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ON A QUARTER-SYMMETRIC NON-METRIC CONNECTION IN A P-SASAKIAN MANIFOLD

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Abstract: The object of the present paper is to study a quarter-symmetric non-metric connection on a P-Sasakian manifold. We have studied some properties of Curvature tensor $\tilde{R}(X,Y)Z$ and Ricci tensor $\tilde{S}(Y,Z)$ with respect to the quarter-symmetric non-metric connection.

Keywords: P-Sasakian, projective Ricci tensor, Einstein manifold andquarter-symmetric non-metric connection.

2010 Mathematical Subject Classification: 53C25.

1. Introduction

In 1975, Golab[5] defined and studied quarter-symmetric connection in a Riemannian manifold with affine connection. This was further developed by Rastogi [10], [11] Mishra andPandey [6] ,Yano and Imai [15], Mukhopadhyay et al.[7], BiswasandDe [3], Sengupta and Biswas [12], singh and Pandey[13], and many other geometers. Recently Prakash and Narain[8], andPrakash and Pandey [9] define a quarter-symmetric non-metric connection in an LP-Sasakian manifold and Kenmotsu manifold.

In this paper, we studied some properties of a quarter-symmetric non-metric connection in a P-Sasakian manifold.Some properties of the curvature tensor and the Ricci tensor of the quarter-symmetric non-metric connection is found. The necessary and sufficient condition for the Ricci tensor and Projective Ricci tensor of $\tilde{\nabla}$ to be skew-symmetric under certain conditions is found. We also find the necessary and sufficient condition for the Einstein manifold of the connection ∇ is equal to the Einstein manifold of the connection$\tilde{\nabla}$ under the certain condition.

2. *P*-Sasakian manifold

Let Mbe an n-dimensional differentiable manifold on which there exists a $(1,1)$ tensor field ϕ, a vector field ξ and 1 −from η satisfying

$$\phi^2 = I - \eta \otimes \xi \qquad \text{.... (1)}$$

$$\eta(\xi) = 1 \qquad \text{... (2)}$$

$$\eta o \phi = 0 \qquad \text{.... (3)}$$

$$\phi\xi = 0 \qquad \text{.... (4)}$$

is called an almost para contact manifold and the structure (ϕ, ξ, η) is called an almost para contact structure.The first and one of the remaining last three above relations imply the other two relations. Let g be a Riemannian metric with(ϕ, ξ, η) -structure such that

$$g(\phi X, \phi Y) = g\ (X, Y) - \eta(X)\eta(Y) \qquad \text{... (5)}$$

or ,equivalently,

$$g(\phi X, Y) = g\ (X, \phi Y) \text{and} g(X, \xi) = \eta(X) \qquad \text{...(6)}$$

for all vector fields X, Y.

Then M is called an almost para contact Riemannian manifold or an almost para contact metric manifold with an almost para contact Riemannian structure-(ϕ, ξ, η, g)[14].

Definition.*An almost para contact Riemannian manifold is called P-Sasakain manifold if*

$$(\nabla_X \phi)(Y) = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \qquad \dots (7)$$

for all vector fields X, Y.

where ∇ *denotes the operator of co-variant differentiation with respect to Riemannian metric* g.

On P-Sasakianmanifold, we have

$$(\nabla_X \eta)(Y) = g(\phi X, Y) = (\nabla_Y \eta)(X) \qquad \dots (8)$$

$$(\nabla_X \eta)(Y) = \Phi(X,Y) \qquad \dots (9)$$

where $\Phi(X,Y) \stackrel{\text{def}}{=} g(\phi X, Y)$ ....(10)

$$(\nabla_X \xi) = \phi X$$

Also in a P- Sasakian manifold M, the curvature tensor R, the Ricci tensor S, and the Ricci orperator Q satisfy

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X \qquad \dots (11)$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi \qquad \dots (12)$$

$$R(\xi,X)\xi = X - \eta(X)\xi \qquad \dots (13)$$

$$S(X,\xi) = -(n-1)\eta(X) \qquad \dots (14)$$

$$Q\xi = -(n-1)\xi \qquad \dots (15)$$

$$\eta(R(X,Y)U) = g(X,U)\eta(Y) - g(Y,U)\eta(X) \qquad \dots (16)$$

$$\eta(R(X,Y)\xi) = 0 \qquad \dots (17)$$

$$\eta(R(\xi,X)Y) = \eta(X)\eta(Y) - g(X,Y) \qquad \dots (18)$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \qquad \dots (19)$$

$$S(X,\phi Y) = S(\phi X, Y) \qquad \dots (20)$$

Definition. *An almost paracontact Riemannian manifold is said to be* η*-Einstein* [2] *if the Ricci tensor* S *satisfy*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \quad \dots (21)$$

where a *and* b *are smooth functions on the manifold .In particular, if* $b = 0$, *then* M *is an Einstein manifold.*

3. Quarter-Symmetric non-metric connection in a *P*-Sasakian manifold

Let (M, g) be a P-Sasakian manifold with Levi-Civita connection ∇. A linear connection $\tilde{\nabla}$ given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X + a(X)\phi Y \dots (22)$$

whereη and a are 1 −froms associated with vector field ξ and A on M given by

$(a)\ \ g(X,\xi) = \eta(X)and(b)\ \ g(X,A) = a(X)$

for all vector fields X. ….(23)

is called a quarter-symmetric non-metric connection[8],if the torsion tensor $\tilde{T}$of the connection $\tilde{\nabla}$ is defined by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \quad \dots (24)$$

for any vector fields X and Y satisfies

$$\tilde{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y + a(X)\phi Y - a(Y)\phi X \quad \dots (25)$$

and

$$\left(\tilde{\nabla}_X g\right)(Y,Z) = -\eta(Y)g(\phi X,Z) - \eta(Z)g(\phi X,Y) \quad \dots (26)$$
$$-2a(X)g(\phi Y,Z)$$

4. Curvature tensor of a *P*-Sasakian manifold with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$

Let $\tilde{R}$ and R be the curvature tensor of the connection $\tilde{\nabla}$ and ∇ respectively then

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z \qquad \text{.... (27)}$$

From (22 and (27), we get

$$\begin{aligned}\tilde{R}(X,Y)Z = {} & \tilde{\nabla}_X(\nabla_Y Z + \eta(Z)\phi Y + a(Y)\phi Z) \\ & - \tilde{\nabla}_Y(\nabla_X Z + \eta(Z)\phi X + a(X)\phi Z) - \nabla_{[X,Y]}Z \\ & - \qquad \eta(Z)\phi([X,Y]) - a(X)\phi Z\end{aligned}$$

which gives on simplification,

$$\tilde{R}(X,Y)Z = R(X,Y,Z) + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + \eta(Z)$$

$$[\eta(Y)X - \eta(X)Y] + [a(X)g(Y,Z) - a(Y)g(X,Z)]\xi$$

$$+3\eta(Z)[a(Y)\eta(X) - a(X)\eta(Y)]\xi + 2a(X)\eta(Z)Y$$

$$-2a(Y)\eta(Z)X + da(X,Y)\phi Z \qquad \text{... (28)}$$

where$R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$

is the curvature tensor of ∇ with respect to the Riemannian connection.
Contracting (28), we get

$$\tilde{S}(Y,Z) = S(Y,Z) - \psi g(\phi Y,Z) + [1 + a(\xi)]g(Y,Z)$$
$$+[n - 2 - 3a(\xi)]\eta(Y)\eta(Z) - 2(n-2)a(Y)\eta(Z) + da(\phi Z,Y) \quad \text{... (29)}$$

and

$$\tilde{r} = r + 2(n-1) + (1-n)a(\xi) + \lambda - \psi^2 \qquad \text{.... (30)}$$

where$\tilde{S}$ and $\tilde{r}$ are the Ricci tensor and scalar curvature with respect to $\tilde{\nabla}$, $\lambda =$ trace $da(\phi Z,Y)$ and $\psi =$trace ϕ
Hence we can state the following theorem:

Theorem 1. *The curvature tensor* $\tilde{R}(X,Y)Z$, *Ricci tensor* $\tilde{S}(Y,Z)$ *and the scalar curvature* $\tilde{r}$ *of a* P*-Sasakian manifold with respect to quarter-symmetric non-metric connection is given by (28), (29) and (30) respectively.*

Let us assume that $\tilde{R}(X,Y)Z = 0$ in (28) and contracting, we get

$$S(Y,Z) = \psi g(\phi Y,Z) - [1 + a(\xi)]g(Y,Z) - [n - 2 - 3a(\xi)]\eta(Y)\eta(Z) + 2(n-2)a(Y)\eta(Z) - da(\phi Z,Y)$$

Again contracting, we have

$$r = 2(1-n) + (n-1)a(\xi) - \lambda + \psi^2$$

Hence we can state the following theorem:

Theorem2.*If a* P*-Sasakian manifold* M *admits a quarter-symmetric non-metric connection whose curvature tensor vanishes, then the scalar curvature* r *is given by*

$$r = 2(1-n) + (n-1)a(\xi) - \lambda + \psi^2.$$

From (28), it follows that

$$'\tilde{R}(X,Y,Z,W) + '\tilde{R}(Y,X,Z,W) = 0 \qquad \dots (31)$$

$$'\tilde{R}(X,Y,Z,W) + '\tilde{R}(Y,Z,X,W) + '\tilde{R}(Z,X,Y,W)$$
$$= da(X,Y)g(\phi Z,\mathrm{W}) + da(Y,Z)g(\phi X,\mathrm{W}) + da(Z,X)$$
$$g(\phi Y,\mathrm{W}) + 2[a(X)\eta(Z) - a(Z)\eta(X)]g(Y,W) - 2[a(Y)\eta(Z) - a(Z)\eta(Y)]g(X,W) + 2[a(Y)\eta(X) - a(X)\eta(Y)]g(Z,W) \qquad \dots (31)$$

If the $1-$ form a is closed and $a \wedge \eta = 0$,then from (32), we have

$$'\tilde{R}(X,Y,Z,W) + '\tilde{R}(Y,Z,X,W) + '\tilde{R}(Z,X,Y,W) = 0$$

Hence we can state the following theorem:

Theorem3. *The curvature tensor of a* P*-Sasakian manifold with respect to the quarter-symmetric non-metric connection* $\tilde{\nabla}$, *satisfy the relation* (31) *and* (32).*In particular, if the* $1-$*form* a *is closed and* $a \wedge \eta = 0$, *then*

$$\tilde{R}(X,Y,Z,W) + '\tilde{R}(Y,Z,X,W) + '\tilde{R}(Z,X,Y,W) = 0$$

5. Skew-symmetric condition of Ricci tensor of $\tilde{\nabla}$ in a P-Sasakian manifold

From(29),we have

$$\tilde{S}(Z,Y) = S(Z,Y) - \psi g(\phi Z,Y) + [1 + a(\xi)]g(Y,Z) \qquad \dots (33)$$
$$+[n - 2 - 3a(\xi)]\eta(Y)\eta(Z) - 2(n-2)a(Z)\eta(Y) + da(\phi Y,Z$$

From (29) and (33),we have

$$\tilde{S}(Y,Z)+\tilde{S}(Z,Y)=2S(Y,Z)-2\psi g(\phi Z,Y)+2[1+a(\xi)]g(Y,Z)+2[n-2-3a(\xi)]\eta(Y)\eta(Z)-2(n-2)[a(Y)\eta(Z)+a(Z)\eta(Y)]+da(\phi Y,Z)+da(\phi Z,Y) \quad \text{…. (34)}$$

If $\tilde{S}(Y,Z)$ is skew-symmetric then the left hand side of (34) vanishes and we get

$$S(Y,Z)=\psi g(\phi Z,Y)-[1+a(\xi)]g(Y,Z)-[n-2-3a(\xi)]\eta(Y)\eta(Z)+(n-2)[a(Y)\eta(Z)+a(Z)\eta(Y)]-\frac{1}{2}[da(\phi Y,Z)+da(\phi Z,Y)] \quad \text{… (35)}$$

Moreover, if $S(Y,Z)$ is given by (35),then from (34), we get

$$\tilde{S}(Y,Z)+\tilde{S}(Z,Y)=0$$

Hence we can state the following theorem:

Theorem 4. *If a P-Sasakian manifold M admits a quarter-symmetric non-metric connection$\tilde{\nabla}$, then a necessary and sufficient condition for the Ricci tensor of $\tilde{\nabla}$ to be skew-symmetric is that the Ricci tensor of the Levi-civita connection ∇ is given by* (35).

6. Skew-symmetric properties of projective Ricci tensor with respect to quarter-symmetric non-metric connection $\tilde{\nabla}$ in a P-Sasakian manifold

Projective Ricci tensor in a Riemannian manifold is defined by (Chaki and Saha [4]) as follows

$$P(X,Y)=\frac{n}{(n-1)}\left[S(X,Y)-\frac{r}{n}g(X,Y)\right] \quad \text{… (36)}$$

Analogous to this definition, we define projective Ricci tensor with respect to thequarter-symmetric non-metric connection$\tilde{\nabla}$, given by

$$\tilde{P}(X,Y)=\frac{n}{(n-1)}\left[\tilde{S}(X,Y)-\frac{\tilde{r}}{n}g(X,Y)\right] \quad \text{… (37)}$$

From (29), (30) and (37),we have

$$\tilde{P}(X,Y)=\frac{n}{(n-1)}\left[S(X,Y)-\psi g(\phi X,Y)+\{1+a(\xi)\}g(X,Y)+\{n-2-3a(\xi)\}\eta(X)\eta(Y)-2(n-2)a(X)\eta(Y)+da(\phi Y,X)-\left\{\frac{r+2(n-1)+(1-n)a(\xi)+\lambda-\psi^2}{n}\right\}g(X,Y)\right] \quad \text{… (38)}$$

From (38), we have

$$\tilde{P}(Y,X)=\frac{n}{(n-1)}[S(Y,X)-\psi g(\phi Y,X)+\{1+a(\xi)\}g(Y,X)+\{n-2-3a(\xi)\}\eta(X)\eta(Y)-2(n-2)a(Y)\eta(X)+da(\phi X,Y)$$

$$-\left\{\frac{r+2(n-1)+(1-n)a(\xi)+\lambda-\psi^2}{n}\right\}g(Y,X)] \quad \dots (39)$$

From (38) and (39),we have

$$\tilde{P}(X,Y)+\tilde{P}(Y,X)=\frac{n}{(n-1)}[2S(X,Y)-\tilde{P}(X,Y)+\tilde{P}(Y,X)$$
$$2\psi g(\phi X,Y)+2\{1+a(\xi)\}g(X,Y)$$
$$+2\{n-2-3a(\xi)\}\eta(X)\eta(Y)-2(n-2)\{a(X)\eta(Y)+a(Y)\eta(X)\}+$$
$$da(\phi X,Y)+da(\phi Y,X)-2\left\{\frac{r+2(n-1)+(1-n)a(\xi)+\lambda-\psi^2}{n}\right\}g(X,Y)] \quad \dots (40)$$

If $\tilde{P}(X,Y)$ is skew-symmetric then the left hand side of (40) vanishes and we get

$$S(X,Y)=\psi g(\phi X,Y)-\{1+a(\xi)\}g(X,Y)$$
$$-\{n-2-3a(\xi)\}\eta(X)\eta(Y)$$
$$+(n-2)\{a(X)\eta(Y)+a(Y)\eta(X)\}-\frac{1}{2}\{da(\phi X,Y)+$$
$$da(\phi Y,X)\}\quad-\left\{\frac{r+2(n-1)+(1-n)a(\xi)+\lambda-\psi^2}{n}\right\}g(X,Y) \quad \dots. (41)$$

Moreover, if $S(X,Y)$ is given by (41), then from (40), we get

$$\tilde{P}(X,Y)+\tilde{P}(Y,X)=0$$

i.e. Projective Ricci tensor of $\tilde{\nabla}$ is skew-symmetric.

Hence we can state the following theorem:

Theorem 5. *If a P-Sasakian manifold$M$ admits a quarter-symmetric non-metric connection$\tilde{\nabla}$, then a necessary and sufficient condition for the projective Ricci tensor of $\tilde{\nabla}$ to be skew-symmetric is that the Ricci tensor of the Levi-civita connection ∇ is given by* (41).

7. Einstein manifold with respect to quarter-symmetric non-metric connection$\tilde{\nabla}$,in a *P*-Sasakian manifold

A Riemannian manifold$M$ is called an Einstein manifold with respect to Riemannian connection if

$$S(X,Y)=\frac{r}{n}g(X,Y) \quad \dots.(42)$$

Analogous to this definition, we define Einstein manifold with respect to thequarter-symmetric non-metric connection$\tilde{\nabla}$, by

$$\tilde{S}(X,Y)=\frac{\tilde{r}}{n}g(X,Y) \quad \dots.(43)$$

From (29), (30) and (43),we have

$$\tilde{S}(X,Y)-\frac{\tilde{r}}{n}g(X,Y)=S(X,Y)-\frac{r}{n}g(X,Y)-\psi g(\phi X,Y)+da(\phi Y,X)$$

$$-\left\{\frac{n-2-(2n-1)a(\xi)+\lambda-\psi^2}{n}\right\}g(X,Y)-2(n-2)a(X)\eta(Y)$$

$$+\{n-2-3a(\xi)\}\eta(X)\eta(Y) \qquad \ldots(44)$$

If

$$\psi g(\phi X,Y)-da(\phi Y,X)=\left\{\frac{n-2-(2n-1)a(\xi)+\lambda-\psi^2}{n}\right\} \qquad \ldots(45)$$

$$g(X,Y)-\{n-2-3a(\xi)\}\eta(X)\eta(Y)+2(n-2)a(X)\eta(Y)$$

Then from (44),we get

$$\tilde{S}(X,Y)-\frac{\tilde{r}}{n}g(X,Y)=S(X,Y)-\frac{r}{n}g(X,Y) \qquad \ldots(46)$$

Hence we can state the following theorem:

Theorem 6.*In a P-Sasakian manifold M with a quarter-symmetric non-metric connection $\tilde{\nabla}$ if the relation* (44) *holds, then the manifold is an Einstein manifold for the Riemannian connection if and only if it is an Einstein manifold for the connection $\tilde{\nabla}$.*

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