIXED POINT

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Abstract: In this paper, we study a subclass of harmonic univalent functions and a new fractional calculus operator. We obtain coefficient conditions, extreme points, distortion bounds, convolution and convex combination for the above class of harmonic univalent functions.

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1. Introduction

A continuous complex-valued function f = u+iv defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$ where h and g are analytic in D. we call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|, z \in D.$ (See Clunie and Sheil-Small [2]).

Denote by S_H the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk U = {z : |z| < 1 }for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g}$ $\in S_H$ we may express the analytic functions h and g as

$$h(z) = a_0 z + \sum_{k=2}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \qquad \ldots(1)$$

The class S_H reduces to class S of normalized analytic univalent functions if co-analytic part of f i.e. $g \equiv 0$, for this class f(z) may be expressed as

$$f(z) = a_0 z + \sum_{k=2}^{\infty} a_k z^k. \qquad \ldots(2)$$

Further, A denotes the class of function of the form (2) which are analytic in the open unit disk U.

The following definitions of fractional derivatives and fractional integrals are due to Owa [6] and Srivastava and Owa [10].

Definition 1. The fractional integral of order λ is defined for a function f(z) of the form 2 by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta, \qquad \ldots(3)$$

where $\lambda > 0$, f(z) is an analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring log(z $-\lambda) > 0$.

Definition 2. The fractional derivative of order λ is defined for a function f(z) of the form (2), by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta, \qquad \ldots(4)$$

where $0 \le \lambda < 1$, f(z) is an analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1 above.

Definition 3. Under the hypothesis of Definition 2, the fractional derivative of order $n+\lambda$ is defined for a function f(z) by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \qquad \ldots(5)$$

where $0 \le \lambda < 1$ and $n \in N_0 = \{0,1,2,\ldots\}$.

For f of the form (2), using the Definition 2 and 3, we study a new fractional derivative operator as

$$\Omega^0 f(z) = f(z)$$

$$\Omega^1 f(z) = \Gamma(1-\lambda) a_0 z^{1+\lambda} D_z^{1+\lambda} f(z)$$

……………………………….

$$\Omega^n f(z) = \Omega(\Omega^{n-1} f(z)).$$

We note that

$$\Omega^n f(z) = a_0 z + \sum_{k=2}^{\infty} [\phi(k,\lambda)]^n a_k z^k,$$

where

$$\phi(k,\lambda) = \frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)}.$$

It is interesting to note that for $\lambda = 0, \Omega^n f(z)$ reduces to familiar Salagean operator defined by Salagean in [7].

From the motivation of the definition of modified Salagean operator

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)} \qquad \ldots(6)$$

for $f = h + \overline{g}$ given by (1) in [4], we define

$$\Omega^n f(z) = \Omega^n h(z) + (-1)^n \overline{\Omega^n g(z)} \qquad \ldots(7)$$

where $\Omega^n h(z) = a_0 z + \sum_{k=2}^{\infty} [\phi(k,\lambda)]^n a_k z^k$ and $\Omega^n g(z) = \sum_{k=1}^{\infty} [\phi(k,\lambda)]^n b_k z^k$.

Now for $0 \le \alpha < 1, 0 \le t \le 1, m \in N, n \in n_0, m > n$ and $z \in U$, suppose that $S_H^{\lambda}(m,n;\alpha;t,z_0)$ denote the family of harmonic functions f of the form (1) such that

$$\operatorname{Re}\left\{\frac{\Omega^m f(z)}{\Omega^n f_t(z)}\right\} > \alpha. \qquad \ldots(8)$$

where $f_t(z) = (1-t)a_0 z + t f(z)$ and $\Omega^m f$ is defined by (7).

Further, let the subclass $\overline{S}_H^{\lambda}(m,n;\alpha,t,z_0)$ consist of harmonic functions $f_m = h + \overline{g_m}$ in $S_H^{\lambda}(m,n;\alpha t,t,z_0)$ so that h and g_m are the form

$$h(z)=a_0 z-\sum_{k=2}^{\infty} |a_k| z^k, g_m(z)=(-1)^{m-1}\sum_{k=1}^{\infty} |b_k| z^k. \qquad \ldots(9)$$

By specializing the parameters in subclass $S_H^{\lambda}(m,n;\alpha,t,z_0)$, we obtain the following known subclasses studied earlier by various researchers.

1. If we put $\lambda = 0$ and $t = 0$ then it reduces to the class $S_H(m,n,\alpha,t,z_0)$ studied by Yalcin [11].
2. If we put $m = 1, n = 0, \lambda = 0, t = 1$ and $m = 2, n = 1, \lambda = 0, t = 1$ then it reduces to the class H S(α) and , H K(α) studied by Jahangiri [4].
3. If we put $m = 1, n = 0, \alpha = 0, \lambda = 0, t = 1$ and $m = 2, n = 1, \alpha = 0, \lambda = 0, t = 1$ with $b_1 = 0$ then it reduces to the class $HS^0(0)$ and H $K^0(0)$ studied by Avci and Zlotkiewicz [1] and Silverman [8].
4. If we put $m = 1, n = 0, \alpha = 0, \lambda = 0, t = 1$ and $m = 2, n = 1, \alpha = 0, t = 1$ then it reduces to the class H S(0) and H K(0) studied by Silverman and Silvia [9], which is an improvement of ([1],[8]).
5. If we put $m = n+1, \lambda = 0, t = 1$ then it reduces to the class H(n,α) studied by Jahangiri et al. [5].

In the present paper, results involving coefficient estimates, extreme points, distortion bounds, convolution condition and convex combinations for the above classes $S_H^{\lambda}(m,n,\alpha,t,z_0)$ and $\overline{S}_H^{\lambda}(m,n,\alpha,t,z_0)$ of harmonic univalent functions have been investigated.

2. Main Results

We begin with a sufficient coefficient condition for function $S_H^{\lambda}(m,n;\alpha,t,z_0)$.

Theorem 1. Let $f=h+\overline{g}$ be such that h and g are given by (1). Furthermore, let

$$\sum_{k=1}^{\infty}\left(\frac{[\phi(k,\lambda)]^m-\alpha[\phi(k,\lambda)]^n}{1-\alpha}|a_k|+\frac{[\phi(k,\lambda)]^m-(-1)^{m-n}\alpha t[\phi(k,\lambda)]^n}{1-\alpha}|b_k|\right)\le 2a_0, \qquad \ldots(10)$$

where

$$a_1=a_0, m\in N, n\in N_0, m>n, 0\le\alpha<1, 0\le\lambda<1, 0\le t\le 1 \text{ and } \phi(k)=\frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)},$$

then f is sense-preserving, harmonic univalent in U and $f\in S_H^{\lambda}(m,n;\alpha,t,z_0)$.

Proof. First we note that f is sense-preserving in U. This is because

$$|h'(z)| \geq a_0 - \sum_{k=2}^{\infty} k|a_k|r^{k-1} > a_0 - \sum_{k=2}^{\infty} k|a_k| \geq a_0 - \sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^m - \alpha t[\phi(k,\lambda)]^n}{1-\alpha}|a_k|$$

$$\geq \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^m - (-1)^{m-n}\alpha t[\phi(k,\lambda)]^n}{1-\alpha}|b_k| \geq \sum_{k=1}^{\infty} k|b_k| > \sum_{k=1}^{\infty} k|b_k|r^{k-1} \geq |g'(z)|.$$

To show that f is univalent U, suppose $z_1, z_2 \in U$ such that $z_1 \neq z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1 - \left|\frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{a_0(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)}\right|$$

$$> 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|} \geq 1 - \frac{\sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^m - (-1)^{m-n}\alpha t[\phi(k,\lambda)]^n}{1-\alpha}}{1 - \sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^m - \alpha t[\phi(k,\lambda)]^n}{1-\alpha}|a_k|} \geq 0.$$

Now, we show that $f \in S_H^{\lambda}(m,n;\alpha,t,z_0)$. Using the fact that Re $\omega \geq \alpha$, if and only if, $|1-\alpha+\omega| \geq |1+\alpha-\omega|$, it suffices to show that

$$|A(z)+(1-\alpha)B(z)| - |A(z)-(1+\alpha)B(z)| \geq 0, \qquad \ldots(11)$$

where $A(z) = \Omega^m f(z)$and $B(z) = \Omega^n f_t(z)$.

Substituting for A(z) and B(z) in Left Hand Side of (11) and making use of (10), we obtain

$$|A(z)+(1-\alpha)B(z)| - |A(z)-(1+\alpha)B(z)|$$

$$= |\Omega^m f(z) + 1(1-\alpha)\Omega^n f(z)| - |\Omega^m f(z) - (1+\alpha)\Omega^n f_t(z)|$$

$$= \left|(2-\alpha)a_0 z + \sum_{k=2}^{\infty} ([\phi(k,\lambda)]^m + (1-\alpha)t[\phi(k,\lambda)]^n)a_k z^k\right.$$

$$+(-1)^n \sum_{k=1}^{\infty} ((-1)^{m-n}[\phi(k,\lambda)]^m + (1-\alpha)t[\phi(k,\lambda)]^n)\overline{b_k z^k}\,\Bigg|$$

$$-\Bigg| -\alpha a_0 z + \sum_{k=2}^{\infty} ([\phi(k,\lambda)]^m - (1+\alpha)t[\phi(k,\lambda)]^n)a_k z^k$$

$$+(-1)^n \sum_{k=1}^{\infty} ((-1)^{m-n}[\phi(k,\lambda)]^m - (1+\alpha)t[\phi(k,\lambda)]^n)\overline{b_k z^k}\,\Bigg|$$

$$\geq 2(1-\alpha)|a_0||z| - 2\sum_{k=2}^{\infty} ([\phi(k,\lambda)]^m - \alpha t[\phi(k,\lambda)]^n)|\alpha_k||z|^k$$

$$-\sum_{k=1}^{\infty} |(-1)^{m-n}[\phi(k,\lambda)]^m + (1-\alpha)t\,[\phi(k,\lambda)]^n||b_k||z|^k$$

$$-\sum_{k=1}^{\infty} |(-1)^{m-n}[\phi(k,\lambda)]^m - (1+\alpha)t\,[\phi(k,\lambda)]^n||b_k||z|^k$$

$$= \begin{cases} 2(1-\alpha)|a_0||z| - 2\sum_{k=2}^{\infty} [(\phi(k,\lambda))^m - \alpha t(\phi(k,\lambda))^n]|a_k||z|^k \\ \quad -2\sum_{k=1}^{\infty} [(\phi(k,\lambda))^m + \alpha t(\phi(k,\lambda))^n]|b_k||z|^k, \text{if } m-n \text{ is odd} \\ 2(1-\alpha)|a_0||z| - 2\sum_{k=2}^{\infty} [(\phi(k,\lambda))^m - \alpha t(\phi(k,\lambda))^n]|b_k||z|^k \\ \quad -2\sum_{k=1}^{\infty} [(\phi(k,\lambda))^m - \alpha t(\phi(k,\lambda))^n]|b_k||z|^k, \text{if } m-n \text{ is even} \end{cases}$$

$$= 2(1-\alpha)|z|\Bigg\{|a_0| - \sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^m - \alpha t[\phi(k,\lambda)]^n}{1-\alpha}|a_k||z|^{k-1}$$

$$-\sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^m - (-1)^{m-n}\alpha t[\phi(k,\lambda)]^n}{1-\alpha}|b_k||z|^{k-1}\Bigg\}$$

$$=2(1-\alpha)\left\{a_0-\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^m-\alpha t[\phi(k,\lambda)]^n}{1-\alpha}|a_k|\right.$$

$$\left.-\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^m-(-1)^{m-n}\alpha t[\phi(k,\lambda)]^n}{1-\alpha}|b_k|\right\}$$

≥ 0 (using (10)).

The coefficient bound (10) is sharp for the function

$$f(z)=a_0z+\sum_{k=2}^{\infty}\frac{1-\alpha}{([\phi(k,\lambda)]^m-\alpha t[\phi(k,\lambda)]^n}x_kz^k$$

$$+\sum_{k=1}^{\infty}\frac{1-\alpha}{([\phi(k,\lambda)]^m-(-1)^{m-n}\alpha t[\phi(k,\lambda)]^n)}\overline{y_kz^k} \quad \ldots(12)$$

where $0\leq\alpha<1, 0\leq\lambda<1, m\in N, n\in N_0, m>n, 0\leq t\leq 1$ and

$$\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1.$$

This completes the proof of theorem.

In the following theorem, it is proved that the condition (10) is also necessary for the functions $f_m=h+\overline{g_m}$, where h and g_m are of the form (9).

Theorem 2. Let $f_m=h+\overline{g}_m$ be given by (9). Then $f_m\in\overline{S}_H^{\lambda}(m,n;\alpha,t,z_0)$ if and only if

$$\sum_{k=1}^{\infty}[\{(\phi(k,\lambda))^m-\alpha t(\phi(k,\lambda))^n\}|a_k|$$

$$+\{(\phi(k,\lambda))^m-(-1)^{m-n}\alpha t(\phi(k,\lambda))^n\}|b_k|]\leq 2(1-\alpha)a_0. \quad \ldots(13)$$

Proof. Since $\overline{S}_H^{\lambda}(m,n;\alpha,t,z_0)\subset S_H^{\lambda}(m,n;\alpha;t,z_0)$, we only need to prove the “only if” part of the theorem. To this end, for function f_m of the form (9), we notice that the condition

$$\operatorname{Re}\left\{\frac{\Omega^m f_m(z)}{\Omega^n f_{(mt)}(z)}\right\} \geq \alpha$$

is equivalent to

$$\operatorname{Re}\left\{\frac{(1-\alpha)a_0 z-\sum_{k=2}^{\infty}\{(\phi(k,\lambda))^m-\alpha t(\phi(k,\lambda))^n\}|a_k|z^k+(-1)^{2m-1}\sum_{k=1}^{\infty}\{(\phi(k,\lambda))^m-(-1)^{m-n}\alpha t(\phi(k,\lambda))^n\}|b_k|\bar{z}^k}{a_0 z-\sum_{k=2}^{\infty}(\phi(k,\lambda))^n t|a_k|z^k+(-1)^{m+n-1}\sum_{k=1}^{\infty}(\phi(k,\lambda))^n t|b_k|\bar{z}^k}\right\} \geq 0.$$

…(14)

The above required condition (14) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\frac{(1-\alpha)a_0-\sum_{k=2}^{\infty}\{(\phi(k,\lambda))^m-\alpha t(\phi(k,\lambda))^n\}|a_k|r^{k-1}-\sum_{k=1}^{\infty}\{(\phi(k,\lambda))^m-(-1)^{m-n}\alpha t(\phi(k,\lambda))^n\}|b_k|r^{k-1}}{a_0-\sum_{k=2}^{\infty}(\phi(k,\lambda))^n t|a_k|r^{k-1}+(-1)^{m+n-1}\sum_{k=1}^{\infty}(\phi(k,\lambda))^n t|b_k|r^{k-1}} \geq 0.$$

…(15)

If the condition (13) does not hold then the numerator in (15) is negative for r sufficiently close to 1. Thus there exist a $z_0 = r_0$ in (0,1) for which the quotient in (15) is negative. This contradicts the required condition for $f_m \in \overline{S}_H^{\lambda}(m,n,\alpha,t,z_0)$ and so the proof is complete.

We prove the following Theorems 3, 4 by using techniques adopted by Yalcin [11].

Theorem 3. Let f_m be given by (9). Then $f_m \in \overline{S}_H^{\lambda}(m,n,\alpha,t,z_0)$ if and only if

$$f_m(z)=\sum_{k=1}^{\infty}(x_k h_k(z)+y_k g_{mk}(z)). \qquad \ldots(16)$$

where

$$h_1(z)=a_0 z,\ h_k(z)=z-\frac{(1-\alpha)a_0}{(\phi(k,\lambda))^m-\alpha t(\phi(k,\lambda))^m}z^k,\ (k=2,3,4,...)\ ,$$

$$g_{mk}(z)=a_0 z+(-1)^{m-1}\frac{(1-\alpha)a_0}{(\phi(k,\lambda))^m-(-1)^{m-n}\alpha t(\phi(k,\lambda))^n}\bar{z}^k\ ,(k=1,2,3,4,...),$$

$x_k\geq 0, y_k\geq 0, \sum_{k-1}^{\infty}(x_k+y_k)=1.$ In particular, the extreme points of $\overline{S}_H^{\lambda}(m,n,\alpha,t,z_0)$ are $\{h_k\}$ and $\{g_{mk}\}$.

Theorem 4. Let $f_m\in\overline{S}_H^{\lambda}(m,n,\alpha,t,z_0)$. Then for | z | = r < 1, we have

$$|f_m(z)|\leq(a_0+|b_1|)r+\left(\frac{1-\lambda}{2}\right)^n\left(\frac{(1-\alpha)a_0}{\left(\frac{2}{1-\lambda}\right)^{m-n}-\alpha t}-\frac{1-(-1)^{m-n}\alpha t}{\left(\frac{2}{1-\lambda}\right)^{m-n}-\alpha t}|b_1|\right)r^2,$$

|z| = r < 1

and

$$|f_m(z)|\geq(a_0+|b_1|)r-\left(\frac{1-\lambda}{2}\right)^n\left(\frac{(1-\alpha)a_0}{\left(\frac{2}{1-\lambda}\right)^{m-n}-\alpha t}-\frac{1-(-1)^{m-n}\alpha t}{\left(\frac{2}{1-\lambda}\right)^{m-n}-\alpha t}|b_1|\right)r^2,$$

|z| = r < 1

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