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SOME CURVATURE PROPERTIES OF LP-SASAKIAN MANIFOLD WITH QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract: Inthis paper we study quarter-symmetric metric connection in LP-Sasakian manifold. Some results related to this connection are obtained and studied. Also some curvature properties of LP-Sasakian manifold with quarter-symmetric metric connection are studied.

Keywords: orentzian Para-Sasakian manifold, Quarter-symmetric metric connection, conformal, con-harmonic, con-circular and projective curvature tensor.

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1. Introduction

In 1975, Golab[5] initiated the study of quarter symmetric linear connection on a differentiable manifold. A linear connection$\widetilde{\nabla}$ in an n dimensional manifold is said to be a quarter-symmetric connection if torsion tensor T is of the form

$$T(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y] = \eta(Y)\phi X - \eta(X)\phi Y$$

where η is a 1-form and ϕ is a tensor of type (1,1). In addition, a quarter-symmetric linear connection $\widetilde{\nabla}$ satisfies the condition $\widetilde{\nabla}_X g = 0$ for all X,Y,Z∈TM, where TM is a lie algebra of vectors fields of the manifold M^n,then $\widetilde{\nabla}$ is said to be quarter-symmetric metric connection. Quarter-symmetric metric connection isalso studied by Biswas and De [1], De and Mondal[2], Singh and Pandey [13], Mishra and Pandey[9],Rastogi[11],Yano and Imai[17],Sular et al. [14],Mukhopadhyay,Roy and Barua[10] and many others.

In particular if $\phi X = X and \phi Y =$Y, then the quarter-symmetric reduces to a semi symmetric connection [4]. The semi symmetric metric connection is generalise case of quarter-symmetric metric connection and it is important in the geometry of Riemannian manifolds.

In 1989, Matsumoto [7] defined the idea of LP-Sasakian manifolds. Then Mihai and Rosca [8] introducedthe same notion independently and they obtained several results on LP-Sasakian manifolds in 1992. LP-Sasakian manifolds are studied bySingh[13],Jaiswal, et al. [6],Saikh and Baishya[12],Erolkihe and Tripathi [3],Tarafrdar and Bhattacharya[15] and many others.

Motivated by the studies of the authors[12],[1] and [13] in this paper we study some properties of the curvature of a LP-Sasakian manifold with a quarter-symmetric metric connection. We also discuss the different type of curvatures with a quarter-symmetric metric connection. This paper organised as follows:After preliminaries, in section 3, we study some properties of curvature tensor of LP-Sasakian manifold with respect to quarter-symmetric metric connection. In section 4, we study the relation between ricci tensor S and ricci tensor $\tilde{S}$with respect to quarter-symmetric metric connection. In section 5,section 6, section 7 and section 8 we study some properties of projective curvature tensor, conformal curvature tensor, concircular curvature tensor and conharmonic curvature tensor respectively.

2 Preliminaries

An n dimensional differentiable manifold M^n is called Lorentzian Para-Sasakian(briefly,LP-Sasakian) manifold[12], if it admits a (1,1) tensor fieldϕ a contravariant vector field ξ, a 1-formη and a Lorentzian metric g which satisfies

$$\eta(\xi) = -1 \quad \ldots(1)$$

$$\phi^2 X = X + \eta(X)\xi \quad \ldots(2)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \quad \ldots(3)$$

$$g(X, \xi) = \eta(X) \quad \ldots(4)$$

$$\nabla_X \xi = \phi X \quad \ldots(5)$$

$$(\nabla_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)(Y)\xi \quad \ldots(6)$$

where∇denotes the covariant differentiation with respect to Lorentzian metric.

It can be easily seen that in an LP-Sasakian manifold the following relation hold:

$$\phi\xi = 0, \quad \eta(\phi)=0 \quad \ldots(7)$$

$$\text{Rank}(\phi) = n-1 \quad \ldots(8)$$

If we put

$$\phi(X,Y) = g(X, \phi Y) \quad \ldots(9)$$

For any vector field X and Y, then the tensor fieldΦ(X,Y) is a symmetric (0,2) tensor field.

Also since the 1-form η is closed in an LP-Sasakian manifold, we have

$$(\nabla_X \eta)(Y) = \Phi(X, Y) \quad , \quad \Phi(X, \xi) = 0 \quad \ldots(10)$$

For all X,Y∈TM.

Also in LP-Sasakian manifold, the following relations hold:

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \quad \ldots(11)$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X \quad \ldots(12)$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \quad \ldots.(13)$$

$$R(\xi,X)\xi = X + \eta(X)\xi \quad \ldots.(14)$$

$$S(X,\xi) = (n-1)\eta(X) \quad \ldots.(15)$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \ldots.(16)$$

For any vector field X,Y and Z, where R and S are the Riemannian Curvature tensor and Ricci tensor of the manifold respectively.

A quarter-symmetric metric connection $\widetilde{\nabla}$ in a L.P.Sasakian manifold can be defined by$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X,Y)\xi$ ….(17)

The curvature tensor $\tilde{R}$ of M^n with respect to quarter-symmetric metric connection $\widetilde{\nabla}$ is defined by

$$\tilde{R}(X,Y)Z = R(X,Y)Z + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + \eta(Z)\{\eta(Y)X - \eta(X)Y\}$$
$$+ \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \quad \ldots(18)$$

The Ricci tensor $\tilde{S}$ and Scalar curvature $\tilde{r}$ of M^n with respect to quarter-symmetric metric connection $\widetilde{\nabla}$ is defined by

$$\tilde{S}(Y,Z) = S(Y,Z) + (n-1)\eta(Y)\eta(Z) \quad \ldots(19) \tilde{r} = \mathrm{r} - (n-1)$$
….(20)

where$\tilde{S}$ and S are the Ricci tensor of the connection $\widetilde{\nabla}$ and ∇ respectively. Similarly $\tilde{r}$ and r are the Scalar curvature of the connection $\widetilde{\nabla}$ and ∇ respectively.

3. SomeCurvature properties of LP-Sasakian manifoldwith respect to quarter-symmetric connection.

Let K and $\tilde{K}$ be the curvature tensor of type (0,4) given by

$K(X,Y,Z,U) = g(R(X,Y)Z,U)$

$\tilde{K}(X,Y,Z,U) = g(\tilde{R}(X,Y)Z,U)$

Theorem1. *In LP-Sasakian manifold with quarter-symmetric metric connection* $\tilde{\nabla}$,*we have*

$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0$ ...*(21)*

$\tilde{K}(X,Y,Z,U) + \tilde{K}(Y,X,Z,U) = 0$ ....*(22)*

$\tilde{K}(X,Y,Z,U) + \tilde{K}(X,Y,U,Z) = 0$ ....*(23)*

$\tilde{K}(X,Y,Z,U) - \tilde{K}(Z,U,X,Y) = 0$ ....(24)

Proof. Using (18) and first Binachi identity

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

with respect to Levi-Civitaconnection∇, we obtain (21)since $\Phi(X, Y)$ is a symmetric.

From (18) we have,

$\tilde{K}(X,Y,Z,U) = K(X,Y,Z,U) + g(\phi Y,Z)g(\phi X,U) - g(\phi X,Z)g(\phi Y,U) + \eta(Z)\{\eta(X)g(Y,U) - \eta(Y)g(X,U)\}$ ....(25)

Since $K(X,Y,Z,U) = -K(Y,X,Z,U)$, we obtain (22).

By using (18), (25) and equation $K(X,Y,Z,U) = -K(X,Y,U,Z)$,

we obtain (23).

Similarly from (18), (25) and the equation$K(X,Y,Z,U)$=$K(U,Z,X,Y)$

we obtain (24).

Theorem2. *Let M^n be an n dimensional LP-Sasakian manifold with the quarter-symmetric metric connection*$\tilde{\nabla}$. *Thenfor all X, Y, Z, ξ ∈TM, we have*

$\tilde{R}(\xi,X)\xi = 0$ ….*(26)*

$\tilde{R}(X,Y)\xi = 0$ ….*(27)*

$\tilde{R}(\xi,X)Y = 0$ ….*(28)*

$\tilde{S}(X,\xi) = 0$ ….*(29)*

$\tilde{S}(\phi X,\phi Y) = S(X,Y)$ *(30)*

Proof. Using (18) and (14), we get (26). Similarly with the help (18) and (13), we obtain(27). Again using (18) and (12), we get(28).

For getting (29), we use (19) and (15). Similarly using (19) and (16) we get (30).

4. η-Einstein manifolds

The Ricci tensor S of LP-Sasakian manifold is said to be $\eta - Einstein$ if its Ricci tensor satisfies the following:

$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$ …(31)

where$a = \frac{r}{(n-1)} - 1$ $and b = \frac{r}{(n-1)} - n$

Then, the Ricci tensor $\tilde{S}$ of an$\eta -$ Einstein quarter-symmetric LP-Sasakian Manifold is given by

$\tilde{S}(X,Y) = \left[\frac{\tilde{r}}{(n-1)} - 1\right] g(X,Y) + \left[\frac{\tilde{r}}{(n-1)} - n\right] \eta(X)\eta(Y)$ ….(32)

From (20), (32) reduces to

$\tilde{S}(X,Y) = \left[\frac{r}{(n-1)} - 2\right] g(X,Y) + \left[\frac{r}{(n-1)} - (n+1)\right] \eta(X)\eta(Y)$ ….(33)

$\tilde{S}(X,Y) = S(X,Y) - g(\phi X, \phi Y)$ ….(34)

Hence we can state following theorem:

Theorem3.*Let (M^n, g) be the LP-Sasakian manifold with almost Lorentzian para contact metric structure (ϕ, η, ξ, g) admitting quarter-symmetric metric connection$\tilde{\nabla}$which satisfies (19) and (20). Then the relation between ricci tensor ofLP-Sasakian manifold and ricci tensor $\tilde{S}$ of quarter symmetric LP-Sasakian manifold is given by (34).*

5. Projective curvature

Let M^n be an n dimensional LP-Sasakian manifold. The projective curvature tensor of M^n with respect to quarter symmetric metric connection $\tilde{\nabla}$ is defined by

$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{(n-1)}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\}$ …(35)

From (18), (19) and (35), we obtain

$$\tilde{P}(X,Y)Z = P(X,Y)Z + g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X$$

$+\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi$ ….(36)

From (36), we obtain

$\tilde{P}(X,Y)Z + \tilde{P}(Y,Z)X + \tilde{P}(Z,X)Y = 0$ ….(37)

Theorem 4. *Let M^n be an n dimensional LP-Sasakian manifold with the quarter-symmetric metric connection$\tilde{\nabla}$, then the projective curvature tensor of M^n with respect to quarter symmetric metric connection $\tilde{\nabla}$ is cyclic.*

6. Conformal curvature tensor

Let M^n be an n dimensional LP-Sasakian manifold. The conformal curvature tensor of M^n with respect to quarter symmetric metric connection $\tilde{\nabla}$ is defined by

$$\tilde{C}(X,Y,Z,U) = \tilde{K}(X,Y,Z,U) - \frac{1}{(n-2)}\{g(Y,Z)\tilde{S}(X,U) - g(X,Z)\tilde{S}(Y,U) + g(X,U)\tilde{S}(Y,Z) - g(Y,U)\tilde{S}(X,Z)\} + \frac{\tilde{r}}{(n-1)(n-2)}\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} \quad \ldots.(38)$$

If $\tilde{S}=0$, (38) gives

$$\tilde{C}(X,Y,Z,U) = \tilde{K}(X,Y,Z,U) \quad \ldots.(39)$$

Theorem 5. *If in a LP-Sasakian manifold the Ricci tensor of a quarter-symmetric metric connection $\tilde{\nabla}$ vanishes, then the curvature tensor of $\tilde{\nabla}$ is equal to the conformal curvature tensor of the quarter-symmetric manifold.*

From (22) and (39), we obtain

$$\tilde{C}(X,Y,Z,U) + \tilde{C}(Y,X,Z,U) = 0 \quad \ldots.(40)$$

7. Concircular curvature tensor

Let M^n be an n dimensional LP-Sasakian manifold. The concircular curvature tensor of M^n with respect to quarter symmetric metric connection $\tilde{\nabla}$ is defined by

$$\tilde{Z}(X,Y)U = \tilde{R}(X,Y)U - \frac{\tilde{r}}{n(n-1)}[g(Y,U)X - g(X,U)Y] \quad \ldots.(41)$$

From (41) and (21), we obtain

$$\tilde{Z}(X,Y)U + \tilde{Z}(Y,U)X + \tilde{Z}(U,X)Y = 0 \quad \ldots(42)$$

Theorem 6. *LetM^n be an n dimensional LP-Sasakian manifold with the quarter-symmetric metric connection$\tilde{\nabla}$, thenthe concircular curvature tensor of M^n with respect to quarter symmetric metric connection $\tilde{\nabla}$ is cyclic.*

8. Conharmonic curvature tensor

Let M^n be an n dimensional LP-Sasakian manifold. The conharmonic curvature tensor of M^n with respect to quarter symmetric metric connection $\tilde{\nabla}$ is defined by

$$\tilde{V}(X,Y,Z,U) = \tilde{K}(X,Y,Z,U) - \frac{1}{(n-2)}[\tilde{S}(Y,Z)g(X,U) - \tilde{S}(X,Z)g(Y,U) + \tilde{S}(X,U)g(Y,Z) - \tilde{S}(Y,U)g(X,Z)] \quad \text{....(43)}$$

If $\tilde{S}$ =0, (43) gives

$$\tilde{V}(X,Y,Z,U) = \tilde{K}(X,Y,Z,U) \quad \text{....(44)}$$

From (22) and (44), we obtain

$$\tilde{V}(X,Y,Z,U) + \tilde{V}(Y,X,Z,U) = 0$$

Again from (39) and (44), we obtain

$$\tilde{V}(X,Y,Z,U) = \tilde{C}(X,Y,Z,U)$$

Theorem 7. *In an LP-Sasakian manifold the Ricci tensor of a quarter symmetric metric connection*$\tilde{\nabla}$ *vanishes, then the conformal curvature tensor is equal to conharmonic curvature tensor of the quarter-symmetric manifold.*

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