

**ON THE n -DIMENSIONAL SAIGO-MAEDA FRACTIONAL
CALCULUS ASSOCIATED WITH n -DIMENSIONAL
 \overline{H} TRANSFORMS**

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Abstract : In this paper, a relation between the n -dimensional \overline{H} -transform and the Weyl type n -dimensional Saigo-Maeda operator of fractional integration is established. Some interesting results associated with the H -functions and generalized Mittag-Leffler functions are deduced from the derived results.

Keywords : Saigo-Maeda operators, fractional calculus, Appell function F_3 , \overline{H} -function, Mittag-Leffler function, Laplace transform.

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1. Introduction

The object of this paper is to establish a theorem on n -dimensional Saigo-Maeda operators of Weyl type. The results obtained are general in nature and include as special cases, the results given earlier by Arora et al. [1], Saxena et al. [12, 13, 14], Saigo et al. [10], Chaurasia and Arya [3] and Chaurasia and Jain [4] etc.

The \overline{H} -function occurring in this paper was introduced by Inayat-Hussain [6] and studied by Buschman and Srivastava [2] and others. It will be defined and represented in the following manner:

$$\overline{H}_{P,Q}^{M,N} \left[z \mid \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right] = \frac{1}{2\pi i} \int_L \overline{\theta}(s) z^s ds \quad \dots (1)$$

where
$$\overline{\theta}(s) = \frac{\prod_{j=1}^M \Gamma(b_j - \beta_j s) \prod_{j=1}^N \left\{ \Gamma(1 - a_j + \alpha_j s) \right\}^{A_j}}{\prod_{j=M+1}^Q \left\{ \Gamma(1 - b_j + \beta_j s) \right\}^{B_j} \prod_{j=N+1}^P \Gamma(a_j - \alpha_j s)} \quad \dots (2)$$

The nature of contour L in (1) and various restrictions on the parameters of the \overline{H} -function can be seen in the paper cited earlier.

If $A_j (j = 1, 2, \dots, N)$ and $B_j (j = M + 1, \dots, Q)$ are equal to unity, the \overline{H} -function reduces to H-function.

The following sufficient conditions for the \overline{H} -function given by (1) have been given by Gupta et al. [5]:

(i) $|\arg(z)| < \frac{1}{2} \Omega \pi$ and $\Omega > 0$

(ii) $|\arg(z)| = \frac{1}{2} \Omega \pi$, $\Omega \geq 0$

and (a) $\zeta \neq 0$ and the contour L is so chosen that $(c\zeta + \delta + 1) < 0$

(b) $\zeta = 0$ and $(\delta + 1) < 0$

where
$$\Omega = \sum_{j=1}^M \beta_j + \sum_{j=1}^N A_j \alpha_j - \sum_{j=M+1}^Q B_j \beta_j - \sum_{j=N+1}^P \alpha_j$$

$$\zeta = \sum_{j=1}^N A_j \alpha_j + \sum_{j=N+1}^P \alpha_j - \sum_{j=1}^M \beta_j - \sum_{j=M+1}^Q B_j \beta_j$$

$$\delta = \operatorname{Re} \left(\sum_{j=1}^M b_j + \sum_{j=M+1}^Q b_j \beta_j - \sum_{j=1}^N a_j A_j - \sum_{j=N+1}^P a_j \right) + \frac{1}{2} \left(\sum_{j=1}^N A_j - \sum_{j=M+1}^Q B_j + P - M - N \right)$$

It may be noted that the conditions of validity given above are more general than those given earlier, see [2].

Let $\alpha, \alpha', \beta, \beta', \gamma \in C$ and $x > 0$ then the generalized fractional integral operators involving Appell function F_3 are defined by Saigo and Maeda [9, p. 393, Eq. (4.12)] by means of the following equations:

$$\begin{aligned} & \left(I_{0,x}^{\alpha, \alpha', \beta, \beta', \gamma} f \right)(x) \\ &= \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3 \left(\alpha, \alpha', \beta, \beta', \gamma; 1-\frac{t}{x}, 1-\frac{x}{t} \right) f(t) dt, \operatorname{Re}(\gamma) > 0 \\ &= \frac{d^k}{dx^k} I_{0,x}^{\alpha, \alpha', \beta+k, \beta', \gamma+k} f, \quad (\operatorname{Re}(\gamma) \leq 0, k = [-\operatorname{Re}(\gamma)] + 1) \end{aligned} \quad \dots (3)$$

$$\begin{aligned} & \text{and } \left(I_{x,\infty}^{\alpha, \alpha', \beta, \beta', \gamma} f \right)(x) \\ &= \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3 \left(\alpha, \alpha', \beta, \beta', \gamma; 1-\frac{x}{t}, 1-\frac{t}{x} \right) f(t) dt, \operatorname{Re}(\gamma) > 0 \\ &= (-1)^k \frac{d^k}{dx^k} I_{x,\infty}^{\alpha, \alpha', \beta, \beta'+k, \gamma+k} f, \quad (\operatorname{Re}(\gamma) \leq 0, k = [-\operatorname{Re}(\gamma)] + 1) \end{aligned} \quad \dots (4)$$

Here, the function $F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y)$ is the familiar Appell hypergeometric function of two variables defined by

$$F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n, \quad (|x| < 1, |y| < 1) \quad \dots (5)$$

The operators (3) and (4) reduce to the Saigo fractional integral operators [8] due to the following relations:

$$I_{0,x}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{0,x}^{\gamma,\alpha-\gamma,-\beta} f(x), \quad (\gamma \in C) \quad \dots (6)$$

$$I_{x,\infty}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{x,\infty}^{\gamma,\alpha-\gamma,-\beta} f(x), \quad (\gamma \in C) \quad \dots (7)$$

2. The n-Dimensional Saigo-Maeda Operator of Weyl Type

Following Miller [[7], p.82], we denote by u_1 the class of function $f(x_1)$ on R_+ which are infinitely differentiable with partial derivatives of any order behaving as $o(|x_1|^{-\xi_1})$ when x_1 tends to ∞ for all ξ_1 . Similarly by u_2 , we denote the class of function $f(x_1, x_2)$ on R_+^2 which are infinitely differentiable with partial derivatives of any order behaving as $o(|x_1|^{-\xi_1} |x_2|^{-\xi_2})$ when x_1 and x_2 both tends to ∞ for all $\xi_i (i = 1, 2)$.

In the same way, we denote the class of functions $f(x_1, x_2, \dots, x_n)$ on R_+^n , which are infinitely differentiable with partial derivatives of any order behaving as $o(|x_1|^{-\xi_1} |x_2|^{-\xi_2} \dots |x_n|^{-\xi_n})$ when $x_i \rightarrow \infty$ for all $\xi_i (i = 1, 2, \dots, n)$ by u_n .

The n-dimensional Saigo-Maeda operator of Weyl type fractional integration is defined in the class u_n by

$$\prod_{i=1}^n \left[\left[I_{p_i, \infty}^{\alpha_i, \alpha_i', \beta_i, \beta_i', \gamma_i} \right] \right] f(p_1, p_2, \dots, p_n) = \prod_{i=1}^n \frac{p_i^{\alpha_i - \gamma_i}}{\Gamma(\gamma_i)} \int_{p_1}^{\infty} \int_{p_2}^{\infty} \dots \int_{p_n}^{\infty} \prod_{i=1}^n \left\{ (t_i - p_i)^{\gamma_i - 1} (t_i)^{-\alpha_i} \right. \\ \left. \times F_3 \left(\alpha_i, \alpha_i', \beta_i, \beta_i', \gamma_i; 1 - \frac{p_i}{t_i}, 1 - \frac{t_i}{p_i} \right) \right\} f(t_1, t_2, \dots, t_n) dt_1 dt_2 \dots dt_n \quad \dots (8)$$

where $\text{Re}(\gamma_i) > 0$, for $i = 1, 2, \dots, n$.

Taking $\alpha_i' = 0, \forall i = 1, 2, \dots, n$ and using (6), the above equation yields the n -dimensional Saigo operator of Weyl type

$$\prod_{i=1}^n \left[\left(I_{p_i, \infty}^{\alpha_i, \beta_i, \eta_i} \right) \right] f(p_1, p_2, \dots, p_n)$$

$$= \prod_{i=1}^n \frac{p_i^{\beta_i}}{\Gamma(\alpha_i)} \int_{p_1}^{\infty} \int_{p_2}^{\infty} \dots \int_{p_n}^{\infty} \prod_{i=1}^n \left\{ (t_i - p_i)^{\alpha_i - 1} (t_i)^{-\alpha_i - \beta_i} \right.$$

$$\left. \times {}_2F_1 \left(\alpha_i + \beta_i, -\eta_i, \alpha_i; 1 - \frac{p_i}{t_i} \right) \right\} f(t_1, t_2, \dots, t_n) dt_1 dt_2 \dots dt_n$$

3. The n -Dimensional Laplace and \bar{H} -Transforms

The n -dimensional Laplace transform $\bar{f}(p_1, p_2, \dots, p_n)$ of a function $f(x_1, x_2, \dots, x_n) \in \mathcal{U}_n$ is defined as

$$\bar{f}(p_1, p_2, \dots, p_n) = L \left[f(x_1, x_2, \dots, x_n); p_1, p_2, \dots, p_n \right]$$

$$= \int_0^{\infty} \int_0^{\infty} \dots \int_0^{\infty} e^{-\sum_{i=1}^n p_i x_i} f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \quad \dots (9)$$

where $\text{Re}(p_i) > 0, i = 1, 2, \dots, n$.

Analogously, the n -dimensional Laplace transform of

$$f \left[u_1 \sqrt{(x_1^2 - \lambda_1^2)} H(x_1 - \lambda_1), u_2 \sqrt{(x_2^2 - \lambda_2^2)} H(x_2 - \lambda_2), \dots, u_n \sqrt{(x_n^2 - \lambda_n^2)} H(x_n - \lambda_n) \right]$$

is defined by the n -dimensional Laplace transform of $F(x_1, x_2, \dots, x_n)$, where

$$F(x_1, x_2, \dots, x_n)$$

$$= f \left[u_1 \sqrt{(x_1^2 - \lambda_1^2)} H(x_1 - \lambda_1), u_2 \sqrt{(x_2^2 - \lambda_2^2)} H(x_2 - \lambda_2), \dots, u_n \sqrt{(x_n^2 - \lambda_n^2)} H(x_n - \lambda_n) \right]$$

$(x_i > \lambda_i > 0; i = 1, 2, \dots, n)$ and $H(t)$ denotes the Heaviside's unit step function.

Definition 1: By n-dimensional \overline{H} -transforms $h(p_1, p_2, \dots, p_n)$ of a function $F(x_1, x_2, \dots, x_n)$, we mean the following repeated integral involving n-different \overline{H} -functions

$$\begin{aligned}
 h(p_1, p_2, \dots, p_n) &= \overline{H}_{P_1, Q_1; P_2, Q_2; \dots; P_n, Q_n}^{M_1, N_1; M_2, N_2; \dots; M_n, N_n} \left[F(x_1, x_2, \dots, x_n); \rho_1, \rho_2, \dots, \rho_n; P_1, P_2, \dots, P_n \right] \\
 &= \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \dots \int_{\lambda_n}^{\infty} \prod_{i=1}^n \left\{ (p_i x_i)^{\rho_i - 1} \overline{H}_{P_i, Q_i}^{M_i, N_i} \left[(p_i x_i)^{k_i} \left| \begin{matrix} (a_{ij}, \alpha_{ij}; A_{ij})_{1, N_i}, (a_{ij}, \alpha_{ij})_{N_i+1, P_i} \\ (b_{ij}, \beta_{ij})_{1, M_i}, (b_{ij}, \beta_{ij})_{M_i+1, Q_i} \end{matrix} \right. \right] \right\} \\
 &\quad \times F(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \quad \dots (10)
 \end{aligned}$$

Here, we assume that $\lambda_i > 0, k_i > 0, (i = 1, 2, \dots, n); h(p_1, p_2, \dots, p_n)$ exists and belongs to U_n . Further, let

$$\left| \arg (p_i^{k_i}) \right| < \frac{1}{2} \phi_i \pi, \quad i = 1, 2, \dots, n$$

where $\phi_i = \sum_{j=1}^{M_i} |\beta_j| + \sum_{j=1}^{N_i} A_j \alpha_j - \sum_{j=M_i+1}^{Q_i} |B_j \beta_j| - \sum_{j=N_i+1}^{P_i} \alpha_j > 0$, for $i = 1, 2, \dots, n$

4. Relationship Between n- Dimensional \overline{H} - Transforms in Terms of n-Dimensional Saigo-Maeda Operator of Weyl Type

To prove the theorem in this section, we need the following n- dimensional \overline{H} - transform $h_1(p_1, p_2, \dots, p_n)$ of $F(x_1, x_2, \dots, x_n)$

$$\begin{aligned}
 h_1(p_1, p_2, \dots, p_n) &= \overline{H}_{P_1+3, Q_1+3; P_2+3, Q_2+3; \dots; P_n+3, Q_n+3}^{M_1+3, N_1; M_2+3, N_2; \dots; M_n+3, N_n} \\
 &\quad \left[F(x_1, x_2, \dots, x_n); \rho_1, \rho_2, \dots, \rho_n; P_1, P_2, \dots, P_n \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \dots \int_{\lambda_n}^{\infty} \prod_{i=1}^n \left\{ (p_i x_i)^{\rho_i - 1} \frac{1}{H_{P_i+3, Q_i+3}} \left[(p_i x_i)^{k_i} \middle|_{(1-\beta_i-\rho_i, k_i), (1+\alpha_i+\alpha'_i-\gamma_i-\rho_i, k_i),} \right. \right. \\
 &\quad \left. \left. (1+\alpha_i+\alpha'_i+\beta'_i-\gamma_i-\rho_i, k_i), (1+\alpha_i-\beta_i-\rho_i, k_i), (1-\rho_i, k_i) \right] \right\} F(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \\
 &\quad \left. (1+\alpha_i+\beta'_i-\gamma_i-\rho_i, k_i), (b_{ij}, \beta_{ij})_{1, M_i}, (b_{ij}, \beta_{ij}; B_{ij})_{M_i+1, Q_i} \right\} \dots (11)
 \end{aligned}$$

where it is assumed that $h_1(p_1, p_2, \dots, p_n)$ exists and belongs to \mathcal{U}_n and $k_i > 0, (i = 1, 2, \dots, n)$ and other conditions on the parameters in which additional parameters $\alpha_i, \alpha'_i, \beta_i, \beta'_i$ and $\gamma_i; i = 1, 2, \dots, n$ included correspond to those in (8).

Theorem 1. Let $h(p_1, p_2, \dots, p_n)$ be given by (10), then for

$\text{Re}(\alpha_i) > 0, \text{Re}(\gamma_i) > 0, \lambda_i > 0, k_i > 0; i = 1, 2, \dots, n$, there holds the formula

$$\prod_{i=1}^n \left[I_{p_i, \infty}^{\alpha_i, \alpha'_i, \beta_i, \beta'_i, \gamma_i} \right] h(p_1, p_2, \dots, p_n) = h_1(p_1, p_2, \dots, p_n) \dots (12)$$

provided that $h_1(p_1, p_2, \dots, p_n)$ exists and belong to \mathcal{U}_n .

Proof. Let $\text{Re}(\alpha_i) > 0, \text{Re}(\gamma_i) > 0, \forall i = 1, 2, \dots, n$, then in view of (8) and (10), we find that

$$\begin{aligned}
 &\prod_{i=1}^n \left[I_{p_i, \infty}^{\alpha_i, \alpha'_i, \beta_i, \beta'_i, \gamma_i} \right] h(p_1, p_2, \dots, p_n) \\
 &= \prod_{i=1}^n \frac{p_i^{\alpha_i - \gamma_i}}{\Gamma(\gamma_i)} \int_{p_1}^{\infty} \int_{p_2}^{\infty} \dots \int_{p_n}^{\infty} \prod_{i=1}^n \left[(t_i - p_i)^{\gamma_i - 1} (t_i)^{-\alpha_i} F_3 \left(\alpha_i, \alpha'_i, \beta_i, \beta'_i, \gamma_i; 1 - \frac{p_i}{t_i}, 1 - \frac{t_i}{p_i} \right) \right] \\
 &\quad \times h(t_1, t_2, \dots, t_n) dt_1 dt_2 \dots dt_n \\
 &= \prod_{i=1}^n \frac{p_i^{\alpha_i - \gamma_i}}{\Gamma(\gamma_i)} \int_{p_1}^{\infty} \int_{p_2}^{\infty} \dots \int_{p_n}^{\infty} \prod_{i=1}^n \left[(t_i - p_i)^{\gamma_i - 1} (t_i)^{-\alpha_i} F_3 \left(\alpha_i, \alpha'_i, \beta_i, \beta'_i, \gamma_i; 1 - \frac{p_i}{t_i}, 1 - \frac{t_i}{p_i} \right) \right] \\
 &\quad \times \left\{ \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \dots \int_{\lambda_n}^{\infty} \prod_{i=1}^n \left[(t_i x_i)^{\rho_i - 1} \frac{1}{H_{P_i, Q_i}} \left[(t_i x_i)^{k_i} \middle|_{(b_{ij}, \beta_{ij})_{1, M_i}, (b_{ij}, \beta_{ij}; B_{ij})_{M_i+1, Q_i}} \right. \right. \right. \\
 &\quad \left. \left. \left. (a_{ij}, \alpha_{ij}; A_{ij})_{1, N_i}, (a_{ij}, \alpha_{ij})_{N_i+1, P_i} \right] \right] \right\} dt_1 dt_2 \dots dt_n \\
 &\quad \times F(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n
 \end{aligned}$$

On interchanging the order of integration which is permissible, we have

$$\begin{aligned}
 &= \prod_{i=1}^n \frac{p_i^{\alpha_i - \gamma_i}}{\Gamma(\gamma_i)} \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \dots \int_{\lambda_n}^{\infty} \prod_{i=1}^n \left[(x_i)^{\rho_i - 1} \right] \left\{ \int_{p_1}^{\infty} \int_{p_2}^{\infty} \dots \int_{p_n}^{\infty} \prod_{i=1}^n \left[(t_i - p_i)^{\gamma_i - 1} (t_i)^{-\alpha_i + \rho_i - 1} \right. \right. \\
 &\times \overline{H}_{P_i, Q_i}^{M_i, N_i} \left[(t_i x_i)^{k_i} \left| \begin{matrix} (a_{ij}, \alpha_{ij}; A_{ij})_{1, N_i}, (a_{ij}, \alpha_{ij})_{N_i + 1, P_i} \\ (b_{ij}, \beta_{ij})_{1, M_i}, (b_{ij}, \beta_{ij}; B_{ij})_{M_i + 1, Q_i} \end{matrix} \right. \right] \\
 &\times F_3 \left(\alpha_i, \alpha'_i, \beta_i, \beta'_i, \gamma_i; 1 - \frac{p_i}{t_i}, 1 - \frac{t_i}{p_i} \right) dt_1 dt_2 \dots dt_n \left. \right\} F(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \quad \dots (13)
 \end{aligned}$$

Now, we evaluate t_1, t_2, \dots, t_n integrals occurring in equation (13) by using the following integral formula

$$\begin{aligned}
 &\int_p^{\infty} t^{-\alpha + \rho - 1} (t - p)^{\gamma - 1} F_3 \left(\alpha, \alpha', \beta, \beta', \gamma; 1 - \frac{p}{t}, 1 - \frac{t}{p} \right) \\
 &\overline{H}_{P, Q}^{M, N} \left[(tx)^k \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1, N}, (a_j, \alpha_j)_{N + 1, P} \\ (b_j, \beta_j)_{1, M}, (b_j, \beta_j; B_j)_{M + 1, Q} \end{matrix} \right. \right] dt \\
 &= \Gamma(\gamma) p^{\rho + \gamma - \alpha - 1} \overline{H}_{P + 3, Q + 3}^{M + 3, N} \left[(px)^k \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1, N}, (a_j, \alpha_j)_{N + 1, P}, \\ (1 - \beta - \rho, k), (1 + \alpha + \alpha' - \gamma - \rho, k), \\ (1 + \alpha + \alpha' + \beta' - \gamma - \rho, k), (1 + \alpha - \beta - \rho, k), (1 - \rho, k) \\ (1 + \alpha + \beta' - \gamma - \rho, k), (b_j, \beta_j)_{1, M}, (b_j, \beta_j; B_j)_{M + 1, Q} \end{matrix} \right. \right]
 \end{aligned}$$

provided that

$$\begin{aligned}
 &\operatorname{Re}(\gamma) > 0, \operatorname{Re}(\rho) < 1 + k \max_{1 \leq j \leq N} \operatorname{Re}\{(1 - a_j) / \alpha_j\} \\
 &+ \min \left[\operatorname{Re}(-\beta), \operatorname{Re}(\alpha + \alpha' - \gamma), \operatorname{Re}(\alpha + \beta' - \gamma) \right] \\
 &|\arg z| < \frac{1}{2} \Omega \pi \text{ (where } \Omega \text{ is given in the definition of } \overline{H} \text{-function).}
 \end{aligned}$$

Finally, interpreting the above result in equation (13), we get

$$\begin{aligned}
 &= \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \dots \int_{\lambda_n}^{\infty} \prod_{i=1}^n \left\{ (p_i x_i)^{\rho_i - 1} \overline{H}_{P_i + 3, Q_i + 3} \left[(p_i x_i)^{k_i} \left| \begin{matrix} (a_{ij}, \alpha_{ij}; A_{ij})_{1, N_i} \\ (1 - \beta_i - \rho_i, k_i) \end{matrix} \right. \right. \right. \\
 &\quad \left. \left. \left. (a_{ij}, \alpha_{ij})_{N_i + 1, P_i}, (1 + \alpha_i + \alpha'_i + \beta'_i - \gamma_i - \rho_i, k_i), \right. \right. \right. \\
 &\quad \left. \left. \left. (1 + \alpha_i + \alpha'_i - \gamma_i - \rho_i, k_i), (1 + \alpha_i + \beta'_i - \gamma_i - \rho_i, k_i), \right. \right. \right. \\
 &\quad \left. \left. \left. (1 + \alpha_i - \beta_i - \rho_i, k_i), (1 - \rho_i, k_i) \right. \right. \right. \\
 &\quad \left. \left. \left. (b_{ij}, \beta_{ij})_{1, M_i}, (b_{ij}, \beta_{ij}; B_{ij})_{M_i + 1, Q_i} \right. \right. \right. \left. \right\} F(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \\
 &= \overline{H}_{M_1 + 3, N_1; M_2 + 3, N_2; \dots; M_n + 3, N_n} \\
 &\quad \left[F(x_1, x_2, \dots, x_n); \rho_1, \rho_2, \dots, \rho_n; P_1, P_2, \dots, P_n \right] \\
 &= h_1(p_1, p_2, \dots, p_n) \quad \text{(by (13))}
 \end{aligned}$$

5. Special cases

If we put $A_{ij} = B_{ij} = 1, \forall i = 1, 2, \dots, n$ in equation (10), then we see that n-dimensional \overline{H} -transform reduces to another n-dimensional H-transform defined as

$$\begin{aligned}
 \hat{H}(p_1, p_2, \dots, p_n) &= H_{P_1, Q_1; P_2, Q_2; \dots; P_n, Q_n} \left[F(x_1, x_2, \dots, x_n); \rho_1, \rho_2, \dots, \rho_n; p_1, p_2, \dots, p_n \right] \\
 &= \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \dots \int_{\lambda_n}^{\infty} \prod_{i=1}^n \left\{ (p_i x_i)^{\rho_i - 1} \overline{H}_{P_i, Q_i} \left[(p_i x_i)^{k_i} \left| \begin{matrix} (a_{ij}, \alpha_{ij})_{1, P_i} \\ (b_{ij}, \beta_{ij})_{1, Q_i} \end{matrix} \right. \right. \right. \\
 &\quad \left. \left. \left. F(x_1, x_2, \dots, x_n) \right. \right. \right. \\
 &\quad \left. \left. \left. \dots \right. \right. \right.
 \end{aligned} \tag{14}$$

Corollary 1. Let $\hat{H}(p_1, p_2, \dots, p_n)$ be given by (14), and for $\Re(\alpha_i) > 0, \lambda_i > 0, k_i > 0; i = 1, 2, \dots, n$, then there holds the formula

$$\prod_{i=1}^n \left[I_{p_i}^{\alpha_i, \alpha'_i, \beta_i, \beta'_i, \gamma_i} \right] \hat{H}(p_1, p_2, \dots, p_n) = \hat{H}_1(p_1, p_2, \dots, p_n) \quad \dots \tag{15}$$

provided that $\hat{H}_1(p_1, p_2, \dots, p_n)$ exists and belongs to \mathcal{U}_n , where

$$\hat{H}_1(p_1, p_2, \dots, p_n) = \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \dots \int_{\lambda_n}^{\infty} \prod_{i=1}^n \left\{ (p_i x_i)^{\rho_i - 1} H_{P_i+3, Q_i+3}^{M_i+3, N_i} \left[(p_i x_i)^{k_i} \right] \right. \\ \left. (a_{ij}, \alpha_{ij})_{1, P_i}, (1 - \rho_i, k_i), \right. \\ \left. (1 - \beta_i - \rho_i, k_i), (1 + \alpha_i + \alpha'_i - \gamma_i - \rho_i, k_i), \right. \\ \left. (1 + \alpha_i + \alpha'_i + \beta'_i - \gamma_i - \rho_i, k_i), (1 + \alpha_i - \beta_i - \rho_i, k_i) \right] \\ \left. (1 + \alpha_i + \beta'_i - \gamma_i - \rho_i, k_i), (b_{ij}, \beta_{ij})_{1, Q_i} \right] F(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n$$

We now deduce the results for the n-dimensional Mittag-Leffler function transform from the above corollary 1.

Definition 2. The generalized Mittag-Leffler function introduced and studied by Salim and Faraj [11] is defined as

$$E_{\sigma, \zeta, \tau}^{\delta, \mu, \nu}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{\nu n}}{\Gamma(\sigma n + \zeta)} \frac{z^n}{(\mu)_{\tau n}} \quad \dots (16)$$

where $\sigma, \zeta, \delta, \mu \in C; \min. \{\Re(\sigma), \Re(\zeta), \Re(\delta), \Re(\mu)\} > 0; \tau, \nu > 0$ and $\nu \leq \tau + \Re(\sigma)$.

Its relation with the H-function is obtained by Salim and Faraj [11] in the following form:

$$E_{\sigma, \zeta, \tau}^{\delta, \mu, \nu}(z) = \frac{\Gamma(\mu)}{\Gamma(\delta)} H_{2,3}^{1,2} \left[-z \middle| \begin{matrix} (0,1), (1-\delta, \nu) \\ (0,1), (1-\zeta, \sigma), (1-\mu, \tau) \end{matrix} \right] \quad \dots (17)$$

Definition 3. By n-dimensional Mittag-Leffler function $\hat{E}(p_1, p_2, \dots, p_n)$ of a function $F(x_1, x_2, \dots, x_n)$, we mean the following repeated integral involving n-different Mittag-Leffler functions.

$$\begin{aligned} \hat{E}(p_1, p_2, \dots, p_n) &= E_{\sigma_1, \varsigma_1, \tau_1; \sigma_2, \varsigma_2, \tau_2; \dots; \sigma_n, \varsigma_n, \tau_n}^{\delta_1, \mu_1, \nu_1; \delta_2, \mu_2, \nu_2; \dots; \delta_n, \mu_n, \nu_n} \\ &\left[F(x_1, x_2, \dots, x_n); \rho_1, \rho_2, \dots, \rho_n; p_1, p_2, \dots, p_n \right] \\ &= \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \dots \int_{\lambda_n}^{\infty} \prod_{i=1}^n \left\{ (p_i x_i)^{\rho_i - 1} E_{\sigma_i, \varsigma_i, \tau_i}^{\delta_i, \mu_i, \nu_i} \left[(p_i x_i)^{k_i} \right] \right\} F(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \dots \end{aligned} \quad (18)$$

where $\sigma_i, \varsigma_i, \delta_i, \mu_i \in C; \min\{\Re(\sigma_i), \Re(\varsigma_i), \Re(\delta_i), \Re(\mu_i)\} > 0; \tau_i, \nu_i, \lambda_i > 0$ and $\nu_i \leq \tau_i + \Re(\sigma_i)$

$\forall i = 1, 2, \dots, n, \hat{E}(p_1, p_2, \dots, p_n)$ exists and belongs to \mathcal{U}_n .

If we use the identity (17) and make suitable changes in the parameters, then n-dimensional H-transform reduces to n-dimensional Mittag-Leffler transform and from Corollary 1, we arrive at the following:

Corollary 2. Let $\hat{E}(p_1, p_2, \dots, p_n)$ be given by (18), then for $\Re(\alpha_i) > 0, \Re(\gamma_i) > 0, \lambda_i > 0, \forall i = 1, 2, \dots, n$, then there holds the formula

$$\prod_{i=1}^n \left[I_{p_i}^{\alpha_i, \alpha'_i, \beta_i, \beta'_i, \gamma_i} \right] \hat{E}(p_1, p_2, \dots, p_n) = \hat{E}_1(p_1, p_2, \dots, p_n) \quad \dots (19)$$

provided that $\hat{E}_1(p_1, p_2, \dots, p_n)$ exists and belongs to \mathcal{U}_n , where

$$\begin{aligned} \hat{E}_1(p_1, p_2, \dots, p_n) &= \frac{\Gamma(\mu_1)}{\Gamma(\delta_1)} \dots \frac{\Gamma(\mu_n)}{\Gamma(\delta_n)} \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \dots \int_{\lambda_n}^{\infty} \prod_{i=1}^n \left\{ (p_i x_i)^{\rho_i - 1} \right. \\ &\left. \begin{aligned} &H_{5, 6}^{4, 2} \left((0, 1), (1 - \rho_i, k_i) \right. \\ &\left. - (p_i x_i)^{k_i} \left| (1 - \beta_i - \rho_i, k_i), (1 + \alpha_i + \alpha'_i - \gamma_i - \rho_i, k_i), \right. \right. \\ &\left. \left. (1 + \alpha_i + \alpha'_i + \beta'_i - \gamma_i - \rho_i, k_i), (1 + \alpha_i - \beta_i - \rho_i, k_i) \right) \right\} \\ &\left. (1 + \alpha_i + \beta'_i - \gamma_i - \rho_i, k_i), (0, 1), (1 - \varsigma_i, k_i), (1 - \mu_i, \tau_i) \right] F(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \end{aligned} \end{aligned}$$

6. Known Results

- (i) If we put $n = 2$ in Theorem 1, then we get known result due to Saxena et al. [12, p. 174, Eq. (4.2)].
- (ii) If we put $n = 2$ in Corollary 1, then we get known result obtained by Saxena et al. [14].
- (iii) If we reduce Saigo-Maeda operator to Saigo operator in Theorem 1, then we get known result obtained by Chaurasia and Arya [3, p.10, Eq. (14)].
- (iv) If we reduce \overline{H} -function to H-function and put $n = 2$ in equation (15), we get known result obtained by Saigo et al. [10, p.67].
- (v) If we reduce Saigo-Maeda operator to Saigo operator and $n = 3$ in Theorem 1 then we get known result obtained by Chaurasia and Jain [4, p.62, Eq. (4.1)].
- (vi) If we put $n = 2$ and $\mu_i = \tau_i = 1, \forall i = 1, 2$ in Corollary 2, we get known result obtained by Saxena et al. [13, p. 275, Eq. (36)].

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References

- [1] Arora, A. K., Raina, R. K. and Koul, C. L. (1985). On the two dimensional Weyl fractional calculus associated with the Laplace transforms, *C.R. Acad. Bulg. Sci.* **38**, 179-182.
- [2] Buschman, R. G. and Srivastava, H. M. (1990). The H-function associated with a certain class of Feynman integrals, *J. Phys. A: Math. Gen.*, **23**, 4707-4710.
- [3] Chaurasia, V. B. L. and Arya, J. C. (2012). On the n -Dimensional Generalized Weyl Fractional Calculus Associated with n -Dimensional \overline{H} -Transform, *IOSR J. of Mathematics*, **1(6)**, 7-11.
- [4] Chaurasia, V. B. L. and Jain M. (2010). Three Dimensional Generalized Weyl Fractional Calculus Pertaining to Three Dimensional \overline{H} -Transform, *Scientia, Series A: Mathematical Sciences*, **19**, 57-68.
- [5] Gupta, K. C., Jain, R. and Agarwal, R. (2007). On existence conditions for generalized Mellin-Barnes type integral, *Natl. Acad. Sci. Lett.*, **30** (5&6), 169-172.
- [6] Inayat-Hussain, A. A. (1987). New Properties of Hypergeometric Series Derivable From Feynman Integrals: II, A Generalization of the H-Function, *J. Phys. A: Math. Gen.*, **20**, 4119-4128.
- [7] Miller, K. S. (1975) The Weyl Fractional Calculus, Fractional Calculus and its Applications, Lecture Notes in Math., **457**, Springer, Berlin-Heidelberg-New York, 80-89.
- [8] Saigo, M. (1978). A Remark on Integral Operators Involving the Gauss Hypergeometric Functions, *Math. Rep., College General Ed. Kyushu Univ.* **11**, 135-143.

- [9] Saigo, M. and Maeda, N. (1996). More Generalization of Fractional Calculus, Transform Methods and Special Functions, Varna, Bulgaria, 386-400.
- [10] Saigo, M., Saxena, R. K. and Ram, J. (1995). On the Two-Dimensional Generalized Weyl Fractional Calculus Associated with Two Dimensional H-Transforms, *Journal of Frac. Cal.*, **8**, 63-73.
- [11] Salim, T. O. and Faraj A.W. (2012). A generalization of Mittag-Leffler function and integral operator associated with fractional calculus, *J. of Fract. Calc. and Appl.*, **3**, 1-13.
- [12] Saxena, R. K., Ram, J. and Chandak, S. (2005). On two dimensional generalized Saigo fractional calculus associated with two dimensional generalized H-transforms, *J. of Indian Acad. Math.*, **27(1)**, 167-180.
- [13] Saxena, R.K., Ram, J. and Kumar, D. (2013). On the Two-Dimensional Saigo-Maeda Fractional Calculus Associated with Two-Dimensional Aleph Transform, *LE Mathematiche*, **28 (2)**, 267-281.
- [14] Saxena, R. K., Ram, J. and Suthar, D. L. (2004). On two dimensional generalized Saigo-Maeda fractional calculus involving two dimensional H-transforms, *Acta Ciencia Indica*, **30(4)**, 813-822.