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AN ANALYSIS OF A FEEDBACK QUEUEING MODELWITH NON IDENTICAL SERVERS AND RENEGING

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Abstract : In this paper, we present the solution of an M/M/2 feedback queueing system with non Identical servers.Customers are assumed to arrive in a poisson fashion and service times follow the exponential distribution. It is also assumed that customers renege according to exponential distribution/fixed probability. Steady-state queue-length probabilities and probability generating function of queue length probabilities for transient-state in terms of their Laplace transformation are obtained. Few interesting cases are derived to match our results with earlier published work.

Keywords : Non-identical servers, poisson arrivals, generating function, feedback, reneging, exponential distribution.

2010 Mathematics Subject Classification : 90B20, 60K25

1. Introduction

The traditional assumption in queueing theory is that the customers arrive, stand in queue and leave the system after getting the service. However, in some realqueueing systems, an arriving unit is given the alternative of re-joining the system with definite probability after being served once. The unit however leaves the system definitely after having received the service for the second time.These types of queues are known as queues with feedback. The concept of feedback was first introduced by Finch[2].In some queueingsituations, customers leave the system without getting service due to impatience. These situations in queueing theory are known as reneging.

Saaty[7] studied a queueing problem with two parallel servers each with different service rate and obtained the steady-state probabilities. Garg [3]studied an M/M/2 queueing system with two servers each having different service rate and obtained the explicit time dependent probabilities.Later, the study was followed by authors like Tackas[10], Maggu [5], Arya [1], Singh [9], Sharma [8] and Montazer-Haghighi [6] who solved the models in the steady-state case.Kumari [4] obtained steady-state probabilities and probability generating function of transient-state queue-length probabilities in terms of their Laplace transformations by considering a feedback queueing system with two non-identical parallel servers.

In this paper,we considered a feedback queueing problem with reneging and two serverseach with different service rate, wherein the customers arrive and join any of the free servers. When the numbers of customers are more than the number of servers, then the arriving customers join the waiting line and wait for their service. Sometimes a customer leaves the queue without getting service due to impatience.

The practical situation which corresponds to the above model can be that of a general shop, wherein the customers arrived for purchasing goods. There are two non-identical servers and customers can join any of the free servers for their service. After getting the first service, some customers are not satisfiedwith the service and so they re-join the queue again. Sometimes customers have not enough time so they renege the queue. The manager of a shop can know the various probabilities of the number of customers to be served by any time.

The queueing system investigated in this paper is governed by the following assumptions

(i) Arrivals are Poisson with parameter λ and the service time are exponentially distributed with parameters μ_1 and μ_2 for the first and second servers respectively.

(ii) When both the servers are empty, an arriving customer joins first channel with probability a_1 and second server with probability a_2, so that $a_1 + a_2 = 1$.

(iii) The probability that the customer re-joins the system is 'p' and that of leaves the system is 'q' for the customers getting first service, so that $p + q = 1$. However the customers will have to leave the system definitely after getting second service.

(iv) The customerdeparts the service channel for the first time with probability c_1 and for the second time with probability c_2, so that $c_1 + c_2 = 1$.

(v) The probability of a customer renege during time Δt, when there are n customers in the queue is $r(n)\,\Delta t$ and also assumed that renege follows the exponential distribution. If any one of the (n-2) customers in the queue renege then the density function for the minimum of (n-2) selections from d(t) becomes

$$d_{n-2}(t) = (n-2)\alpha e^{-(n-2)\alpha t}$$

Also $$r(n)=\begin{cases} 0, & 0 < n \leq 2 \\ (n-2)\alpha, & n > 2 \end{cases}$$

(vi) The waiting space is infinite.

(vii) The stochastic processes involved, viz

(a) Arrival of units (b) Departure of units

are statistically independent.

2. Definitions

$P_1^{(k)}(1,0,t)$ = Probability that customer is in the first server at time t and next customeris to depart for the first time or second time according as k=0 or 1.

$P_1^{(k)}(0,1,t)$ = Probability that customer is in the second server at time t and next customer is to depart for the first time or second time according as k=0 or 1.

$P_n^{(k)}(t)$ = Probability that there are n customers in the system at time t and next unit is todepart for the first time or second time according as k=0 or 1.

$P_n(t)$ =Probability that there are n units in the system at time t.

We

$$P_1^{(k)}(t) = P_1^{(k)}(1,0,t) + P_1^{(k)}(0,1,t), \quad k = 0,1$$
$$P_n(t) = P_n^{(0)}(t) + P_n^{(1)}(t), \qquad n \geq 0 \qquad \ldots(1)$$

Initially $P_0^{(0)}(t) = 1$ and $P_0^{(1)}(t) = 0$, $t \geq 0$

The difference – differential equations describing the system are

$$\frac{d}{dt}P_0^{(0)}(t) = -\lambda P_0^{(0)}(t) + \mu_1\left\{qP_1^{(0)}(1,0,t) + P_1^{(1)}(1,0,t)\right\} + \mu_2\left\{qP_1^{(0)}(0,1,t) + P_1^{(1)}(0,1,t)\right\} \qquad \ldots(2)$$

$$\frac{d}{dt}P_1^{(0)}(1,0,t) = -(\lambda+\mu_1)P_1^{(0)}(1,0,t) + \lambda a_1 P_0^{(0)}(t) + \mu_2 q c_1 P_2^{(0)}(t) + \mu_2 c_1 P_2^{(1)}(t) \ldots(3)$$

$$\frac{d}{dt}P_1^{(1)}(1,0,t) = -(\lambda+\mu_1)P_1^{(1)}(1,0,t) + \mu_1 pa_1 P_1^{(0)}(1,0,t) + \mu_2 pa_1 P_1^{(0)}(0,1,t)$$

$$+ \mu_2 qc_2 P_2^{(0)}(t) + \mu_2 c_2 P_2^{(1)}(t) \quad \ldots(4)$$

$$\frac{d}{dt}P_1^{(0)}(0,1,t) = -(\lambda+\mu_2)P_1^{(0)}(0,1,t) + \lambda a_2 P_0^{(0)}(t) + \mu_1 qc_1 P_2^{(0)}(t) + \mu_1 c_1 P_2^{(1)}(t) \ldots(5)$$

$$\frac{d}{dt}P_1^{(1)}(0,1,t) = -(\lambda+\mu_2)P_1^{(1)}(0,1,t) + \mu_1 pa_2 P_1^{(0)}(1,0,t) + \mu_2 pa_2 P_1^{(0)}(0,1,t)$$

$$+\mu_1 qc_2 P_2^{(0)}(t) + \mu_1 c_2 P_2^{(1)}(t) \quad \ldots(6)$$

$$\frac{d}{dt}P_n^{(0)}(t) = -(\lambda+\mu_1+\mu_2+(n-2)\alpha)P_n^{(0)}(t) + \lambda P_{n-1}^{(0)}(t) + (\mu_1+\mu_2)pc_1 P_n^{(0)}(t)$$

$$+ (\mu_1+\mu_2)qc_1 P_{n+1}^{(0)}(t) + (n-1)\alpha P_{n+1}^{(0)}(t) + (\mu_1+\mu_2)c_1 P_{n+1}^{(1)}(t), n \geq 2 \quad \ldots(7)$$

$$\frac{d}{dt}P_n^{(1)}(t) = -(\lambda+\mu_1+\mu_2+(n-2)\alpha)P_n^{(1)}(t) + \lambda P_{n-1}^{(0)}(t) + (\mu_1+\mu_2)c_2 P_{n+1}^{(1)}(t)$$

$$+ (n-1)\alpha P_{n+1}^{(1)}(t) + (\mu_1+\mu_2)pc_2 P_n^{(0)}(t) + (\mu_1+\mu_2)qc_2 P_{n+1}^{(0)}(t), (n \geq 2) \quad \ldots(8)$$

The steady- state difference equations describing the system are

$$\lambda P_0^{(0)} = \mu_1 \left\{qP_1^{(0)}(1,0) + P_1^{(1)}(1,0)\right\} + \mu_2 \left\{qP_1^{(0)}(0,1) + P_1^{(1)}(0,1)\right\} \quad \ldots(9)$$

$$(\lambda+\mu_1)P_1^{(0)}(1,0) = \lambda a_1 P_0^{(0)} + \mu_2 qc_1 P_2^{(0)} + \mu_2 c_1 P_2^{(1)} \quad \ldots(10)$$

$$(\lambda+\mu_1)P_1^{(1)}(1,0) = \mu_1 pa_1 P_1^{(0)}(1,0) + \mu_2 pa_1 P_1^{(0)}(0,1) + \mu_2 qc_2 P_2^{(0)} + \mu_2 c_2 P_2^{(1)} \ldots(11)$$

$$(\lambda+\mu_2)P_1^{(0)}(0,1) = \lambda a_2 P_0^{(0)} + \mu_1 qc_1 P_2^{(0)} + \mu_1 c_1 P_2^{(1)} \quad \ldots(12)$$

$$(\lambda+\mu_2)P_1^{(1)}(0,1) = \mu_1 pa_2 P_1^{(0)}(1,0) + \mu_2 pa_2 P_1^{(0)}(0,1) + \mu_1 qc_2 P_2^{(0)} + \mu_1 c_2 P_2^{(1)} \ldots(13)$$

$$(\lambda+\mu_1+\mu_2+(n-2)\alpha)P_n^{(0)} = \lambda P_{n-1}^{(0)} + (\mu_1+\mu_2)pc_1 P_n^{(0)} + (\mu_1+\mu_2)qc_1 P_{n+1}^{(0)}$$

$$+(n+1-2)\alpha P_{n+1}^{(0)} + (\mu_1+\mu_2)c_1 P_{n+1}^{(1)}, (n \geq 2) \quad \ldots(14)$$

$$(\lambda+\mu_1+\mu_2+(n-2)\alpha)P_n^{(1)} = \lambda P_{n-1}^{(0)} + (\mu_1+\mu_2)c_2 P_{n+1}^{(1)} + (n-1)\alpha P_{n+1}^{(1)}$$

$$+ (\mu_1+\mu_2)pc_2 P_n^{(0)} + (\mu_1+\mu_2)qc_2 P_{n+1}^{(0)}, (n \geq 2) \quad \ldots(15)$$

3. Steady - State Solution of the Problem

Using E f(x) = f(x+1), Eqs. (14) and (15) for n > 2 give

$$[\{(\mu_1+\mu_2)c_1 q + (n-1)\alpha\}E^2 + \{(\mu_1+\mu_2)c_1 p - (\lambda+\mu_1+\mu_2+(n-2)\alpha)\}E + \lambda]P_n^{(0)} + (\mu_1+\mu_2)c_1 E^2 P_n^{(1)} = 0, (n > 2)$$

$$\ldots(16)$$

$$[\{(\mu_1+\mu_2)c_2+(n-1)\alpha\}E^2-\{\lambda+\mu_1+\mu_2+(n-2)\alpha\}E+\lambda]P_n^{(1)}$$

$$+\{(\mu_1+\mu_2)c_2qE^2+(\mu_1+\mu_2)c_2pE\}P_n^{(0)} = 0,\ (n>2) \qquad \ldots(17)$$

Solving (16) and (17) with help of determinants, we have

$$[(n-1)\alpha E^2-\{\lambda+\mu_1+\mu_2+(n-2)\alpha\}E+\lambda][\{(\mu_1+\mu_2)c_1q+(\mu_1+\mu_2)c_2$$

$$+(n-1)\alpha\}E^2+\{(\mu_1+\mu_2)c_1p-(\lambda+\mu_1+\mu_2+(n-2)\alpha)\}E+\lambda=0,(n>2) \quad \ldots(18)$$

The two roots of(18) are obtained by solving its first factor and we get

$$E=\frac{\lambda+\mu_1+\mu_2+(n-2)\alpha \pm\sqrt{\{\lambda+\mu_1+\mu_2+(n-2)\alpha\}^2-4\lambda(n-1)\alpha}}{2(n-1)\alpha}$$

$$=\frac{\lambda+\mu_1+\mu_2+(n-2)\alpha \pm\sqrt{\{\lambda+\mu_1+\mu_2-(n-2)\alpha\}^2+4\alpha\{(\mu_1+\mu_2)(n-2)-\lambda\}}}{2(n-1)\alpha}$$

$$=\frac{\lambda+\mu_1+\mu_2+(n-2)\alpha \pm\{\lambda+\mu_1+\mu_2-(n-2)\alpha+\delta\}}{2(n-1)\alpha}$$

whereδ is a small +ve quantity.

Therefore $z_0=\frac{2(n-2)\alpha-\delta}{2(n-1)\alpha}$ and $z_1=\frac{2(\lambda+\mu_1+\mu_2)+\delta}{2(n-1)\alpha}$ ...(19)

$z_0$is always less than 1 whatever may be the value of various parameters but z_1 is less than 1 only when $2(\lambda+\mu_1+\mu_2)+\delta<2(n-1)\alpha$. The other two roots z_2, z_3 are obtained from the equation

$$\{(\mu_1+\mu_2)c_1q+(\mu_1+\mu_2)c_2+(n-1)\alpha\}E^2 + \{(\mu_1+\mu_2)c_1p-(\lambda+\mu_1+\mu_2+(n-2)\alpha)\}E+\lambda=0$$

after putting the values of the parameters λ,μ_1, μ_2, c_1, c_2,p, and q, in the quadratic equation. After evaluatingz_2, z_3, for the convergence of solution, any root ≥ 1 must be rejected.

The value of $P_n^{(0)}$and $P_n^{(1)}$ are given by

$$P_n^{(0)}=\sum_{i=0}^{3} a_i z_i^n \text{ and } P_n^{(1)}=\sum_{i=0}^{3} b_i z_i^n, \quad (n>2)$$

wherez_0, z_1, z_2, z_3 are the roots of (18) and a_i, b_i(i=0, 1, 2, 3) are arbitrary constants to be evaluated. . In case $z_i \geq 1$ take $a_i, b_i = 0$; i = 1, 2, 3. From (14) for n=2, we can get probability $P_2^{(0)}$ in terms of $P_1^{(0)}(0,1), P_1^{(0)}(1,0)$. From (13) we can get probability $P_2^{(1)}$ in terms of $P_1^{(1)}(0,1), P_1^{(0)}(0,1), P_0^{(0)}(1,0)$. From (12) we can get probability $P_1^{(0)}(1,0)$ in terms of $P_1^{(1)}(0,1), P_1^{(0)}(0,1), P_0^{(0)}$. From (11) we can get probability $P_1^{(0)}(0,1)$ in terms of

$P_1^{(1)}(0,1), P_1^{(1)}(1,0), P_0^{(0)}$. From (10) we can get probability $P_1^{(1)}(1,0)$ in terms of $P_1^{(1)}(0,1), P_0^{(0)}$. From equation (9) we can get probability $P_1^{(1)}(0,1)$ in terms of $P_0^{(0)}$.

Therefore

$$P_1^{(1)}(0,1) = \frac{\lambda[(\lambda+\mu_1)D + A\{-c_2q(\lambda+\mu_1) + pc_2(\mu_2a_2-\mu_1a_1)\}]P_0^{(0)}}{\mu_2H}$$

$$P_1^{(1)}(1,0) = \frac{\lambda[(\lambda+\mu_2)E - A\{c_2q(\lambda+\mu_2) + pc_2(\mu_2a_2-\mu_1a_1)\}]P_0^{(0)}}{\mu_1H}$$

$$P_1^{(0)}(0,1) = \frac{\lambda[(\lambda+\mu_1)F + c_1(\mu_2a_2-\mu_1a_1)\{pA + q(\lambda+\mu_1)(\lambda+\mu_2)\}]P_0^{(0)}}{\mu_2H}$$

$$P_1^{(0)}(1,0) = \frac{\lambda[(\lambda+\mu_2)G - c_1(\mu_2a_2-\mu_1a_1)\{pA + q(\lambda+\mu_1)(\lambda+\mu_2)\}]P_0^{(0)}}{\mu_1H}$$

$$P_2^{(1)} = \frac{qJ}{B} + \frac{\lambda}{\mu_1\mu_2HB}[B(\lambda+\mu_1)^2(\lambda+\mu_2)^2 - AB\{pA + q(\lambda+\mu_1)(\lambda+\mu_2)\} -$$
$$\mu_1\lambda q(\lambda+\mu_1)F + \mu_2\lambda q(\lambda+\mu_2)G + \lambda q(\mu_1-\mu_2)c_1(\mu_2a_2-\mu_1a_1)$$
$$\{pA + q(\lambda+\mu_1)(\lambda+\mu_2)\}]P_0^{(0)}$$

$$P_2^{(0)} = \frac{J}{B} + \frac{\lambda^2}{\mu_1\mu_2HB}[\mu_1(\lambda+\mu_1)F + \mu_2(\lambda+\mu_2)G$$
$$+ (\mu_1-\mu_2)c_1(\mu_2a_2-\mu_1a_1)\{pA + q(\lambda+\mu_1)(\lambda+\mu_2)\}]P_0^{(0)}$$

where $A = \mu_2a_2(\lambda+\mu_1) + \mu_1a_1(\lambda+\mu_2)$

$B = \lambda + (\mu_1+\mu_2)(1-pc_1)$

$D = c_2(\lambda+\mu_1)(\lambda+\mu_2) + \mu_2c_1pa_2(2\lambda+\mu_1+\mu_2)$

$E = c_2(\lambda+\mu_1)(\lambda+\mu_2) + \mu_1c_1pa_1(2\lambda+\mu_1+\mu_2)$

$F = c_1(\lambda+\mu_1)(\lambda+\mu_2) + \mu_2a_2c_2(2\lambda+\mu_1+\mu_2)$

$G = c_1(\lambda+\mu_1)(\lambda+\mu_2) + \mu_1a_1c_2(2\lambda+\mu_1+\mu_2)$

$H = (2\lambda+\mu_1+\mu_2)[pc_1A + (c_2+c_1q)(\lambda+\mu_1)(\lambda+\mu_2)]$

and $J = \{(\mu_1+\mu_2)c_1q + \alpha\}\sum_{i=0}^{3} a_iz_i^3 + (\mu_1+\mu_2)\sum_{i=0}^{3} b_iz_i^3$

Eight unknown $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3$ (two unknowns a_0, b_0 in case$z_i \geq 1$; i = 1, 2, 3) can be evaluated from (14) for n= 3, 4, 5, 6 and (15) for n=2, 3, 4, 5 in terms of $P_0^{(0)}$and the value of $P_0^{(0)}$ can be found by using the relation

$$P_0^{(0)} = 1 - \sum_{n=1}^{\infty} (P_n^{(0)} + P_n^{(1)})$$

Hence by using the value of $a_0, a_1, a_2, a_3, b_0 b_1, b_2, b_3$ and $P_0^{(0)}$, the probabilities $P_n^{(0)}$ and $P_n^{(1)}$ are completely known for various value of n.

4. Special cases

(i) When there is no reneging i.e. feedback queueing problem with two non-identical servers.

Putting α=0 in (18), we get

$$\{-(\lambda + \mu_1 + \mu_2)E + \lambda\}[(\mu_1 + \mu_2)(c_1 q + c_2)E^2 + \{(\mu_1 + \mu_2)c_1 p - (\lambda + \mu_1 + \mu_2)\}E + \lambda] = 0 \ , (n > 2)$$

After simplification we get

$$(E - 1)[\{(\mu_1 + \mu_2)(\lambda + \mu_1 + \mu_2)(c_1 q + c_2)\}E^2 - \lambda\{\lambda + 2(\mu_1 + \mu_2) - (\mu_1 + \mu_2)c_1 p\}E + \lambda^2] = 0$$

Taking common $(\mu_1 + \mu_2)^2$ and tubg $\rho = \frac{\lambda}{(\mu_1 + \mu_2)}$ we get

$$(E - 1)[\{(\rho + 1)(c_1 q + c_2)\}E^2 - \rho\{\rho + 2 - c_1 p\}E + \rho^2] = 0 \quad , (n \geq 2) \qquad \ldots(20)$$

This coincides with equation (3.25) of Kumari[4].

(ii) When there is no feedback and service rate of both servers are equal i.e. M/M/2 with reneging.

Putting q=1, p=0, $c_1 = 1, c_2 = 0, P_n^{(0)} = P_n , P_n^{(1)} = 0$ then (9)-(15) reduces to

$$\lambda P_0 = \mu P_1 \qquad \ldots(21)$$

$$(\lambda + \mu)P_1 = \lambda P_0 + 2\mu P_2 \qquad \ldots(22)$$

$$\{\lambda + 2\mu + (n - 2)\alpha\}P_n = \lambda P_{n-1} + \{ 2\mu + (n + 1 - 2)\}P_{n+1}, n \geq 2 \qquad \ldots(23)$$

Solving the above equationsrecursively and get

$$P_1 = \frac{\lambda}{\mu} P_0, P_2 = \frac{\lambda^2}{2\mu^2} P_0 \text{and } P_n = \frac{\lambda_n}{2\mu^2 \prod_{i=1}^{n-2} (2\mu + i\alpha)} P_o, \quad (n > 2) \qquad \ldots(24)$$

(iii) When there is no reneging.

Puting α=0 in (24) we get

$$P_n = 2\left(\frac{\lambda}{2\mu}\right)^n P_0 \quad \text{and} \quad P_0 = \frac{2\mu-\lambda}{2\mu+\lambda}$$

$$\text{Therefore} \quad P_n = \frac{2(2\mu-\lambda)}{(2\mu+\lambda)}\left(\frac{\lambda}{2\mu}\right)^n, n > 2 \qquad (25)$$

This coincides with M/M/2 classical model.

5. Now we assume that customers renege according to fixed probability α. Then

$$r(n) = \begin{cases} 0, & 0 < n \le 2 \\ \alpha, & n > 2 \end{cases}$$

and the steady- state difference equations describing the system will be

$$\lambda P_0^{(0)} = \mu_1\left\{qP_1^{(0)}(1,0) + P_1^{(1)}(1,0)\right\} + \mu_2\left\{qP_1^{(0)}(0,1) + P_1^{(1)}(0,1)\right\} \dots (26)$$

$$(\lambda+\mu_1)P_1^{(0)}(1,0) = \lambda a_1 P_0^{(0)} + \mu_2 q c_1 P_2^{(0)} + \mu_2 c_1 P_2^{(1)} \dots (27)$$

$$(\lambda+\mu_1)P_1^{(1)}(1,0) = \mu_1 p a_1 P_1^{(0)}(1,0) + \mu_2 p a_1 P_1^{(0)}(0,1) + \mu_2 q c_2 P_2^{(0)} + \mu_2 c_2 P_2^{(1)} \dots (28)$$

$$(\lambda+\mu_2)P_1^{(0)}(0,1) = \lambda a_2 P_0^{(0)} + \mu_1 q c_1 P_2^{(0)} + \mu_1 c_1 P_2^{(1)} \dots (29)$$

$$(\lambda+\mu_2)P_1^{(1)}(0,1) = \mu_1 p a_2 P_1^{(0)}(1,0) + \mu_2 p a_2 P_1^{(0)}(0,1) + \mu_1 q c_2 P_2^{(0)} + \mu_1 c_2 P_2^{(1)} \dots (30)$$

$$(\lambda+\mu_1+\mu_2+\alpha(1-\delta_{n,2}))P_n^{(0)} = \lambda P_{n-1}^{(0)} + (\mu_1+\mu_2)pc_1 P_n^{(0)} +$$

$$(\mu_1+\mu_2)qc_1 P_{n+1}^{(0)} + \alpha P_{n+1}^{(0)} + (\mu_1+\mu_2)c_1 P_{n+1}^{(1)}, n \ge 2) \qquad \dots (31)$$

$$(\lambda+\mu_1+\mu_2+\alpha(1-\delta_{n,2}))P_n^{(1)} = \lambda P_{n-1}^{(0)}$$

$$+(\mu_1+\mu_2)c_2 P_{n+1}^{(1)} + (\mu_1+\mu_2)pc_2 P_n^{(0)} + \alpha P_{n+1}^{(1)} + (\mu_1+\mu_2)qc_2 P_{n+1}^{(0)}, (n \ge 2) \dots (32)$$

$$\text{where} \delta_{n,2} = \begin{cases} 1, & \text{for } n = 2 \\ 0, & \text{0therwise} \end{cases}$$

6. Steady-State Solution of the Problem

Using E f(x) = f(x+1), (31) and (32) for n > 2 we get.

$$[\{(\mu_1+\mu_2)c_1 q + \alpha\}E^2 + \{(\mu_1+\mu_2)c_1 p - (\lambda+\mu_1+\mu_2+\alpha)\}E + \lambda]P_n^{(0)}$$
$$+ (\mu_1+\mu_2)c_1 E^2 P_n^{(1)} = 0, (n > 2) \qquad \dots (33)$$

$$[\{(\mu_1+\mu_2)c_2 + \alpha\}E^2 - \{\lambda+\mu_1+\mu_2+\alpha\}E + \lambda]P_n^{(1)} + \{(\mu_1+\mu_2)c_2 q E^2$$

$$+(\mu_1+\mu_2)c_2 p E\}P_n^{(0)} = 0, (n > 2) \qquad \dots (34)$$

Solving (33) and (34) with help of determinants, we have

$$[\{(\mu_1+\mu_2)c_1q+(\mu_1+\mu_2)c_2+\alpha\}E^2+\{(\mu_1+\mu_2)c_1p-(\lambda+\mu_1+\mu_2+\alpha)\}E+\lambda]$$

$$[\alpha E^2-(\lambda+\mu_1+\mu_2+\alpha)E+\lambda]=0,\ \ (n>2) \qquad \ldots(35)$$

The two roots of the equation (35) are obtained by solving the first factor of (35) which is $z_0=1$ and

$$z_1=\frac{\lambda}{(\mu_1+\mu_2)-(\mu_1+\mu_2)c_1p+\alpha}$$

The other two roots of (35) are obtained from the second factor of the (35)

$$E=\frac{(\lambda+\mu_1+\mu_2+\alpha)\ \pm\sqrt{\{\lambda+(\mu_1+\mu_2)+\alpha\}^2\ -4\lambda\alpha}}{2\alpha}$$

$$=\frac{(\lambda+\mu_1+\mu_2+\alpha)\ \pm\sqrt{\{\lambda+(\mu_1+\mu_2)-\alpha\}^2\ +4(\mu_1+\mu_2)\alpha}}{2\alpha}$$

$$=\frac{(\lambda+\mu_1+\mu_2+\alpha)\ \pm(\lambda+\mu_1+\mu_2-\alpha+\delta)}{2\alpha}$$

Therefore $z_2=\dfrac{2\alpha-\delta}{2\alpha}$ and $z_3=\dfrac{2(\lambda+\mu_1+\mu_2)+\delta}{2\alpha}$

$z_2$is always less than 1 but $z_1$and z_3 areless than 1 only if

$\lambda<(\mu_1+\mu_2)-(\mu_1+\mu_2)c_1p+\alpha$ and $\lambda<\alpha-(\mu_1+\mu_2)-\delta/2$

respectively. To have convergence of solution, any root ≥1 must be rejected. Thus rejectz_0=1.

The value of $P_n^{(0)}$and $P_n^{(1)}$ are given by

$$P_n^{(0)}=\sum_{i=1}^{3}a_iz_i^n \ \text{ and } \ P_n^{(1)}=\sum_{i=1}^{3}b_iz_i^n,\quad (n>2)$$

wherez_1,z_2,z_3 are the roots of (35) and a_I,b_i(i= 1, 2, 3) are arbitrary constants to be evaluated. In case z_1 and $z_3\geq 1$ take $a_1,a_3,b_1,b_3=0$. From equation (31) for n=2, we can get probability $P_2^{(0)}$ in terms of $P_1^{(0)}(0,1),P_1^{(0)}(1,0)$. From (30) we can get probability $P_2^{(1)}$ in terms of $P_1^{(1)}(0,1),P_1^{(0)}(0,1),P_1^{(0)}(1,0)$. From (29) we can get probability $P_1^{(0)}(1,0)$ in terms of $P_1^{(1)}(0,1),P_1^{(0)}(0,1),P_0^{(0)}$. From (28) we can get probability $P_1^{(0)}(0,1)$ in terms of $P_1^{(1)}(0,1),P_1^{(1)}(1,0),P_0^{(0)}$. From (27) we can get probability $P_1^{(1)}(1,0)$ in terms of $P_1^{(1)}(0,1),P_0^{(0)}$. From (26) we can get probability $P_1^{(1)}(0,1)$ in terms of $P_0^{(0)}$.Hence we get probabilities $P_2^{(0)},P_2^{(1)},P_1^{(0)}(1,0),P_1^{(0)}(0,1),P_1^{(1)}(1,0),P_1^{(1)}(0,1)$in terms of $P_0^{(0)}$ and a_i's, b_i's . Eight unknown $a_0,a_1,a_2,a_3,b_0,b_1,b_2,b_3$ (two unknowns a_0,b_0 in case$z_i\geq 1$; i = 1, 2, 3) can be evaluated from equations (31) for n= 3, 4, 5, 6 and equation (32) for n=2, 3, 4, 5 in terms of $P_0^{(0)}$and the value of $P_0^{(0)}$ can be found by using the relation

$$P_0^{(0)} = 1 - \sum_{n=1}^{\infty} (P_n^{(0)} + P_n^{(1)})$$

Hence by using the value of $a_0, a_1, a_2, a_3, b_0 b_1, b_2, b_3$ and $P_0^{(0)}$, the probabilities $P_n^{(0)}$ and $P_n^{(1)}$ are completely known for various value of n.

7. Special Cases

(i) When there is no feedback and service rate of both servers are equal i.e. M/M/2 with reneging.

Puting q=1, p=0, $c_1 = 1, c_2 = 0, P_n^{(0)} = P_n, P_n^{(1)} = 0$ then (9)-(15) reduces to

$$\lambda P_0 = \mu P_1 \quad \text{....(36)}$$

$$(\lambda + \mu)P_1 = \lambda P_0 + 2\mu P_2 \quad \text{...(37)}$$

$$\{\lambda + 2\mu + (n-2)\alpha\}P_n = \lambda P_{n-1} + \{ 2\mu + (n+1-2)\}P_{n+1}, \quad (n \geq 2) \quad \text{... (38)}$$

Solving the above equationsrecursively and get

$$P_1 = \frac{\lambda}{\mu} P_0, \; P_2 = \frac{\lambda^2}{2\mu^2} P_0 \text{ and} P_n = \frac{\lambda^n}{2\mu^2 (2\mu+\alpha)^{n-2}}, n > 2 \quad \text{...(39)}$$

(ii) When there is no reneging.

Put α=0 in (39) we get

$$P_n = 2\left(\frac{\lambda}{2\mu}\right)^n P_0 \quad \text{and} \quad P_0 = \frac{2\mu - \lambda}{2\mu + \lambda}$$

Therefore $P_n = \frac{2(2\mu-\lambda)}{(2\mu+\lambda)}\left(\frac{\lambda}{2\mu}\right)^n, \; (n > 2)$ ....(40)

This coincides with M/M/2 classical model.

Taking the Laplace transformation $\overline{P}_n(s) = \int_0^{\infty} e^{-st} P_n(t)dt$; (Re s>0) of (2)-(8)for $r(n) = \begin{cases} 0, & 0 < n \leq 2 \\ \alpha, & n > 2 \end{cases}$, we have

$$(s+\lambda)\overline{P}_0^{(0)}(s) = 1 + \mu_1\{q\overline{P}_1^{(0)}(1,0,s) + \overline{P}_1^{(1)}(1,0,s)\}$$

$$+\mu_2\{q\overline{P}_1^{(0)}(0,1,s) + \overline{P}_1^{(1)}(0,1,s)\} \quad \text{...(41)}$$

$$(s+\lambda+\mu_1)\overline{P}_1^{(0)}(1,0,s) = \lambda a_1 \overline{P}_0^{(0)}(s) + \mu_2 q c_1 \overline{P}_2^{(0)}(s) + \mu_2 c_1 \overline{P}_2^{(1)}(s) \quad \text{...(42)}$$

$$(s+\lambda+\mu_1)\overline{P}_1^{(1)}(1,0,s) = \mu_1 p a_1 \overline{P}_1^{(0)}(1,0,s) + \mu_2 p a_1 \overline{P}_1^{(0)}(0,1,s)$$

$$+\mu_2 qc_2 \overline{P}_2^{(0)}(s) + \mu_2 c_2 \overline{P}_2^{(1)}(s) \quad \ldots(43)$$

$$\left(s+\lambda+\mu_2\right)\overline{P}_1^{(0)}(0,1,s) = \lambda a_2 \overline{P}_0^{(0)}(s) + \mu_1 qc_1 \overline{P}_2^{(0)}(s) + \mu_1 c_1 \overline{P}_2^{(1)}(s) \quad \ldots(44)$$

$$\left(s+\lambda+\mu_2\right)\overline{P}_1^{(1)}(0,1,s) = \mu_1 pa_2 \overline{P}_1^{(0)}(1,0,s) + \mu_2 pa_2 \overline{P}_1^{(0)}(0,1,s)$$

$$+\mu_1 qc_2 \overline{P}_2^{(0)}(s) + \mu_1 c_2 \overline{P}_2^{(1)}(s) \quad \ldots(45)$$

$$\left(s+\lambda+\mu_1+\mu_2+\alpha\left(1-\delta_{n,2}\right)\right)\overline{P}_n^{(0)}(s) = \lambda \overline{P}_{n-1}^{(0)}(s) + \left(\mu_1+\mu_2\right)pc_1 \overline{P}_n^{(0)}(s)$$

$$+\alpha \overline{P}_{n+1}^{(0)}(s) + \left(\mu_1+\mu_2\right)qc_1 \overline{P}_{n+1}^{(0)}(s) + \left(\mu_1+\mu_2\right)c_1 \overline{P}_{n+1}^{(1)}(s), \quad (n \geq 2) \quad \ldots(46)$$

$$\left(s+\lambda+\mu_1+\mu_2+\alpha\left(1-\delta_{n,2}\right)\right)\overline{P}_n^{(1)}(s) = \lambda \overline{P}_{n-1}^{(0)}(s) + \left(\mu_1+\mu_2\right)c_2 \overline{P}_{n+1}^{(1)}(s)$$

$$+\alpha \overline{P}_{n+1}^{(1)}(s) + \left(\mu_1+\mu_2\right)pc_2 \overline{P}_n^{(0)}(s) + \left(\mu_1+\mu_2\right)qc_2 \overline{P}_{n+1}^{(0)}(s), (n \geq 2) \ldots(47)$$

If we define $P^{(k)}(z.t) = \sum_{n=0}^{\infty} P_n^{(k)}(t)z^n$, $\overline{P}(k)(z,s) = \int_o^{\infty} e^{-st}P^{(k)}(z,t)dt$

$$P(z,t) = P^{(0)}(z,t) + P^{(1)}(z,t) \text{ and } \overline{P}(z,s) = \int_o^{\infty} e^{-st}P^{(k)}(z,t)dt$$

for all $k = 0$ or 1 with $|z| \leq 1$
thenLaplace transformation of probability generating function of transient – state queue length probabilities gives

$$\overline{P}^{(0)}(z,s) = \frac{1}{Q(z)}[z[\{K(z)(q-z) - \left(\mu_1+\mu_2\right)(c_2(q-z) - c_1pz)\}$$

$$\left\{\mu_1\overline{P}_1^{(0)}(1,0,s) + \overline{P}_1^{(0)}(0,1,s)\right\} + \{K(z) - \left(\mu_1+\mu_2\right)(c_2 - c_1z\}$$

$$\left\{\mu_1\overline{P}_1^{(1)}(1,0,s) + \mu_2\overline{P}_1^{(1)}(0,1,s)\right\} + \left(\mu_1+\mu_2\right)c_1\{-K(z) + \alpha(z-1) + \left(\mu_1+\mu_2\right)z\}\overline{P}_1^1(s)$$

$$-z(1-z)\alpha\left\{\left(K(z) - \left(\mu_1+\mu_2\right)c_2\right)\overline{P}_2^{(0)}(s) + \left(\mu_1+\mu_2\right)c_1\overline{P}_2^{(1)}(s)\right\} + \{K(z) -$$
$$\left(\mu_1+\mu_2\right)c_2\}] + \{(K(z) - \left(\mu_1+\mu_2\right)c_2)(\alpha(z-1) + \left(\mu_1+\mu_2\right)z) - K(z)\left(\mu_1 +\right.$$
$$\left.\mu_2\right)c_1(q+pz)\}\{z\overline{P}_1^{(0)}(s) + \overline{P}_0^{(0)}(s)\}]$$

$$\overline{P}^{(1)}(z,s) = \frac{1}{Q(z)}[z[\{K(z)pz + \left(\mu_1+\mu_2\right)(q+pz)(c_2(q-z) - c_1pz)\}$$

$$\left\{\mu_1\overline{P}_1^{(0)}(1,0,s) + \mu_2\overline{P}_1^{(0)}(0,1,s)\right\} + \{-K(z)z + \left(\mu_1+\mu_2\right)(q+pz)(c_2 + c_1z)\}$$

$$\left\{\mu_1\overline{P}_1^{(1)}(1,0,s) + \mu_2\overline{P}_1^{(1)}(0,1,s)\right\} + \{\left(K(z) - \left(\mu_1+\mu_2\right)c_1(q+pz)\right)$$

$$(\alpha(z-1) + \left(\mu_1+\mu_2\right)z) - K(z)\left(\mu_1+\mu_2\right)c_2\}\overline{P}_1^1(s) - z(1-z)\alpha$$

$\{(\mu_1+\mu_2)c_2(q+pz)\overline{P}_2^{(0)}(s)+(K(z)-(\mu_1+\mu_2)c_1(q+pz))\overline{P}_2^{(1)}(s)\}$

$+(\mu_1+\mu_2)c_2(q+pz)]+\ \{(\mu_1+\mu_2)c_2(q+pz)\{-K(z)+\alpha(z-1)+(\mu_1+\mu_2)z\}$

$\{z\overline{P}_1^{(0)}(s)+\overline{P}_0^{(0)}(s)$

and $\overline{P}(z,s)=\frac{1}{Q(z)}[z(1-z)[\{K(z)q-(\mu_1+\mu_2)p(c_2(q-z)-c_1pz)\}\{\mu_1\overline{P}_1^{(0)}(1,0,s)+$

$\mu_2\overline{P}_1^{(0)}(0,1,s)\}+\{K(z)-(\mu_1+\mu_2)p(c_2+c_1z)\}\{\mu_1\overline{P}_1^{(1)}(1,0,s)+\mu_2\overline{P}_1^{(1)}(0,1,s)\}-$

$\alpha z\{L\overline{P}_2^{(0)}(s)+M\overline{P}_2^{(1)}(s)\}]+\{L(\alpha(z-1)+\ (\mu_1+\ \mu_2)z)-K(z)(\mu_1+\mu_2)(q+pz)\}$

$\{z\overline{P}_1^{(0)}(s)+\overline{P}_0^{(0)}(s)\}+\ z\{M(\alpha(z-1)+\ (\mu_1+\mu_2)z)-K(z)(\mu_1+\ \mu_2)\})$

$\overline{P}_1^1(s)+Lz]\,,\lambda<(\mu_1+\mu_2)\ ;\ |(z\ |\leq 1$ ...(48)

where $K(z)=\ -\lambda z^2+(s+\lambda+\mu_1+\mu_2+\alpha)z-\alpha,$

$Q(z)=\{K(z)-(\mu_1+\mu_2)c_2\}\{K(z)-(\mu_1+\mu_2)c_1(q+pz)\}-(\mu_1+\mu_2)^2c_1c_2(q+pz),$

$L\ =K(z)-(\mu_1+\mu_2)c_2p(1-z)$, and $M\ =K(z)+(\mu_1+\mu_2)c_1p(1-z)$

Let $N(z)\ =\ K_1(z)K_2(z)-(\mu_1+\mu_2)^2c_1c_2(q+pz)$

where $K_1(z)\ =\ (-\lambda z^2+(s+\lambda+\mu_1+\mu_2+\alpha)z-((\mu_1+\mu_2)c_2+\alpha)$ and

$K_2(z)\ =\ (-\lambda z^2+(s+\lambda+\mu_1+\mu_2+\alpha-(\mu_1+\mu_2)c_1p)z-((\mu_1+\mu_2)c_1q+\alpha)$

Obviously $K_1(z)$ and $K_2(z)$ have two zeros inside the unit circle.

Let $f(z)\ =K_1(z)K_2(z)$ and $g(z)\ =(\mu_1+\mu_2)^2c_1c_2(q+pz)$, then

$|f(z)|\ =|K_1(z)\,K_2(z)|$

$=\ |(-\lambda z^2+(s+\lambda+\mu_1+\mu_2+\alpha)z-\{(\mu_1+\mu_2)c_2+\alpha\}|$

$|(-\lambda z^2+(s+\lambda+\mu_1+\mu_2+\alpha-(\mu_1+\mu_2)c_1p)z-\{(\mu_1+\mu_2)c_1q+\alpha\}|$

$\geq(\xi+(\mu_1+\mu_2)c_1)(\xi+(\mu_1+\mu_2)c_2)$ for $s=\xi+i\,\eta\,,\ |z|=1$

$\geq(\mu_1+\mu_2)^2c_1c_2\geq\ |g(z)|$

Hence $|\,f(z)|\ \geq\ |g(z)|$ on $|z|\ =\ 1$

Since all the condition of Rouche's theorem are satisfied, so $N(z)$ has two zeroes inside the unit circle. Let these zeroes be $z_m(m=0,1)$. Numerator must also vanish for these two zeroes since $\overline{P}(z,s)$ is analytical function of z. These two equations along with equation (41), (42) (43), (44) and (45)will determine the seven unknowns $P_2^{(0)}(s)$, $P_2^{(1)}(s)$, $P_1^{(0)}(1,0,s)$, $P_1^{(0)}(0,1,s)$, $P_1^{(1)}(1,0,s)$, $P_1^{(1)}(0,1,s)$, and $P_0^{(0)}(s)$Hence the generating function $\overline{P}(z,s)$ is completely known.

$\overline{P}_n(s)$ can be obtained by using the following formula

$$\overline{P}_n(s) = \frac{1}{n!}\frac{d^n}{dz^n}\overline{P}(z,s) \qquad \text{at } z=0$$

$P_n(t)$ can be obtained by inverting the Laplace transform $\overline{P}_n(s)$.

Further $\overline{P}(1,s) = 1/s$ and $\overline{P}(0,s) = \overline{P}_0^{(0)}(s)$, as desired

9. Special cases

(i) When there is no reneging i.e. M/M/2 feedback queueing system with unequal service rate.

Putting α=0 in (48) we get

$$\overline{P}(z,s) = \frac{1}{Q(z)}\Big[z(1-z)[\{K(z)'q - (\mu_1+\mu_2)p(c_2(q-z) - c_1pz)\}\Big\{\mu_1\overline{P}_1^{(0)}(1,0,s) + \mu_2\overline{P}_1^{(0)}(0,1,s)\Big\} + \{K(z)' - (\mu_1+\mu_2)p(c_2+c_1z)\}\Big\{\mu_1\overline{P}_1^{(1)}(1,0,s) + \mu_2\overline{P}_1^{(1)}(0,1,s)\Big\}] + (\mu_1+\mu_2)\{Lz - K(z)'(q+pz)\}\Big\{z\overline{P}_1^{(0)}(s) + \overline{P}_0^{(0)}(s)\Big\} + z(\mu_1+\mu_2)\{M\,z - K(z)'\}\overline{P}_1^{1}(s) + Lz\Big], \ \lambda < (\mu_1+\mu_2)\ ;\ |z| \le 1 \qquad \ldots(49)$$

where $K(z)' = -\lambda z^2 + (s+\lambda+\mu_1+\mu_2)z$

$$Q(z) = \{K(z)' - (\mu_1+\mu_2)c_2\}\{K(z)' - (\mu_1+\mu_2)c_1(q+pz)\} - (\mu_1+\mu_2)^2c_1c_2(q+pz)$$

$$L = K(z)' - (\mu_1+\mu_2)c_2p(1-z), \quad M = K(z)' + (\mu_1+\mu_2)c_1p(1-z)$$

(ii) M/M/2 feedback queueing system with equal service rate.

Putting $\mu_1 = \mu_2 = \mu$ in (49) we get

$$\overline{P}(z,s) = \frac{1}{Q(z)^\#}z\{K(z)^\# + 2\mu c_2p(z-1)\} + 2\mu(z-1)\{qK(z)^\# + 2\mu c_2p\}\overline{P}_0^{(0)}(s) + \mu z(z-1)[\{K(z)^\#q + 2\mu p(c_2q + c_2z - c_1pz)\}\overline{P}_1^{(0)}(s) + \{K(z)^\# + 2\mu p(c_2 - c_1z)]$$

where $K(z)^\# = -\lambda z^2 + (s+\lambda+2\mu)z$ and

$$Q(z)^\# = \{K(z)^\# - 2\mu c_2\}\{K(z)^\# - 2\mu c_1(q+pz)\} - 4\mu^2c_1c_2(q+pz)$$

This coincides with (3.32) of Kumari[4] for $K(z)^\# = \frac{F'^*}{2\mu}$.

(iii) When there is no feedback, no reneging with unequal service rate.

Putting q=1, p=0, $c_1 = 1$, $c_2 = 0$, $\overline{P}^{(1)}(z,s) = 0$, $\overline{P}^{(0)}(z,s) = \overline{P}(z,s)$,

$P_0^{(0)}(s) = P_0(s)$, and $P_n^{(0)}(s) = P_n(s)$, in(49), we get

$$\overline{P}(z,s) = \frac{z + z(z-1)\{-\mu_1\overline{P}_1(1,0,s) - \mu_2\overline{P}_1(0,1,s) + (\mu_1+\mu_2)\overline{P}_1(s)\} + (z-1)(\mu_1+\mu_2)\overline{P}_0(s)}{\{K(z)' - (\mu_1+\mu_2)\}},$$

$$,\lambda < (\mu_1+\mu_2)\ ;\ |z| \le 1 \quad \ldots(50)$$

(iv) When there is no feedback &no reneging with equal service rate.

Putting$\mu_1 = \mu_2 = \mu$ in (50), we get

$$\overline{P}(z,s) = \frac{z+z(z-1)\{z\mu\overline{P}_1(s)\}+2\mu\overline{P}_0(s)\}}{\{K(z)^{\#}-2\mu\}}, \lambda < 2\mu; \ |z| \le 1 \qquad \ldots(51)$$

where$K(z)^{\#} = -\lambda z^2 + (s+\lambda+2\mu)z$

(v) When there is one server and no feedback, no reneging.

Putting$\mu_1 = \mu, \mu_2 = 0$ in (50), we get

$$\overline{P}(z,s) = \frac{z+z(z-1)\mu\overline{P}_0(s)}{-\lambda z^2+(s+\lambda+\mu)z-\mu}, \lambda < \mu; \ |z| \le 1 \qquad \ldots(52)$$

This coincides with M/M/1 transient –state.

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