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A COMMON FIXED POINT THEOREM IN COMPLETE FUZZY METRIC SPACES

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Abstract : Mishra et al. (2008) proved a common fixed point theorem in fuzzy metric space by introducing reciprocal continuity. In this paper we have extended and generalize the above result for Fuzzy 3 matric space.

Keywords: Fuzzy 2-metric space, Fuzzy 3-metric space, common fixed point

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1. Introduction

Fixed point theorems in Fuzzy metric spaces satisfying some contractive conditionis a central area of research now days. The concept of Fuzzy sets wasintroduced by Zadeh[10] in 1965.After this Fuzzy set theory was further developedand a series of research were done by several Mathematicians. Kramosiland Michlek [5] introduced the concept of Fuzzy metric space in 1975 and fixedpoint theorems for Fuzzy metric space was first obtained by Helpern [4] in 1981.Later in 1994, George and Veeramani [3] modified the notion of Fuzzy metricspace with the help of t-norm. Some fixed point theorems in metric space are generalized to Fuzzy metric space by several authors.

There are various ways to define a Fuzzy metric space, here we adopt thenotion that, the distance between objects is Fuzzy, the objects themselves maybe Fuzzy or not.

Gahler [1], [2] investigated the properties of 2-metric space in his papers,and many authors investigated contraction mappings in 2-metric spaces. Succeedingthis, the notion of 3-metric space was also introduced. We know that2-metric space is a real valued function of a point triples on a set *X*, whichabstract properties were suggested by the area function in the Eucledian space,whereas the 3-metric space was suggested by the volume function.The idea of Fuzzy 2-metric space and Fuzzy 3-metric space were used by Sharma [8] and obtained some fruitful results.Motivated by Sharma [9], we prove some common fixed point theoremin Fuzzy 2-metric space and Fuzzy 3-metric space by employing the notionof reciprocal continuity, of which we can widen the scope of many interestingfixed point theorems in Fuzzy metric space.

2. Preliminary Results

Definition 2.1. A triangular norm *(shortly t-norm) is a binary operationon the unit interval [0, 1] such that for all a, b, c, d $\in$[0, 1] the followingconditions are satisfied:

(i) $a*1 = a$;

(ii) $a*b = b*a$;

(iii) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$

(iv) $a*(b*c) = (a*b)*c$.

Definition 2.2. The 3-tuple (X,M, *) is called a fuzzy metric space, if Xis an arbitrary set, *is a continuous t-norm and M is a fuzzy set in $X^2 \times [0,\infty]$satisfying the following conditions: for all x, y, z $\in$X and s, t >0

(i) $M(x, y, 0) = 0$

(ii) $M(x, y, t) = 1$, for all $t > 0$, if and only if $x = y$

(iii)M(x, y, t) = M(y, x, t)

(iv)M(x, y, t) *M(y, z, s) ≤ M(x, z, t + s)

(v) M(x, y, .) : [0,∞) → [0, 1] is left continuous,

(vi) $\lim_{t\to\infty} M(x, y, t) = 1$

Example 2.3. Let (X, d) be a metric space. Define a*b = ab (or a *b = min{a, b}) and for all x, y ∈X and t >0, M(x, y, t) = $\frac{t}{t+d(x,y)}$. Then (X,M, *) is a Fuzzy metric space and this metric d is the standard Fuzzy metric.

Definition 2.4. A sequence {x_n} in a Fuzzy metric space (X,M, *) is saidto converge to x in X if and only if M(x_n, x, t) = 1 for each t >0.

Definition 2.5. Let (X,M, *) be a Fuzzy metric space A sequence {x_n} in Xis called Cauchy sequence if and only if M(x_{n+p},x_n, t) = 1 for each p >0, t >0.

Definition 2.6. A Fuzzy metric space (X,M, *) is said to be complete ifand only if every Cauchy sequence in X is convergent in X.

Definition 2.7. A pair (f, g) of self maps of a Fuzzy metric space (X,M, *)is said to be reciprocal continuous if $\lim_{n\to\infty} fgx_n = fx$ and $\lim_{n\to\infty} gfx_n = gx$ whenever there exist sequences{x_n} such that $\{x_n\} such that \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x∈X.

Definition 2.8. Two self maps A and B of a fuzzy metric space (X,M, *)are said to be weak compatible if they commute at their coincidence points, that is Ax= Bx Implies ABx= BAx.

Definition 2.9. A pair (A, S) of self maps of a Fuzzy metric space (X,M, *) is said to be semi-compatible if $\lim_{n\to\infty} ASx_n = Sx$ whenever there exists a sequence {x_n} in X suchthat $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some x ∈X.

Definition 2.10. A binary operation *: [0, 1] × [0, 1] × [0, 1] → [0, 1] iscalled a continuous t-norm if ([0, 1]), *) is an abelian topological monoid withunit 1 such that $a_1*b_1*c_1 \leq a_2*b_2*c_2$whenever $a_1 \leq a_2$, $b_1 \leq b_2$,$c_1 \leq c_2$forall a_1, a_2, b_1, $b_2$and c_1, $c_2$are in [0, 1].

Definition 2.11. The 3-tuple (X,M, *) is called a Fuzzy 2-metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a Fuzzy set in $X^3 \times [0,\infty]$satisfying the following conditions: for all x, y, z, u $\in$X and t_1, t_2, t_3>0.

(i) M(x, y, z, 0) = 0,

(ii) M(x, y, z, t) = 1, t >0 and when at least two of the three points are equal,

(iii) M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)

(Symmetry about three variables)

(iv) $M(x, y, z, t_1+ t_2+ t_3) \geq M(x, y, u, t_1) *M(x, u, z, t_2) *M(u, y, z, t_3)$

(This corresponds to tetrahedron inequality in 2-metric space)The function value M(x,y,z,t) may be interpreted as the probability thatthe area of triangle is less than t.

(v) M(x, y, z, .) : $[0,\infty) \rightarrow [0, 1]$ is left continuous.

Definition 2.12. A sequence $\{x_n\}$ in a Fsuzzy 2-metric space (X,M, *) issaid to converge to x in X if and only if $\lim_{n\to\infty} M(x_n, x, a, t)1$ for all a $\in$ X and t >0.

Definition 2.13. Let (X, M,*) be a Fuzzy 2-metric space. A sequence $\{x_n\}$in X is called Cauchy sequence, if and only if $\lim_{n\to\infty} M(x_{n+p}, x_n, a, t)1$ for all a $\in$ X and p>0, t >0 .

Definition 2.14. A Fuzzy 2-metric space (X,M, *) is said to be completeif and only if everyCauchy sequence in X is convergent in X.

Definition 2.15. A binary operation *: $[0, 1]\times[0, 1]\times[0, 1]\times[0, 1] \rightarrow [0, 1]$is called a continuous t-norm if ([0, 1]), *) is an abelian topological monoid withunit 1 such that $a_1*b_1*c_1*d_1 \leq a_2*b_2*c_2*d_2$whenever $a_1 \leq a_2$, $b_1 \leq b_2$,$c_1 \leq c_2$and $d_1 \leq d_2$for all a_1, a_2, b_1, b_2, c_1, $c_2$and d_1, $d_2$are in [0, 1].

Definition 2.16. The 3-tuple (X,M, *) is called a Fuzzy 3-metric space if Xis an arbitrary set, *is a continuous t-norm and M is a fuzzy set in $X^4 \times [0,\infty]$satisfying thefollowing conditions: for all x, y, z, w, u$\in$X and t_1, t_2, t_3, t_4>0.

(i) M(x, y, z,w, 0) = 0,

(ii) M(x, y, z,w, t) = 1, for all t >0 ,

(Only when the three simplex < x,y,z,w > degenerate)

(iii) M(x, y, z,w, t) = M(x,w, z, y, t) = M(y, z,w, x, t) = M(z,w, x, y, t) =……

(iv) M(x, y, z,w, $t_1+t_2+t_3+t_4$) ≥ M(x, y, z, u, t_1)*M(x, y, u,w, t_2)*M(x, u, z,w, t_3)

M(u, y, z,w, t_4)

(v) M(x, y, z,w,t) : [0,∞) → [0, 1] is left continuous.

Definition 2.17. A sequence {x_n} in a Fuzzy 3-metric space (X,M, *) issaid to converge to x in X if and only if $\lim_{n\to\infty} M(x_n, x, a, t)1$ *for all* $a, b \in X$ *and* $t > 0$. for all a, b∈X and t > 0.

Definition 2.18.Let (X,M, *) be a Fuzzy 3-metric space. A sequence {x_n}in X is called Cauchy sequence, if and only if $\lim_{n\to\infty} M(x_{n+p}, x_n, a, t)1$ for all a, b∈X , p > 0,and t > 0.

Definition 2.19. A Fuzzy 3-metric space (X,M, *) is said to be completeif and only if every Cauchy sequence in X is convergent in X.

3. Main Results

Theorem 3.1. *Let A,B,C,R, S, T be self maps on a complete Fuzzy 3-metricspace (X,M, *) where * is a continuous t-norm, satisfying*

T-1 AX ⊆TX, BX ⊆SX, CX ⊆RX.

T-2 (B, S) and (C,T)are weakly compatible and reciprocally continuous,

T-3 For each x,y,z ∈X and t > 0,M (Ax,By,Cz,u,t) ≥ Φ (M (Rx,Sy,Tz,u,t)),

where Φ :[0, 1] → [0, 1] is a continuous function such that Φ(1) = 1,Φ(0) = 0 and Φ(a) >a for each 0 < a < 1.If (A, R) is semi compatible and reciprocally continuous, then A,B, C, R, S, Thave a common fixed point.

Proof. Suppose x_0 ∈ X be an arbitrary point. Then there existsx_1,x_2,x_3∈Xsuch thatAx_0= Tx_1,Bx_1=$Sx_2$and Cx_2=Rx_3. Thus we canform sequences{z_n},{y_n}and

$\{x_n\}$in X such that$y_{2n+1} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}, y_{2n+3} = Cx_{2n+2} = Rx_{2n+3}$for n = 0,1, ...

$$M(y_{2n+1}, y_{2n+2}, y_{2n+3}, u, t) = M(Ax_{2n}, Bx_{2n+1}, Cx_{2n+2}, u, t)$$
$$\geq \Phi(M(Rx_{2n}, Sx_{2n+1}, Tx_{2n+2}, u, t))$$
$$> \Phi(M(y_{2n}, y_{2n+1}, y_{2n+2}, u, t))$$

Similarly $M(y_{2n+3}, y_{2n+4}, y_{2n+5}, u, t) > \Phi(M(y_{2n+2}, y_{2n+3}, y_{2n+4}, u, t))$.

More generally $M(y_{n+1}, y_n, y_{n-1}, u, t) > \Phi(M(y_n, y_{n-1}, y_{n-2}, u, t))$

Therefore $\{M(y_{n+1}, y_n, y_{n-1}, u, t)\}$ is an increasing sequence of positive real numbers In [0,1] and tendstolimitl ≤ 1.We claimhat l = 1. If l < 1then

$M(y_{n+1}, y_n, y_{n-1}, u, t) > \Phi(M(y_n, y_{n-1}, y_{n-2}, u, t))$. On letting $n \to \infty$ we get

$$\lim_{n\to\infty} M(y_{n+1}, y_n, y_{n-1}, u, t) > \lim_{n\to\infty} M(y_n, y_{n-1}, y_{n-2}, u, t)).$$

that is $l \geq \Phi(l) > l$ a contradiction. Now for any positive integer p.

$$M(y_n, y_{n+1}, y_{n+p}, u, t) \geq M(y_n, y_{n+1}, y_{n+2}, y_{n+p}, \frac{t}{3(p-1)+1)})$$
$$*M(y_{n+1}, y_{n+2}, y_{n+3}, y_{n+p}, \frac{t}{3(p-1)+1)})$$
$$*M(y_{n+2}, y_{n+3}, y_{n+4}, y_{n+p}, \frac{t}{3(p-1)+1)})$$
$$*\ldots\ldots*M(y_{n+p-3}, y_{n+p-2}, y_{n+p-1}, y_{n+p}, \frac{t}{3(p-1)+1)})$$
$$*M(y_n, y_{n+1}, y_{n+2}, z, \frac{t}{3(p-1)+1)})$$
$$*M(y_{n+1}, y_{n+2}, y_{n+3}, z, \frac{t}{3(p-1)+1)})$$
$$*M(y_{n+2}, y_{n+3}, y_{n+4}, z, \frac{t}{3(p-1)+1)})$$
$$*\ldots\ldots*M(y_{n+p-2}, y_{n+p-1}, y_{n+p}, z, \frac{t}{3(p-1)+1)})$$

$$*M(y_{n+p-2},y_{n+p-1},y_{n+p},z,\frac{t}{3(p-1)+1})$$

Taking limits

$$\lim_{n\to\infty}M(y_n,y_{n+1},y_{n+p},u,t)\geq \lim_{n\to\infty}M(y_n,y_{n+1},y_{n+2},y_{n+p},\frac{t}{3(p-1)+1})$$

$$*\lim_{n\to\infty}M(y_{n+1},y_{n+2},y_{n+3},y_{n+p},\frac{t}{3(p-1)+1})$$

$$*\lim_{n\to\infty}M(y_{n+2},y_{n+3},y_{n+4},y_{n+p},\frac{t}{3(p-1)+1})$$

$$*\ldots\ldots*\lim_{n\to\infty}M(y_{n+p-3},y_{n+p-2},y_{n+p-1},y_{n+p},\frac{t}{3(p-1)+1})$$

$$*\lim_{n\to\infty}M(y_n,y_{n+1},y_{n+2},z,\frac{t}{3(p-1)+1})$$

$$*\lim_{n\to\infty}M(y_{n+1},y_{n+2},y_{n+3},z,\frac{t}{3(p-1)+1})$$

$$*\lim_{n\to\infty}M(y_{n+2},y_{n+3},y_{n+4},z,\frac{t}{3(p-1)+1})$$

$$*\ldots\ldots*\lim_{n\to\infty}M(y_{n+p-2},y_{n+p-1},y_{n+p},z,\frac{t}{3(p-1)+1})$$

that is $\lim_{n\to\infty}M(y_n,y_{n+1},y_{n+p},u,t)\geq 1*1*1*1*\ldots\ldots*1=1$

which means $\{y_n\}$ is a Cauchy sequence in X. Since X is complete $y_n\to w$ in X. that is $\{Ax_{2n}\},\{Tx_{2n+1}\},\{Bx_{2n+1}\},\{Sx_{2n+2}\},\{Cx_{2n+2}\},\{Rx_{2n+3}\}$ also converges to w in X.

That is $\lim_{n\to\infty}Rx_{2n}\to w$ and $\lim_{n\to\infty}Ax_{2n}\to w$.

Since (A,R) is semi-compatible, $\lim_{n\to\infty}Ax_{2n}=Rw$

Also (A,R) is reciprocal continuous also, therefore $\lim_{n\to\infty}Ax_{2n}=Aw$

Combining this process we get $Aw = Rw$.Now to prove that $Aw = w$,for if we consider that $Aw \neq w$.Then by the contractive condition,

$M(Aw, Bx_{2n+1}, Cx_{2n+2}, u, t) \geq \Phi(M(Rw, Sx_{2n+1}, Tx_{2n+2}, u, t))$

Letting $n \to \infty$,

$M(Aw, w, w, u, t) \geq \Phi(M(Rw, w, w, u, t)) > M(Aw, w, w, u, t)$

a contradiction. Therefore $Aw = w = Rw$.

Since (B, S) and (C,T) are weakly compatible and reciprocally continuous, as above we get

$Bw = w = Sw$ & $Cw = w = Tw$.

Therefore A, B, C, R, S and T has a common fixed point.

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