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A STUDY OF UNIFIED FINITE INTEGRAL INVOLVING HYPERGEOMETRIC FUNCTION AND GENERAL CLASS OF FUNCTIONS-II

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Abstract: In the present paper, we evaluate an interesting integral involving hypergeometric function and a general class of functions having general arguments. This integral is quite general in nature and yields a large number of useful functions merely by specializing the parameters involved therein and on considering all possible values of i and j (occurring in the integrand of our main integral) from the set $\{0, \pm 1, \pm 2\}$.

Keywords and Phrases: Generalized hypergeometric functions, General class of functions, Bessel function, Struve's function, Hurwitz – Lerch Zeta function.

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1. Introduction

The general class of functions is defined as under [5, p. 223, Eq. (1)]

$$V_n^{h_m, d, g_l}\left[p, \tau, k, w, q, k_m, a_l, b_\upsilon, \alpha_1, \beta_1, \delta; z\right]$$

$$= \lambda \sum_{n=0}^{\infty} \frac{(-p)^n \prod_{m=1}^{t}\left[\left(h_m\right)_{n+k_m}\right]\left(d+\alpha_1 n+\beta_1\right)^{-\tau}\left(\frac{z}{2}\right)^{nk+dw+q}}{\prod_{l=1}^{s}\left[\left(g_l\right)_{n+a_l}\right]\prod_{\upsilon=1}^{u}\left[(d)_{\alpha_1 n\delta+b_\upsilon}\right]} \tag{1}$$

where

(i) $p, k, w, q, \beta_1, \delta, a_l, b_\upsilon, k_m \ (l = 1, \ldots, s; \upsilon = 1, \ldots, u; m = 1, \ldots, t)$ are real numbers,

(ii) t, s and u are natural numbers,

(iii) $g_l, h_m \geq 1 (l = 1, \ldots, s; m = 1, \ldots, t)$,

(iv) $\alpha_1 > 0, \mathrm{Re}(\tau) > 0, \mathrm{Re}(d) > 0, z$ is a variable and λ is an arbitrary constant.

2. Result Required

We shall require the following known result to evaluate the main integral in this paper:

(i)
$$\int_a^b (x-a)^{\rho-1}(b-x)^{\rho+j-1}\left[A(x-a)+B(b-x)+C\right]^{-2\rho-j}$$

$$ {}_2F_1\left[\begin{matrix}\alpha, \beta \\ \frac{\alpha+\beta+i+1}{2};\end{matrix} \frac{\left[A(b-a)+C\right](x-a)}{(b-a)\left[A(x-a)+B(b-x)+C\right]}\right]dx$$

$$= \frac{\Gamma(\rho)\ \Gamma(\rho+j)\ A_{i,j}\ 2^{\alpha+\beta+i-2}\ (b-a)^{2\rho+j-1}\ \Gamma\left(\frac{1}{2}(\alpha+\beta+i+1)\right)}{\Gamma(2\rho+j)\ \sqrt{\pi}\ \left[A(b-a)+C\right]^{\rho}\ \left[B(b-a)+C\right]^{\rho+j}\ \Gamma(\alpha)\ \Gamma(\beta)}$$

$$\times\ \Gamma\left(\rho+\left[\frac{j}{2}\right]+\frac{1}{2}\right)\Gamma\left(\rho-\frac{1}{2}(\alpha+\beta+|i+j|-j-1)\right)$$

$$\times\left[\frac{B_{i,j}\ \Gamma\left(\frac{\alpha}{2}+\frac{1-(-1)^i}{4}\right)\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\rho-\frac{\alpha}{2}+\frac{1}{2}+\left[\frac{j}{2}\right]-\frac{(-1)^j}{4}\left(1-(-1)^i\right)\right)\Gamma\left(\rho-\frac{\beta}{2}+\frac{1}{2}+\left[\frac{j}{2}\right]\right)}\right.$$

$$\left.+\frac{C_{i,j}\ \Gamma\left(\frac{\alpha}{2}+\frac{1+(-1)^i}{4}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\rho-\frac{\alpha}{2}+\left[\frac{j+1}{2}\right]+\frac{(-1)^j}{4}\left(1-(-1)^i\right)\right)\Gamma\left(\rho-\frac{\beta}{2}+\left[\frac{j+1}{2}\right]\right)}\right] \quad (2)$$

$$\forall i, j \in \{0, \pm 1, \pm 2\}$$

where $[x]$ is the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficients $A_{i,j}$, $B_{i,j}$ *and* $C_{i,j}$ are given in [[9], pp.24-25; see also [6]] tabular form for different values of i and j.

The conditions for validity of (2) are

$$\operatorname{Re}(\alpha)>0,\ \operatorname{Re}(\beta)>-1, \operatorname{Re}(2\rho-\alpha-\beta)>-i-1-2j \quad \text{for} \quad i,j\in\{0,\pm1,\pm2\};$$

$$\operatorname{Re}(\alpha+\beta+i+1)>0 \quad \text{for } i\in\{0,\pm1,\pm2\} \text{ and } \operatorname{Re}(\rho)>0 \text{ for } j\in\{0,1,2\};$$

$$\operatorname{Re}(\rho)>-j \quad \text{for} \quad j\in\{-2,-1\}$$

3. Main Integral

$$\int_a^b (x-a)^{\rho-1}(b-x)^{\rho+j-1}\left[A(x-a)+B(b-x)+C\right]^{-2\rho-j}$$

$${}_2F_1\left(\begin{matrix}\alpha,\ \beta \\ \frac{\alpha+\beta+i+1}{2}\end{matrix};\frac{\{A(b-a)+C\}(x-a)}{(b-a)\{A(x-a)+B(b-x)+C\}}\right)$$

$$\times\ V_n^{h_m,d,g_l}\left[p,\tau,k,w,q,k_m,a_l,b_\upsilon,\alpha_1,\beta_1,\delta;z\left(\frac{(x-a)(b-x)}{\{A(x-a)+B(b-x)+C\}^2}\right)^{\sigma}\right]dx$$

$$=\frac{A_{i,j}\ 2^{\alpha+\beta+i-2}\ \Gamma\left(\frac{\alpha+\beta+i+1}{2}\right)}{\sqrt{\pi}\ \Gamma(\alpha)\Gamma(\beta)}\lambda$$

$$\sum_{n=0}^{\infty}\frac{(-p)^n\prod_{m=1}^{t}\left[\left(h_m\right)_{n+k_m}\right]\left(d+\alpha_1 n+\beta_1\right)^{-\tau}}{\prod_{l=1}^{s}\left[\left(g_l\right)_{n+a_l}\right]\prod_{\upsilon=1}^{u}\left[(d)_{\alpha_1 n\delta+b_\upsilon}\right]}\left(\frac{z}{2}\right)^{nk+dw+q}$$

$$\times\frac{(b-a)^{2\{\rho+\sigma(nk+dw+q)\}+j-1}\ \Gamma\{\rho+\sigma(nk+dw+q)\}}{\left[A(b-a)+C\right]^{\rho+\sigma(nk+dw+q)}\ \left[B(b-a)+C\right]^{\rho+\sigma(nk+dw+q)+j}}$$

$$\times\frac{\Gamma\{\rho+\sigma(nk+dw+q)+j\}}{\Gamma\{2\rho+2\sigma(nk+dw+q)+j\}}$$

$$\times\,\Gamma\left(\rho+\sigma(nk+dw+q)+\left[\frac{j}{2}\right]+\frac{1}{2}\right)\Gamma\left(\rho+\sigma(nk+dw+q)-\frac{1}{2}\left(\alpha+\beta+|i+j|-j-1\right)\right)$$

$$\times\left[\frac{B_{i,j}\;\Gamma\left(\frac{\alpha}{2}+\frac{1-(-1)^{i}}{4}\right)\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\rho+\sigma(nk+dw+q)-\frac{\alpha}{2}+\frac{1}{2}+\left[\frac{j}{2}\right]-\frac{(-1)^{j}\left(1-(-1)^{i}\right)}{4}\right)\Gamma\left(\rho+\sigma(nk+dw+q)-\frac{\beta}{2}+\frac{1}{2}+\left[\frac{j}{2}\right]\right)}\right.$$

$$\left.+\frac{C_{i,j}\;\Gamma\left(\frac{\alpha}{2}+\frac{1+(-1)^{i}}{4}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\rho+\sigma(nk+dw+q)-\frac{\alpha}{2}+\left[\frac{j+1}{2}\right]+\frac{(-1)^{j}\left(1-(-1)^{i}\right)}{4}\right)\Gamma\left(\rho+\sigma(nk+dw+q)-\frac{\beta}{2}+\left[\frac{j+1}{2}\right]\right)}\right] \tag{3}$$

where $[y]$ is the greatest integer less than or equal to y and its modulus is denoted by $|y|$. The coefficients $A_{i,j}$, $B_{i,j}$ *and* $C_{i,j}$ are given in [[9], pp. 24-25; see also[6]] tabular form for different values of i and j.

The conditions for validity of (3) are

(i) $\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>-1, \mathrm{Re}(\alpha+\beta+i+1)>0$ for $i\in\{0,\pm1,\pm2\}$;

(ii) $\mathrm{Re}(2\rho-\alpha-\beta)+2\sigma(nk+dw+q)>-i-1-2j$ for $i,j\in\{0,\pm1,\pm2\}$;

(iii) $\mathrm{Re}(\rho)+\sigma(nk+dw+q)>0$ for $j\in\{0,1,2\}$;

(iv) $\mathrm{Re}(\rho)+\sigma(nk+dw+q)>-j$ for $j\in\{-1,-2\}$;

(v) $p,k,w,q,\beta_1,\delta,a_l,b_\upsilon,k_m\ (l=1,\ldots,s;\upsilon=1,\ldots,u;m=1,\ldots,t)$ are real numbers.

(vi) t,s and u are natural numbers.

(vii) $g_l,h_m\geq1(l=1,\ldots,s;m=1,\ldots,t)$

(viii) $\alpha_1>0, \mathrm{Re}(\tau)>0, \mathrm{Re}(d)>0, z$ is a variable and λ is an arbitrary constant.

Proof: In order to prove the integral (3), we first express the general class of functions in terms of series given by (1), Now changing the order of integration and summation which is permissible under the conditions, we have

$$\int_a^b (x-a)^{\rho-1}(b-x)^{\rho+j-1}[A(x-a)+B(b-x)+C]^{-2\rho-j}$$

$${}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \frac{\alpha+\beta+i+1}{2}\end{matrix};\frac{\{A(b-a)+C\}(x-a)}{(b-a)\{A(x-a)+B(b-x)+C\}}\right)$$

$$\times V_n^{h_m,d,g_l}\left[p,\tau,k,w,q,k_m,a_l,b_\upsilon,\alpha_1,\beta_1,\delta;z\left(\frac{(x-a)(b-x)}{\{A(x-a)+B(b-x)+C\}^2}\right)^{\sigma}\right]dx$$

$$=\lambda\sum_{n=0}^{\infty}\frac{(-p)^n\prod_{m=1}^{t}\left[\left(h_m\right)_{n+k_m}\right]\left(d+\alpha_1 n+\beta_1\right)^{-\tau}}{\prod_{l=1}^{s}\left[\left(g_l\right)_{n+a_l}\right]\prod_{\upsilon=1}^{u}\left[(d)_{\alpha_1 n\delta+b_\upsilon}\right]}\left(\frac{z}{2}\right)^{nk+dw+q}$$

$$\times\int_a^b\Big\{(x-a)^{\rho+\sigma(nk+dw+q)-1}(b-x)^{\rho+\sigma(nk+dw+q)+j-1}$$

$$\{A(x-a)+B(b-x)+C\}^{-2\{\rho+\sigma(nk+dw+q)\}-j}$$

$$\times\ {}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \frac{\alpha+\beta+i+1}{2}\end{matrix};\frac{\{A(b-a)+C\}(x-a)}{(b-a)\{A(x-a)+B(b-x)+C\}}\right)\Big\}dx$$

To evaluate the above integral with the help of known result (2), where $\rho=\rho+\sigma(nk+dw+q)$, then we get the required integral (3).

If we choose specific values of i and j from the set $\{0,\pm1,\pm2\}$ and use the tables for the coefficients $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ are given [[9], pp.24-25; see also [6]] in tabular form for different values of i and j, we get twenty five integrals in this manner.

For the sake of illustration, we mention below one such integral:

Taking i=0, j=0 and using $A_{0,0} = B_{0,0} = 1$ and $C_{0,0} = 0$ in equation (3) as given in table [[9], pp.24-25; see also [6]], we arrive at the following interesting integral:

$$\int_a^b (x-a)^{\rho-1} (b-x)^{\rho-1} \{A(x-a)+B(b-x)+C\}^{-2\rho}$$

$$ {}_2F_1\left(\begin{matrix} \alpha, \beta \\ \frac{\alpha+\beta+1}{2} \end{matrix} ; \frac{\{A(b-a)+C\}(x-a)}{(b-a)\{A(x-a)+B(b-x)+C\}} \right)$$

$$\times V_n^{h_m, d, g_l}\left[p,\tau,k,w,q,k_m,a_l,b_\upsilon,\alpha_1,\beta_1,\delta; z\left(\frac{(x-a)(b-x)}{\{A(x-a)+B(b-x)+C\}^2} \right)^{\sigma} \right] dx$$

$$= \frac{2^{\alpha+\beta-2}\, \Gamma\left(\frac{\alpha+\beta+1}{2}\right) \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\beta}{2}\right)}{\sqrt{\pi}\; \Gamma(\alpha)\; \Gamma(\beta)} \lambda$$

$$\sum_{n=0}^{\infty} \frac{(-p)^n \prod_{m=1}^{t} \left[(h_m)_{n+k_m} \right] \left(d+\alpha_1 n+\beta_1 \right)^{-\tau}}{\prod_{l=1}^{s} \left[(g_l)_{n+a_l} \right] \prod_{\upsilon=1}^{u} \left[(d)_{\alpha_1 n\delta + b_\upsilon} \right]} \left(\frac{z}{2}\right)^{nk+dw+q}$$

$$\times \frac{(b-a)^{2\{\rho+\sigma(nk+dw+q)\}-1} \left(\Gamma\{\rho+\sigma(nk+dw+q)\} \right)^2}{[A(b-a)+C]^{\rho+\sigma(nk+dw+q)}\; [B(b-a)+C]^{\rho+\sigma(nk+dw+q)}\; \Gamma\{2\rho+2\sigma(nk+dw+q)\}}$$

$$\times \frac{\Gamma\left\{\rho+\sigma(nk+dw+q)+\frac{1}{2}\right\} \Gamma\left(\rho+\sigma(nk+dw+q)-\frac{1}{2}(\alpha+\beta-1) \right)}{\Gamma\left\{\rho+\sigma(nk+dw+q)-\frac{\alpha}{2}+\frac{1}{2}\right\} \Gamma\left\{\rho+\sigma(nk+dw+q)-\frac{\beta}{2}+\frac{1}{2}\right\}}$$

provided that the conditions easily obtainable from those mentioned after (3) are satisfied.

4. Special Cases

(A). Taking $\alpha = \alpha + 2r$ and $\beta = -2r$ in L.H.S. of equation (3) (where r is zero or a positive integer), we arrive at the following integral:

$$\int_a^b (x-a)^{\rho-1}(b-x)^{\rho+j-1}\{A(x-a)+B(b-x)+C\}^{-2\rho-j}$$

$${}_2F_1\left(\begin{matrix}\alpha+2r,\ -2r\\ \frac{\alpha+i+1}{2}\end{matrix};\frac{\{A(b-a)+C\}(x-a)}{(b-a)\{A(x-a)+B(b-x)+C\}}\right)$$

$$\times\, V_n^{h_m,d,g_l}\left[p,\tau,k,w,q,k_m,a_l,b_\upsilon,\alpha_1,\beta_1,\delta;z\left(\frac{(x-a)(b-x)}{\{A(x-a)+B(b-x)+C\}^2}\right)^{\sigma}\right]dx$$

$$=\frac{D_{i,j}\,(1/2)_r}{\left(\frac{\alpha}{2}+\frac{1+(-1)^i}{4}\right)_r}\lambda\sum_{n=0}^{\infty}\frac{(-p)^n\prod_{m=1}^{t}\left[(h_m)_{n+k_m}\right]\left(d+\alpha_1 n+\beta_1\right)^{-\tau}}{\prod_{l=1}^{s}\left[(g_l)_{n+a_l}\right]\prod_{\upsilon=1}^{u}\left[(d)_{\alpha_1 n\delta+b_\upsilon}\right]}\left(\frac{z}{2}\right)^{nk+dw+q}$$

$$\times\frac{(b-a)^{2\{\rho+\sigma(nk+dw+q)\}+j-1}\,\Gamma\{\rho+\sigma(nk+dw+q)\}}{\left[A(b-a)+C\right]^{\rho+\sigma(nk+dw+q)}\left[B(b-a)+C\right]^{\rho+\sigma(nk+dw+q)+j}}$$

$$\times\frac{\Gamma\{\rho+\sigma(nk+dw+q)+j\}\left(\frac{\alpha}{2}-\{\rho+\sigma(nk+dw+q)\}+\frac{3}{4}-\frac{(-1)^i}{4}-\left[\frac{j}{2}+\frac{1-(-1)^i}{4}\right]\right)_r}{\Gamma\{2\rho+2\sigma(nk+dw+q)+j\}\left(\rho+\sigma(nk+dw+q)+\frac{1}{2}+\left[\frac{j}{2}\right]\right)_r}$$

(4)

where the value of $D_{i,j}$ are given in [[9], p.25; see also [6]] tabular form. Thus by taking values of i and j from the set $\{0,\pm1,\pm2\}$ and using the corresponding values of $D_{i,j}$. We can easily obtain twenty five integral from equation (4).

For example, taking i=0, j=0 and using $D_{0,0}=1$ in equation (4) as given in table [[9], p. 25; see also [6]], we get the following integral:

$$\int_a^b (x-a)^{\rho-1}(b-x)^{\rho-1}\{A(x-a)+B(b-x)+C\}^{-2\rho}$$

$${}_2F_1\left(\begin{matrix}\alpha+2r,\ -2r\\(\alpha+1)/2\end{matrix};\frac{\{A(b-a)+C\}(x-a)}{(b-a)\{A(x-a)+B(b-x)+C\}}\right)$$

$$\times V_n^{h_m,d,g_l}\left[p,\tau,k,w,q,k_m,a_l,b_\upsilon,\alpha_1,\beta_1,\delta;z\left(\frac{(x-a)(b-x)}{\{A(x-a)+B(b-x)+C\}^2}\right)^\sigma\right]dx$$

$$=\frac{(1/2)_r}{\left(\frac{\alpha+1}{2}\right)_r}\lambda\sum_{n=0}^{\infty}\frac{(-p)^n\prod_{m=1}^{t}\left[(h_m)_{n+k_m}\right]\left(d+\alpha_1 n+\beta_1\right)^{-\tau}}{\prod_{l=1}^{s}\left[(g_l)_{n+a_l}\right]\prod_{\upsilon=1}^{u}\left[(d)_{\alpha_1 n\delta+b_\upsilon}\right]}\left(\frac{z}{2}\right)^{nk+dw+q}$$

$$\times\frac{(b-a)^{2\{\rho+\sigma(nk+dw+q)\}-1}}{\left[A(b-a)+C\right]^{\rho+\sigma(nk+dw+q)}\left[B(b-a)+C\right]^{\rho+\sigma(nk+dw+q)}}$$

$$\times\frac{\left(\Gamma\{\rho+\sigma(nk+dw+q)\}\right)^2\left(\frac{\alpha}{2}-\{\rho+\sigma(nk+dw+q)\}+\frac{1}{2}\right)_r}{\Gamma\{2\rho+2\sigma(nk+dw+q)\}\left(\rho+\sigma(nk+dw+q)+\frac{1}{2}\right)_r}\qquad(5)$$

Now, choosing A=B=1, C=0, a=0, and b=1 in equation (5), we get

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1}\ {}_2F_1\left(\begin{matrix}\alpha+2r,\ -2r\\(\alpha+1)/2\end{matrix};x\right)$$

$$V_n^{h_m,d,g_l}\left[p,\tau,k,w,q,k_m,a_l,b_\upsilon,\alpha_1,\beta_1,\delta;z\{x(1-x)\}^\sigma\right]dx$$

$$= \frac{(1/2)_r}{\left(\frac{\alpha+1}{2}\right)_r} \lambda \sum_{n=0}^{\infty} \frac{(-p)^n \prod_{m=1}^{t}\left[(h_m)_{n+k_m}\right] \left(d+\alpha_1 n+\beta_1\right)^{-\tau}}{\prod_{l=1}^{s}\left[(g_l)_{n+a_l}\right] \prod_{\upsilon=1}^{u}\left[(d)_{\alpha_1 n\delta+b_\upsilon}\right]} (z/2)^{nk+dw+q}$$

$$\times \frac{\left(\Gamma\{\rho+\sigma(nk+dw+q)\}\right)^2 \left(\frac{\alpha}{2}-\{\rho+\sigma(nk+dw+q)\}+\frac{1}{2}\right)_r}{\Gamma\{2\rho+2\sigma(nk+dw+q)\} \left(\rho+\sigma(nk+dw+q)+\frac{1}{2}\right)_r} \quad (6)$$

Particular Cases:

(i) If we take $n=0, d=1, m=1, l=1, \upsilon=1, \tau=1, w=0, q=0, k_1=0,$ $a_1=0, b_1=0,\ \beta_1=0, \delta=0$ and $\lambda=1$ in the above equation (6) then general class of functions reduces to unity, we obtain

$$\int_0^1 (x)^{\rho-1}(1-x)^{\rho-1}\ {}_2F_1\left(\begin{matrix}\alpha+2r,\ -2r \\ \frac{\alpha+1}{2}\end{matrix};\ x\right) dx$$

$$= \frac{(\Gamma(\rho))^2 \left(\frac{\alpha+1}{2}-\rho\right)_r \left(\frac{1}{2}\right)_r}{\Gamma(2\rho) \left(\rho+\frac{1}{2}\right)_r \left(\frac{\alpha+1}{2}\right)_r}$$

(ii) If we take $m=1, l=1, \upsilon=1, h_1=h, g_1=1, p=-2, k=1, w=0, q=0, k_1=0, a_1=0,$ $b_1=0, \alpha_1=1,\ \beta_1=0, \delta=0$ and $\lambda=1$ in an equation (6), then general class of functions reduces to unified Riemann –zeta function [3, p.100, Eq. (1.5); see also [4] and [10]] as follows:

$$\int_0^1 (x)^{\rho-1}(1-x)^{\rho-1}\ {}_2F_1\left(\begin{matrix}\alpha+2r,\ -2r \\ \frac{\alpha+1}{2}\end{matrix};\ x\right) \phi_h^*\left(z\{x(1-x)\}^{\sigma}, \tau, d\right) dx$$

$$= \frac{(1/2)_r}{\left(\frac{\alpha+1}{2}\right)_r} \sum_{n=0}^{\infty} \frac{(h)_n \, (d+n)^{-\tau} \, z^n \left(\frac{\alpha}{2}-\rho-\sigma n+\frac{1}{2}\right)_r \{\Gamma(\rho+\sigma n)\}^2}{n! \;\; \Gamma(2(\rho+\sigma n)) \left(\rho+\sigma n+\frac{1}{2}\right)_r} \tag{7}$$

Further, taking h=1 in the above equation (7), then unified Riemann zeta function reduces to Hurwitz-Lerch zeta function [1, p.27, Eq. (1)] as follows:

$$\int_0^1 (x)^{\rho-1}(1-x)^{\rho-1} \; {}_2F_1\left(\begin{matrix}\alpha+2r, \; -2r \\ \frac{\alpha+1}{2}\end{matrix}; \; x\right) \phi\left(z\{x(1-x)\}, \tau, d\right) \, dx$$

$$= \frac{(1/2)_r}{\left(\frac{\alpha+1}{2}\right)_r} \sum_{n=0}^{\infty} \frac{(d+n)^{-\tau} \, z^n \left(\frac{\alpha}{2}-\rho-\sigma n+\frac{1}{2}\right)_r \{\Gamma(\rho+\sigma n)\}^2}{\Gamma(2(\rho+\sigma n))\left(\rho+\sigma n+\frac{1}{2}\right)_r}$$

(B) Similarly, taking $\beta = -2r-1$ and $\alpha = \alpha + 2r + 1$ in L.H.S. of Equation (3) (where r is zero or a positive integer), we arrive at the following integral:

$$\int_a^b (x-a)^{\rho-1} \, (b-x)^{\rho+j-1} \, \{A(x-a)+B(b-x)+C\}^{-2\rho-j}$$

$${}_2F_1\left(\begin{matrix}\alpha+2r+1, \; -2r-1 \\ \frac{\alpha+i+1}{2}\end{matrix}; \frac{\{A(b-a)+C\}(x-a)}{(b-a)\{A(x-a)+B(b-x)+C\}}\right)$$

$$\times V_n^{h_m, d, g_l}\left[p, \tau, k, w, q, k_m, a_l, b_\upsilon, \alpha_1, \beta_1, \delta; z\left(\frac{(x-a)(b-x)}{\{A(x-a)+B(b-x)+C\}^2}\right)^{\sigma}\right] dx$$

$$= \frac{E_{i,j} \; (3/2)_r}{\left(\frac{\alpha}{2}+\frac{3-(-1)^i}{4}\right)_r} \lambda \sum_{n=0}^{\infty} \frac{(-p)^n \prod_{m=1}^{t}\left[(h_m)_{n+k_m}\right] \left(d+\alpha_1 n+\beta_1\right)^{-\tau}}{\prod_{l=1}^{s}\left[(g_l)_{n+a_l}\right] \prod_{\upsilon=1}^{u}\left[(d)_{\alpha_1 n\delta+b_\upsilon}\right]} \left(\frac{z}{2}\right)^{nk+dw+q}$$

$$\times \frac{(b-a)^{2\{\rho+\sigma(nk+dw+q)\}+j-1} \; \Gamma\{\rho+\sigma(nk+dw+q)\}}{\left[A(b-a)+C\right]^{\rho+\sigma(nk+dw+q)} \; \left[B(b-a)+C\right]^{\rho+\sigma(nk+dw+q)+j}}$$

$$\times \frac{\Gamma\left\{\rho+\sigma\left(nk+dw+q\right)+j\right\}\left(\frac{\alpha}{2}-\left\{\rho+\sigma\left(nk+dw+q\right)\right\}+\frac{5}{4}+\frac{(-1)^i}{4}-\left[\frac{j}{2}+\frac{\left(1-(-1)^i\right)}{4}\right]\right)_r}{\Gamma\left\{2\rho+2\sigma\left(nk+dw+q\right)+j\right\}\left(\rho+\sigma\left(nk+dw+q\right)+\frac{1}{2}+\left[\frac{j+1}{2}\right]\right)_r} \tag{8}$$

where the value of $E_{i,j}$ are given in [[9], p.26]; see also [6]] tabular form for different values of i and j.

For example, taking $i=-1,\ j=0$ and using $E_{-1,0}=-1/\alpha$ in equation (8) as given in table [[9], p. 26; see also [6]], we get the following integral:

$$\int_a^b (x-a)^{\rho-1}\ (b-x)^{\rho-1}\ \left\{A(x-a)+B(b-x)+C\right\}^{-2\rho}$$

$$ {}_2F_1\left(\begin{matrix}\alpha+2r+1,\ -2r-1\\ \alpha/2\end{matrix};\frac{\left\{A(b-a)+C\right\}(x-a)}{(b-a)\left\{A(x-a)+B(b-x)+C\right\}}\right)$$

$$\times\ V_n^{h_m,d,g_l}\left[p,\tau,k,w,q,k_m,a_l,b_\upsilon,\alpha_1,\beta_1,\delta;z\left(\frac{(x-a)(b-x)}{\left\{A(x-a)+B(b-x)+C\right\}^2}\right)^{\sigma}\right]dx$$

$$=\frac{(-1/\alpha)\ (3/2)_r}{\left(\frac{\alpha}{2}+1\right)_r}\lambda\sum_{n=0}^{\infty}\frac{(-p)^n\prod_{m=1}^{t}\left[\left(h_m\right)_{n+k_m}\right]\left(d+\alpha_1 n+\beta_1\right)^{-\tau}}{\prod_{l=1}^{s}\left[\left(g_l\right)_{n+a_l}\right]\prod_{\upsilon=1}^{u}\left[(d)_{\alpha_1 n\delta+b_\upsilon}\right]}\left(\frac{z}{2}\right)^{nk+dw+q}$$

$$\times\frac{(b-a)^{2\left\{\rho+\sigma\left(nk+dw+q\right)\right\}-1}}{\left[A(b-a)+C\right]^{\rho+\sigma\left(nk+dw+q\right)}\ \left[B(b-a)+C\right]^{\rho+\sigma\left(nk+dw+q\right)}}$$

$$\times \frac{\left(\Gamma\left\{\rho+\sigma\left(nk+dw+q\right)\right\}\right)^{2} \left(\frac{\alpha}{2}-\left\{\rho+\sigma\left(nk+dw+q\right)\right\}+1\right)_{r}}{\Gamma\left\{2\rho+2\sigma\left(nk+dw+q\right)\right\} \left(\rho+\sigma\left(nk+dw+q\right)+\frac{1}{2}\right)_{r}} \tag{9}$$

Now, choosing A=B=1, C=0, a=0, and b=1 in equation (9), we get

$$\int_{0}^{1} x^{\rho-1} (1-x)^{\rho-1} {}_{2}F_{1}\left(\begin{matrix}\alpha+2r+1, \ -2r-1\\ \alpha/2\end{matrix}; x\right)$$

$$V_{n}^{h_m, d.g_l}\left[p,\tau,k,w,q,k_m,a_l,b_\upsilon,\alpha_1,\beta_1,\delta; z\left\{x(1-x)\right\}^{\sigma}\right] dx$$

$$= \frac{(-1/\alpha)\ (3/2)_r}{\left(\frac{\alpha}{2}+1\right)_r}\lambda \sum_{n=0}^{\infty} \frac{(-p)^n \prod_{m=1}^{t}\left[\left(h_m\right)_{n+k_m}\right]\left(d+\alpha_1 n+\beta_1\right)^{-\tau} (z/2)^{nk+dw+q}}{\prod_{l=1}^{s}\left[\left(g_l\right)_{n+a_l}\right] \prod_{\upsilon=1}^{u}\left[(d)_{\alpha_1 n\delta+b_\upsilon}\right]}$$

$$\times \frac{\left(\Gamma\left\{\rho+\sigma\left(nk+dw+q\right)\right\}\right)^{2} \left(\frac{\alpha}{2}-\left\{\rho+\sigma\left(nk+dw+q\right)\right\}+1\right)_{r}}{\Gamma\left\{2\rho+2\sigma\left(nk+dw+q\right)\right\} \left(\rho+\sigma\left(nk+dw+q\right)+\frac{1}{2}\right)_{r}} \tag{10}$$

Particular Cases:

(i) If we take $m=1, l=2, \upsilon=1, h_1=1, g_1=1, g_2=1, p=1, \tau=1, k=2, w=1, q=0,$ $k_1=0,\ \ a_1=0, a_2=0, b_1=0, \alpha_1=1, \beta_1=0, \delta=1$ and $\lambda=1/\Gamma(d)$ in the above equation (10), then general class of functions reduces to Bessel's function [8, p.109] as follows:

$$\int_{0}^{1} x^{\rho-1} (1-x)^{\rho-1} {}_{2}F_{1}\left(\begin{matrix}\alpha+2r+1, \ -2r-1\\ \alpha/2\end{matrix}; x\right) J_d\left[z\left\{x(1-x)\right\}^{\sigma}\right] dx$$

$$= \frac{(-1/\alpha)\ (3/2)_r}{\left(\frac{\alpha}{2}+1\right)_r} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\Gamma\{\rho+\sigma(2n+d)\}\right)^2 \left(\frac{\alpha}{2}-(\rho+\sigma(2n+d))+1\right)_r}{\lfloor n \ \Gamma(d+n+1)\ \Gamma\{2(\rho+\sigma(2n+d))\} \left(\rho+\sigma(2n+d)+\frac{1}{2}\right)_r} (z/2)^{2n+d}$$

(ii) If we take $m = 1, l = 2, \upsilon = 1, h_1 = 1, g_1 = 3/2, g_2 = 1, p = 1, \tau = 1, k = 2, w = 1, q = 1,$ $k_1 = 0, a_1 = 0,$ $a_2 = 0, b_1 = 1/2, \alpha_1 = 1, \beta_1 = 1/2, \delta = 1$ and $\lambda = \{\Gamma(d)\Gamma(3/2)\}^{-1}$ in equation (10), then general class of functions reduces to Struve's function [[7], p.143; see also [2]] as follows:

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1}\, {}_2F_1\left(\begin{matrix}\alpha+2r+1, & -2r-1\\ \alpha/2 & \end{matrix}; x\right) H_d\left[z\{x(1-x)\}^{\sigma}\right]dx$$

$$=\frac{(-1/\alpha)\ (3/2)_r}{\left(\frac{\alpha}{2}+1\right)_r}\sum_{n=0}^{\infty}\frac{(-1)^n\ \{\Gamma(\rho+\sigma(2n+d+1))\}^2\left(\frac{\alpha}{2}-\rho-\sigma(2n+d+1)+1\right)_r\ (z/2)^{2n+d+1}}{\Gamma\left(\frac{3}{2}+n\right)\ \Gamma\left(d+n+\frac{3}{2}\right)\ \Gamma\{2(\rho+\sigma(2n+d+1))\}\left(\rho+\sigma(2n+d+1)+\frac{1}{2}\right)_r}$$

(11)

(iii) If we take $m = 1, l = 2, \upsilon = 1, h_1 = 1, g_1 = 3/2, g_2 = 1, p = 1, \tau = 1, k = 2, w = 1, q = 1,$ $k_1 = 0, a_1 = 0,$ $a_2 = 0, b_1 = 1/2, \alpha_1 = 1, \beta_1 = 1/2, \delta = 1$ and

$\lambda = 2^d\,\Gamma\left(d+\frac{1}{2}\right)\{\Gamma(d)\}^{-1}$ in equation (10), then general class of functions reduces to Lommel's function [[7], p.144; see also [2]] as follows:

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1}\, {}_2F_1\left(\begin{matrix}\alpha+2r+1, & -2r-1\\ \alpha/2 & \end{matrix}; x\right) S_{d,d}\left[z\{x(1-x)\}^{\sigma}\right]dx$$

$$=\frac{2^d\,\Gamma\left(d+\frac{1}{2}\right)\left(\frac{-1}{\alpha}\right)\left(\frac{3}{2}\right)_r}{\left(\frac{\alpha}{2}+1\right)_r}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\Gamma\{\rho+\sigma(2n+d+1)\}\right)^2\left(\frac{\alpha}{2}-\{\rho+\sigma(2n+d+1)\}+1\right)_r}{\Gamma\{2\rho+2\sigma(2n+d+1)\}\ \Gamma\left(d+n+\frac{3}{2}\right)\left(\frac{3}{2}\right)_n\left(\rho+\sigma(2n+d+1)+\frac{1}{2}\right)_r}$$

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