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FEKETE-SZEGO PROBLEMS AND COEFFICIENT ESTIMATES OF QUASI-SUBORDINATION CLASSES

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Abstract: In the present paper, we defined certain subclasses of analytic univalent functions associated with quasi-subordination and the bounds for the Fekete-Szego coefficient functional $|a_3 - \mu a_2^2|$ for functions belonging to the subclasses are derived. We also obtain the coefficient estimates of $|a_2|$ and $|a_3|$.

Keywords : Univalent functions, quasi-subordination, Fekete-Szego coefficient, coefficient estimates.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by the following Taylor-Maclaurin series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \; (z \in \mathbb{D}) \qquad \ldots(1)$$

which are analytic in the open unit disk

$$\mathbb{D} = \{z : z \in \mathbb{C}; |z| < 1\}$$

Some of the important and well-investigated subclasses of univalent function class involves the class $S^*(\alpha)$ of starlike functions of order α and $K(\alpha)$ of convex functions of order α in $\mathbb{D}$. By definition, we have

$$S^*(\alpha)=\left\{f : f\in\mathcal{A}\text{ and }\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\quad (z\in\mathbb{D} : 0\leq\alpha<1)\right\} \qquad \ldots (2)$$

and

$$K(\alpha)=\left\{f : f\in\mathcal{A}\text{ and }\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha\quad (z\in\mathbb{D} : 0\leq\alpha<1)\right\}. \qquad \ldots (3)$$

Obviously

$f(z)\in K(\alpha)\Leftrightarrow zf'(z)\in S^*(\alpha).$

An analytic function $f(z)$ is subordinate to an analytic function $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{D}$ satisfying $w(0)=0$, $|w(z)|<1 (z\in\mathbb{D})$ and $f(z)=g(w(z))$. We denote this subordination by

$$f(z)\prec g(z)\quad (z\in\mathbb{D})\qquad \text{[cf. 2, p. 226].} \qquad \ldots (4)$$

Further, a function $f(z)$ is said to be quasi-subordinate to $g(z)$ in the open unit disk $\mathbb{D}$ if there exists an analytic function $\varphi(z)$ such that

$f(z)/\varphi(z)$ is analytic in $\mathbb{D}$,

$$\frac{f(z)}{\varphi(z)}\prec g(z)\quad (z\in\mathbb{D}) \qquad \ldots (5)$$

and $|\varphi(z)|\leq 1 \quad (z\in\mathbb{D})$. We also denote this quasi-subordination by

$$f(z)\prec_q g(z)\quad (z\in\mathbb{D}). \qquad \ldots (6)$$

Note that the quasi-subordination (6) is equivalent to

$$f(z)=\varphi(z)g(w(z))\quad (z\in\mathbb{D}) \qquad \ldots (7)$$

where $|\varphi(z)|\leq 1 \ (z\in\mathbb{D})$.

In the quasi-subordination (7), if $\varphi(z) \equiv 1$, then it becomes the subordination (4). For analytic functions $f(z)$ and $g(z)$ in $\mathbb{D}$, we say $f(z)$ is majorized by $g(z)$ and write $f(z) \ll g(z) \quad (z \in \mathbb{D})$, if there exists an analytic function $\varphi(z)$ in $\mathbb{D}$ satisfying $|\varphi(z)| \leq 1$ and $f(z) = \varphi(z)g(z)\ (z \in \mathbb{D})$. If we take $w(z) = z$ in (7), then quasi-subordination (6) becomes the majorization. Hence quasi-subordination is a generalization of subordination as well as majorization. (See [3,4,10,11] for works related to quasi-subordination).

Subsequently, Brannan et al. [1] conjuctured that

$$|a_2| < \sqrt{2}$$

Netanyahu [8], on the other hand, showed for class Σ of biunivalent functions

$$\max_{f \in \Sigma} |a_2| = 4/3 .$$

The coefficient estimate problem for each of the following Taylor Maclaurin coefficients $|a_n| \quad \left(n \in \mathbb{N} \setminus \{1,2,3\} : \mathbb{N} = \{1,2,3,4,...\}\right)$ $ is presently still an open problem.

Throughout this paper it is assumed that h is analytic in $\mathbb{D}$ with $h(0) = 1$ and let these functions are of the form:

$$\varphi(z) = A_0 + A_1 z + A_2 z^2 + ..., \qquad ... (8)$$

$$h(z) = 1 + B_1 z + B_2 z^2 + ..., \quad B_1 > 0, B_1 \in \mathbb{R}^+. \qquad ... (9)$$

Motivated by earlier works ([3], [7]) on quasi-subordination we define the following classes:

Definition 1. *A function f given by (1) is said to be in the class $\mathcal{M}^q_{\gamma,\tau}(h)$ $(0 \leq \gamma < 1, \tau \in \mathbb{C} / \{0\})$ if the following quasi-subordination holds:*

$$\frac{1}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right) \prec_q h(z) - 1 \qquad ... (10)$$

Definition 2. *A function f given by (1) is said to be in the class $\mathcal{N}^{q}_{\beta}(h)$ $(\beta \geq 0)$, if it satisfies the following condition*

$$\left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^{\beta} -1 \prec_q h(z)-1 \qquad \ldots (11)$$

In this paper, the coefficient bounds of $|a_2|$ and $|a_3|$ and Fekete-Szego coefficient functional $|a_3-\mu a_2^2|$ for functions in the classes $\mathcal{M}^{q}_{\gamma,\tau}(h)$ and $\mathcal{N}^{q}_{\beta}(h)$ are obtained .

Lemma 1.1[5,6 *] If $p(z)=1+c_1z+c_2z^2+\ldots\in\mathcal{P}$, then for any complex number μ*

$$|c_2-\mu c_1^2|\leq 2max.\{1;|2v-1|\} \qquad \ldots (12)$$

Lemma 1.2[9] *If $p(z)=1+c_1z+c_2z^2+\ldots\in\mathcal{P}$ is an analytic function in $\mathbb{D}$ with positive real part, then*

$$|c_n|\leq 2, \qquad n\in\mathcal{N} \qquad \ldots (13)$$

Main Results

Theorem 1. *If f given by (1) be in the class $\mathcal{M}^{q}_{\gamma,\tau}(h)$, then*

$$|a_3-\mu a_2^2|\leq\frac{|\tau|B_1}{3(1+2\gamma)}\left[1+max\left\{1,\frac{3}{4}B_1|\mu\tau|\frac{(1+2\gamma)}{(1+\gamma)^2}+\left|\frac{B_2}{B_1}\right|\right\}\right],(\mu\in\mathbb{C}) \quad \ldots (14)$$

$$|a_2|\leq\frac{|\tau|B_1}{2(1+\gamma)}, \qquad \ldots (15)$$

and

$$|a_3|\leq\frac{|\tau|B_1}{3(1+2\gamma)}\left[1+max\left\{1,\left|\frac{B_2}{B_1}\right|\right\}\right] \qquad \ldots (16)$$

Proof. Let $f\in M^{q}_{\gamma,\tau}(h)$, then there exists analytic functions φ in $\mathbb{D}$ with $|\varphi(z)|\leq 1$ and $u:\mathbb{D}\to\mathbb{D}$ with $u(0)=0$ and $|u(z)|<1$ such that

$$\frac{1}{\tau}\left(f'(z)+\gamma zf''(z)-1\right)=\varphi(z)\left(h\left(u(z)\right)-1\right) \qquad \ldots (17)$$

Define the function p by

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+c_1z+c_2z^2+\ldots \qquad \ldots (18)$$

or equivalently

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left[c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\ldots\right] \qquad \ldots (19)$$

Then p is analytic in $\mathbb{D}$ with $\mathrm{p}(0)=1.$ Since $u:\mathbb{D}\to\mathbb{D}$, the functions p has a positive real part in $\mathbb{D}$, and $|c_i|\le 2(i=1,2)$. In view of (17)-(19), and we have

$$\frac{1}{\tau}\left(f'(z)+\gamma zf''(z)-1\right)=\varphi(z)\left(h\left(\frac{p(z)-1}{p(z)+1}\right)-1\right) \qquad \ldots (20)$$

Now using (19) together with (8) and (9), it is evident that

$$\varphi(z)\left[h\left(\frac{p(z)-1}{p(z)+1}\right)-1\right]=\frac{1}{2}A_0B_1c_1z+\left\{\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2}{4}c_1^2A_0\right\}z^2+\ldots$$

$$\ldots (21)$$

$$\frac{1}{\tau}\left(f'(z)+\gamma zf''(z)-1\right)=\frac{1}{\tau}\left[2a_2(1+\gamma)z+3a_3(1+2\gamma)z^2+\ldots\right] \qquad \ldots (22)$$

It follows from (20), (21) and (22) that

$$2a_2\left(\frac{1+\gamma}{\tau}\right)=\frac{1}{2}A_0B_1C_1 \qquad \ldots (23)$$

and

$$3a_3\left(\frac{1+2\gamma}{\tau}\right)=\frac{1}{2}A_1B_1C_1+\frac{1}{2}B_1A_0\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2}{4}c_1^2A_0 \qquad \ldots (24)$$

Since $\varphi(z)$ is analytic and bounded in $\mathbb{D}$, we have [9, page 172].

$$|A_n|\le 1-|A_0|^2\le 1 \quad (n>0) \qquad \ldots (25)$$

and using the well-known inequalities $|c_i| \le 2 \ (i = 1,2)$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (15).

Further,

$$a_3 - \mu a_2^2 = \frac{\tau}{6(1+2\gamma)}\left[A_1B_1C_1 + A_0B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2}{2}A_0c_1^2\right] - \frac{\mu}{16}\frac{\tau^2}{(1+\gamma)^2}A_0^2B_1^2c_1^2$$

which implies

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{6(1+2\gamma)}\left[\left|A_1B_1C_1\right| + \left|A_0B_1\right|\left|c_2 - \left(\frac{1}{2} - \frac{B_2}{2B_1} + \frac{3\mu}{8}\frac{\tau(1+2\gamma)}{(1+\gamma)^2}A_0B_1\right)c_1^2\right|\right]$$

Now using the inequalities $|c_i| \le 2 \ (i = 1,2)$ and (25), we get

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau| B_1}{3(1+2\gamma)}\left[1 + \frac{1}{2}\left|c_2 - \left(\frac{3\mu}{8}\frac{\tau(1+2\gamma)}{(1+\gamma)^2}A_0B_1 + \frac{1}{2}\left(1 - \frac{B_2}{B_1}\right)\right)c_1^2\right|\right] \quad \ldots(26)$$

By using the estimate (12)

the inequaity (26) reduces to the desired Fekete-Szego inequality (14).

If we put $\mu = 0$ in above inequality, we get desired estimate for $|a_3|$ as asserted in (16).

Corollary 2.1. *For the class of strongly starlike function*

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \ldots (0 < \alpha \le 1)$$

the inequalities (15) and (16) reduces to

$$|a_2| \le \frac{|\tau|\alpha}{(1+\gamma)}, \qquad |a_3| \le \frac{4|\tau|\alpha}{3(1+2\gamma)}$$

and for $\varphi(z) = 1$ *and*

$$h(z) = \frac{1+(1-2\gamma)z}{1-z} = 1 + 2(1-\gamma)z + 2(1-\gamma)z^2 + \ldots$$

which gives $B_1 = B_2 = 2(1-\alpha)$, *then estimates for* $|a_2|$ *and* $|a_3|$ *reduces to*

$$|a_2| \le \frac{|\tau|(1-\alpha)}{(1+\gamma)}, \qquad |a_3| \le \frac{2|\tau|(1-\alpha)}{3(1+2\gamma)} \qquad \ldots (27)$$

Corollary 2.2 *For* $\varphi(z)=1$ *and* $\tau=1, \gamma=0$, *we get the estimate for* $|a_2|$ *and* $|a_3|$ *as*

$$|a_2| \le \frac{B_1}{2}, \qquad |a_3| \le \frac{B_1}{3} max.\left\{1, \frac{|B_2|}{B_1}\right\} \qquad \ldots (28)$$

Theorem 2. *If f given by (1) be in the class* $\mathcal{N}_\beta^q(h)$ $(\beta \ge 0)$*, then*

$$|a_3-\mu a_2^2| \le \frac{B_1}{(\beta+2)}\left[1+max.\left\{1; \frac{(\beta+2)B_1}{(\beta+1)^2}\left|\frac{\beta-1}{2}+\mu\right| + \frac{|B_2|}{B_1}\right\}\right], (\mu \in \mathbb{C}) \ldots (29)$$

$$|a_2| \le \frac{B_1}{(\beta+1)} \qquad \ldots (30)$$

and $$|a_3| \le \frac{B_1}{(\beta+2)}\left[1+max\left\{1; \frac{(\beta+2)B_1}{(\beta+1)^2}\frac{|\beta-1|}{2} + \frac{|B_2|}{B_1}\right\}\right] \qquad \ldots (31)$$

Proof. Let $f \in \mathcal{N}_\beta^q(h)$, then there exists analytic function φ in $\mathbb{D}$ with $|\varphi(z)| \le 1$ and $u : \mathbb{D} \to \mathbb{D}$, with $u(0)=0$ such that

$$\left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^\beta - 1 = \varphi(z)\big(h(u(z)-1\big) \qquad \ldots (32)$$

If we define the function $p(z)$ by (18) (or (19), then we have

$$\left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^\beta = \phi(z)\left(h\left(\frac{p(z)-1}{p(z)+1}\right)-1\right) \qquad \ldots (33)$$

Also

$$\left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^\beta - 1 = (\beta+1)a_2 z + \left((\beta+2)a_3 + (\beta-1)\left(\frac{\beta+2}{2}\right)a_2^2\right)z^2 + \ldots$$

$$\ldots (34)$$

Now using (21), (33), and (34), we get

$$a_2(\beta+1)=\frac{A_0B_1c_1}{2}$$

and
$$a_3(\beta+2)+a_2^2(\beta-1)\frac{(\beta+2)}{2}=\frac{A_1B_1c_1}{2}+\frac{A_0B_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2}{4}A_0c_1^2$$

Since $\varphi(z)$ is analytic and bounded in $\mathbb{D}$, therefore using the well-known inequalities $|c_i|\leq 2\ \ (i=1,2)$ and (25) for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (30).

Further,

$$\left|a_3-\mu a_2^2\right|\leq\frac{1}{2(\beta+2)}\left[\left|A_1B_1c_1\right|+\left|A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2}{2}A_0c_1^2-\right.\right.$$

$$\left.\left.\frac{(\beta-1)A_0^2B_1^2c_1^2(\beta+2)}{4(\beta+1)^2}-\frac{\mu A_0^2B_1^2c_1^2(\beta+2)}{2(\beta+1)^2}\right|\right]$$

Again applying $\left|A_n\right|\leq 1$ and $\left|c_n\right|\leq 2$ for $n\geq 1,$ we have

$$\left|a_3-\mu a_2^2\right|\leq\frac{B_1}{2(\beta+2)}\left[2+\left|c_2-c_1^2\left\{\frac{(\beta+2)B_1A_0}{2(\beta+1)^2}\left(\frac{\beta-1}{2}+\mu\right)+\frac{1}{2}\left(1-\frac{B_2}{B_1}\right)\right\}\right|\right]$$

Applying lemma (1.1), we get the desired inequality (29).

Now put $\mu=0$ in above inequality, we get desired estimate on $|a_3|$ as asserted in (31).

Corollary 3.1. *For* $\beta=1,$ *the coefficient estimates becomes*

$$\left|a_2\right|\leq\frac{B_1}{2}\ \ \textit{and}\ \ \left|a_3\right|\leq\frac{B_1}{3}\left[1+max.\left\{1,\frac{|B_2|}{B_1}\right\}\right]$$

Corollary 3.2. *For* $\beta=0,$ *the coefficient estimates becomes*

$$\left|a_2\right|\leq B_1,\ \ \left|a_3\right|\leq\frac{B_1}{2}\left[1+max.\left\{1,B_1+\frac{|B_2|}{B_1}\right\}\right]$$

In subordinate case, the above inequality reduces to

(i) For $\beta = 1$

$$|a_2| \le \frac{B_1}{2} \text{ and } |a_3| \le \frac{B_1}{3}\left[max.\left\{1, \frac{|B_2|}{B_1}\right\}\right]$$

(ii) For $\beta = 0$

$$|a_2| \le B_1 \text{ and } |a_3| \le \frac{B_1}{2}\left[max.\left\{1, B_1 + \frac{|B_2|}{B_1}\right\}\right]$$

Corollary 3.3. *For* $\varphi(z) \equiv 1, \beta = 0$ *and*

$$h(z) = \frac{1+(1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)z + 2(1-\alpha)z^2 + \ldots$$

We have

$$|a_2| \le 2(1-\alpha) \quad and \quad |a_3| \le (1-\alpha)(3-2\alpha)$$

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