

## ON A SUBCLASS OF ANALYTIC FUNCTION WITH NEGATIVE COEFFICIENTS DEFINED BY GENERALIZED SALAGEAN OPERATOR

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**Abstract :** In this paper, we introduce and study a new subclass of analytic function with negative coefficients which is defined by using generalized Salagean operator in the unit disc. By giving specific values of we obtain important classes studied by various researchers in earlier works. In fact, an attempt has been made to have a unified and detailed study of uniformly convex functions and starlike functions corresponding to uniformly convex functions. The results presented here included coefficient estimates, distortion theorem, closure theorem etc. of several functions belonging to this class.

**Key Words:** Analytic, Univalent, Salagean operator, uniformly convex functions, starlike functions.

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### 1. Introduction

Let  $A_j$ , ( $j \in N = \{1, 2, 3, \dots\}$ ) denote the class of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k$$

which are analytic in the unit disk  $E = \{z \in C : |z| < 1\}$ . For function  $f(z)$  in  $A_j$ , Al-Oboudi [1] defined

$$D_\lambda^0 f(z) = f(z)$$

$$D_\lambda^1 f(z) = (1 - \lambda)f(z) + \lambda zf(z) = D_\lambda f(z) \lambda \geq 0$$

$$D_\lambda^n f(z) = D_\lambda D_\lambda^{n-1} f(z)$$

$$\text{Then } D_\lambda^n f(z) = z + \sum_{k=j+1}^{\infty} \{1 + (k-1)\lambda\}^n \alpha_k z^k.$$

If we put  $\lambda = 1$ , we have Salagean operator introduced by Salagean [8].

With the help of differential operator  $D_\lambda^n$ , we say that the function  $f(z)$  belonging to  $A_j$  is in  $S(n, m, j, \alpha, \lambda)$ , if and only if

$$\operatorname{Re} \left\{ \frac{D_\lambda^{n+m} f(z)}{D_\lambda^n f(z)} \right\} \geq \alpha \left| \frac{D_\lambda^{n+m} f(z)}{D_\lambda^n f(z)} - 1 \right|$$

for some  $\alpha \geq 0$  and for all  $z \in E$ .

Let  $T_j$  denote the subclass of consisting of functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} \alpha_k z^k, \quad (\alpha_k \geq 0). \quad (1)$$

Further, we define the class  $T(n, m, j, \alpha, \lambda)$  by

$$T(n, m, j, \alpha, \lambda) = S(n, m, j, \alpha, \lambda) \cap T_j.$$

By giving specific values of  $n, m, j, \alpha$  and  $\lambda$ , we obtain important classes studied by various researchers in earlier works (see [2], [3], [4], [5], [7], [10]).

In view of this remark we see that a study of the class  $T(n, m, j, \alpha, \lambda)$  leads to unified results on properties of various subclasses of univalent functions.

## 2. Main Results

In this section we give some results for the class. Our first result is contained in the following theorem.

**Theorem 2.1.** *Let the function  $f(z)$  be defined by (1), then  $f(z)$  belongs to*

*$T(n, m, j, \alpha, \lambda)$ , if and only if*

$$\sum_{k=j+1}^{\infty} [1 + (k-1)\lambda]^n [(\alpha+1)\{1 + (k-1)\lambda\}^m - \alpha] \alpha_k \leq 1. \quad (2)$$

*The result is sharp.*

**Proof.** Assume that  $f(z) \in T(n, m, j, \alpha, \lambda)$ , then by definition

$$\operatorname{Re} \left\{ \frac{D_\lambda^{n+m} f(z)}{D_\lambda^n f(z)} \right\} \geq \alpha \left| \frac{D_\lambda^{n+m} f(z)}{D_\lambda^n f(z)} - 1 \right|, \quad z \in E.$$

Equivalently

$$\begin{aligned} \operatorname{Re} \left\{ \frac{1 - \sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^{n+m} a_k z^{k-1}}{1 - \sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n a_k z^{k-1}} \right\} &\geq \alpha \left| \frac{1 - \sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^{n+m} a_k z^{k-1}}{1 - \sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n a_k z^{k-1}} - 1 \right| \\ &= \alpha \left| \frac{\sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^{n+m} a_k z^{k-1} - \sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n a_k z^{k-1}}{1 - \sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n a_k z^{k-1}} \right| \end{aligned} \tag{3}$$

Choosing values of  $z$  on the real axis so that left side of (3) is real and letting  $z \rightarrow 1$ , we get

$$\left[ 1 - \sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^{n+m} a_k \right] \geq \alpha \sum_{k=j+1}^{\infty} \left[ [1+(k-1)\lambda]^{n+m} - [1+(k-1)\lambda]^n \right] a_k$$

which yields

$$\sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n [(\alpha+1)\{1+(k-1)\lambda\}^m - \alpha] a_k \leq 1.$$

Conversely, suppose that (2) is true for  $z \in E$ , then

$$\operatorname{Re} \left\{ \frac{D_\lambda^{n+m} f(z)}{D_\lambda^n f(z)} \right\} - \alpha \left| \frac{D_\lambda^{n+m} f(z)}{D_\lambda^n f(z)} - 1 \right| \geq 0.$$

$$\left[ 1 - \sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^{n+m} a_k \right] \geq \alpha \sum_{k=j+1}^{\infty} \left[ [1+(k-1)\lambda]^{n+m} - [1+(k-1)\lambda]^n \right] a_k$$

If

$$\left\{ \frac{1 - \sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^{n+m} \alpha_k |z|^{k-1}}{1 - \sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n \alpha_k |z|^{k-1}} \right\} - \alpha \frac{\sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n \left[ [1+(k-1)\lambda]^m - 1 \right] \alpha_k |z|^{k-1}}{1 - \sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n \alpha_k |z|^{k-1}} \geq 0.$$

That is, if

$$\sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^n [(\alpha+1)\{1+\lambda(k-1)\}^m - \alpha] \alpha_k \leq 1.$$

This completes the proof.

**Corollary 2.1.** Let the function  $f(z)$  defined by (1), is in the class  $T(n, m, j, \alpha, \lambda)$  then

$$0 \leq \alpha_k \leq \frac{1}{[1+\lambda(k-1)]^n [(\alpha+1)\{1+\lambda(k-1)\}^m - \alpha]}, \quad k \geq j+1.$$

The result is sharp for the functions

$$f(z) = z - \frac{z^k}{[1+\lambda(k-1)]^n [(\alpha+1)\{1+\lambda(k-1)\}^m - \alpha]}. \quad (4)$$

**Theorem 2.2.** Let  $0 \geq \alpha_1 \leq \alpha_2$ , Then  $T(n, m, j, \alpha_1, \lambda) \supseteq T(n, m, j, \alpha_2, \lambda)$ .

**Proof.** Let the function  $f(z)$  be defined by (1), be in the class  $T(n, m, j, \alpha_2, \lambda)$ , then by Theorem 2.1, we have

$$\sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^n [(\alpha_2+1)\{1+\lambda(k-1)\}^m - \alpha_2] \alpha_k \leq 1.$$

Consequently

$$\begin{aligned} \sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^n [(\alpha_1+1)\{1+\lambda(k-1)\}^m - \alpha_1] \alpha_k \\ \leq \sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^n [(\alpha_2+1)\{1+\lambda(k-1)\}^m - \alpha_2] \alpha_k \end{aligned}$$

This completes the proof of Theorem 2.2 with the help of Theorem 2.1.

**Theorem 2.3.** For  $\alpha \geq 0$ ,  $T(n+1, m, j, \alpha, \lambda) \subseteq T(n, m, j, \alpha, \lambda)$ .

**Proof.** Let the function  $f(z)$  defined by (1), be in the class  $T(n+1, m, j, \alpha, \lambda)$  then by Theorem 2.1, we have

$$\sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^{n+1} [(\alpha + 1)\{1 + \lambda(k-1)\}^m - \alpha] \alpha_k \leq 1.$$

Consequently

$$\begin{aligned} \sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^n [(\alpha + 1)\{1 + \lambda(k-1)\}^m - \alpha] \alpha_k &\leq 1 \\ &\leq \sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^{n+1} [(\alpha + 1)\{1 + \lambda(k-1)\}^m - \alpha] \alpha_k \leq 1. \end{aligned}$$

This completes the proof of Theorem 2.3 with the aid of Theorem 2.1.

Similarly we can prove

**Theorem 2.4.** For  $0 < \lambda_1 \leq \lambda_2$ ,  $T(n, m, j, \alpha, \lambda_2) \subseteq T(n, m, j, \alpha, \lambda_1)$ .

**Theorem 2.5.**  $T(n, m, j, \alpha, \lambda)$  is a convex set.

**Proof .** Let the function

$$f(z) = z - \sum_{k=j+1}^{\infty} a_{k,\nu} z^k, \quad (a_k, \nu \geq 0; \nu = 1, 2) \tag{5}$$

be in the class  $T(n, m, j, \alpha, \lambda)$ . It is sufficient to show that the function  $h(z)$  defined by

$$h(z) = \mu f_1(z) + (1 - \mu) f_2(z), \quad (0 \leq \mu \leq 1)$$

is also in the class  $T(n, m, j, \alpha, \lambda)$ . Since  $0 \leq \mu \leq 1$

$$h(z) = z - \sum_{k=j+1}^{\infty} [\mu a_{k,1} + (1 - \mu) a_{k,2}] z^k$$

with the aid of Theorem 2.1, we obtain

$$\sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^n [(\alpha + 1)\{1 + \lambda(k-1)\}^m - \alpha] [\mu \alpha_{k_1} + (1 - \mu) \alpha_{k_2}] \leq 1$$

which implies that  $h(z) \in T(n, m, j, \alpha, \lambda)$ .

Hence  $T(n, m, j, \alpha, \lambda)$  is a convex set.

**Theorem 2.6.** Let the function  $f(z)$  defined by (1), be in the class  $T(n, m, j, \alpha, \lambda)$  then

for  $|z| = r < 1$

$$|D_{\lambda}^i f(z)| \geq r - \frac{r^{j+1}}{(1+\lambda)^{n-i}[(\alpha+1)(1+\lambda)^m - \alpha]}, \quad (6)$$

and

$$|D_{\lambda}^i f(z)| \leq r + \frac{r^{j+1}}{(1+\lambda)^{n-i}[(\alpha+1)(1+\lambda)^m - \alpha]}, \quad (7)$$

for  $z \in E$  and  $0 \leq i \leq n$ .

**Proof.** Note that  $f(z) \in T(n, m, j, \alpha, \lambda)$ . if and only if

$D_{\lambda}^i f(z) \in T(n-i, m, j, \alpha, \lambda)$  and that

$$D_{\lambda}^i f(z) = z - \sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^i \alpha_k z^k. \quad (8)$$

By Theorem 2.1, we know that

$$\begin{aligned} (1+\lambda)^{n-i}[(\alpha+1)(1+\lambda)^m - \alpha] \sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^i \alpha_k \\ \leq \sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^m [(\alpha+1)\{1 + \lambda(k-1)\}^m - \alpha] \alpha_k \\ \leq 1. \end{aligned}$$

That is

$$\sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^i \alpha_k \leq \frac{1}{(1+\lambda)^{n-i}[(\alpha+1)(1+\lambda)^m - \alpha]} \quad (9)$$

$$|D_{\lambda}^i f(z)| \leq |z| + r^{j+1} \sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^i \alpha_k \leq r + r^{j+1} \frac{1}{(1+\lambda)^{n-i}[(\alpha+1)(1+\lambda)^m - \alpha]},$$

and

$$|D_{\lambda}^i f(z)| \geq r - \frac{r^{j+1}}{(1+\lambda)^{n-i}[(\alpha+1)(1+\lambda)^m - \alpha]}$$

This completes the proof of Theorem 2.6.

**Corollary 2.2.** *Let the function  $f(z)$  defined by (1) be in the class  $T(n, m, j, \alpha, \lambda)$ . then for  $|z| = r < 1$ .*

$$|f(z)| \geq r - \frac{r^{j+1}}{(1+\lambda)^n [(\alpha+1)(1+\lambda)^m - \alpha]} \tag{10}$$

and

$$|f(z)| \leq r + \frac{r^{j+1}}{(1+\lambda)^n [(\alpha+1)(1+\lambda)^m - \alpha]}, \quad (z \in E). \tag{11}$$

The equalities in (10) and (11) are attained for the function given by

$$f(z) = z - \frac{z^{j+1}}{(1+\lambda)^n [(\alpha+1)(1+\lambda)^m - \alpha]}$$

**Proof.** Taking  $i = 0$  in Theorem 2.6, we immediately obtain (10) and (11).

**Theorem 2.7.** *Let  $f_j(0) = z$  and*

$$f_k(z) = z - \frac{1}{(1+(k-1)\lambda)^n [(\alpha+1)(1+(k-1)\lambda)^m - \alpha]} z^k \quad (k \geq j+1; n \in N)$$

for  $\alpha \geq 0$ . Then  $f(z)$  is in the class  $T(n, m, j, \alpha, \lambda)$  if and only if it can be expressed in the form

$$f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z) \quad \text{where } \mu_k \geq 0 \quad \text{and} \quad \sum_{k=j}^{\infty} \mu_k = 1 \tag{12}$$

**Proof.** Assume that

$$f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z) = z - \sum_{k=j+1}^{\infty} \frac{1}{(1+(k-1)\lambda)^n [(\alpha+1)(1+(k-1)\lambda)^m - \alpha]} \mu_k z^k.$$

Then it follows that

$$\begin{aligned} & \sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^n [(\alpha+1)\{1+\lambda(k-1)\}^m - \alpha] \frac{1}{(1+(k-1)\lambda)^n [(\alpha+1)(1+(k-1)\lambda)^m - \alpha]} \mu_k \\ &= \sum_{k=j+1}^{\infty} \mu_k = 1 - \mu_j \leq 1. \end{aligned}$$

So, by Theorem 2.1,  $f(z) \in T(n, m, j, \alpha, \lambda)$ .

Conversely, assume that the function defined by (1) belongs to class  $\mathcal{S}$ . Then

$$\alpha_k \leq \frac{1}{(1+(k-1)\lambda)^n [(\alpha+1)(1+(k-1)\lambda)^m - \alpha]},$$

$$(k \geq j+1, n \in N_0).$$

Setting

$$\mu_k = [1+(k-1)\lambda]^n [(\alpha+1)(1+(k-1)\lambda)^m - \alpha] \alpha_k, \quad (k \geq j+1, n \in N_0)$$

$$\text{and } \mu_j = 1 - \sum_{k=j+1}^{\infty} \mu_k.$$

We can see that  $f(z)$  can be expressed in the form (12). This completes the proof of Theorem 2.7.

**Theorem 2.8.** *Let the function  $f(z)$  defined by (1) be in the class  $T(n, m, j, \alpha, \lambda)$  then  $f(z)$  is close to convex of order  $\rho$  ( $0 \leq \rho < 1$ ) in  $|z| < r$ , where*

$$r_1 = r_1(n, m, \alpha, \lambda, \rho) = \inf_k \left[ \left( \frac{1-\rho}{k} \right) \{1+(k-1)\lambda\}^n \left[ (\alpha+1) \{1+(k-1)\lambda\}^m - \alpha \right] \right]^{\frac{1}{k-1}} \quad (13)$$

The result is sharp with the extremal function  $f(z)$  given by (4).

**Proof.** We must show that  $|f'(z)-1| \leq (1-\rho)$  for  $|z| < r_1(n, m, \alpha, \lambda, \rho)$  where  $r_1(n, m, \alpha, \lambda, \rho)$  is given by (13). Indeed we find from (1) that

$$|f'(z)-1| \leq \sum_{k=j+1}^{\infty} k \alpha_k |z|^{k-1}.$$

Thus  $|f'(z)-1| \leq (1-\rho)$ ,

$$\text{if } \sum_{k=j+1}^{\infty} \frac{k}{1-\rho} \alpha_k |z|^{k-1} \leq 1. \quad (14)$$

But by Theorem 2.1, (14) will be true if

$$k/(1-\rho) |z| (k-1) \leq (1+(k-1)\lambda)^n [(\alpha+1)(1+(k-1)\lambda)^m - \alpha]$$

that is, if

$$|z| \leq \left[ \left( \frac{1-\rho}{k} \right) (1+(k-1)\lambda)^n [(\alpha+1)(1+(k-1)\lambda)^m - \alpha] \right]^{\frac{1}{k-1}} \tag{15}$$

Theorem 2.8 follows easily from (15).

**Theorem 2.9.** *Let the function  $f(z)$  be defined by (1) be in the class  $T(n, m, j, \alpha, \lambda)$ . Then  $f(z)$  is starlike of order  $\rho(0 \leq \rho < 1)$ , in  $|z| < r_2$ , where*

$$r_2 = r_2(n, m, \alpha, \lambda, \rho) = \inf_k \left[ \left( \frac{1-\rho}{k-\rho} \right) \{1+(k-1)\lambda\}^n [(\alpha+1)\{1+(k-1)\lambda\}^m - \alpha] \right]^{\frac{1}{k-1}} \tag{16}$$

The result is sharp with the extremal function  $f(z)$  given by (4).

**Proof.** It is sufficient to show that

$$|zf'(z)/(f(z)) - 1| \leq (1-\rho),$$

for  $|z| < r_2(n, m, \alpha, \lambda, \rho)$  where  $r_2$  is given by (16). Indeed, we find again from (1), that

$$\sum_{k=j+1}^{\infty} \frac{k-\rho}{1-\rho} a_k |z|^{k-1} \leq 1. \tag{17}$$

But, by the Theorem 2.1, (17) will be true if

$$\left( \frac{1-\rho}{k-\rho} \right) |z|^{k-1} \leq \{1+(k-1)\lambda\}^n [(\alpha+1)\{1+\lambda(k-1)\}^m - \alpha]$$

That is, if

$$|z| < \left[ \left( \frac{1-\rho}{k-\rho} \right) \{1+(k-1)\lambda\}^n [(\alpha+1)\{1+(k-1)\lambda\}^m - \alpha] \right]^{\frac{1}{k-1}} \tag{18}$$

Theorem 2.9 follows easily from (18).

**Theorem 2.10.** *Let the function  $f(z)$  defined by (1) be in the class*

*$T(n, m, j, \alpha, \lambda)$ . Then  $f(z)$  is convex of order  $\rho(0 \leq \rho < 1)$  in  $|z| < r_3$*

where

$$r_3 = r_3(n, m, \alpha, \lambda, \rho)$$

$$= \inf_k \left[ \left( \frac{1-\rho}{k(k-\rho)} \right) \{1 + (k-1)\lambda\}^n \left[ (\alpha+1) \{1 + (k-1)\lambda\}^m - \alpha \right] \right]^{\frac{1}{k-1}}.$$

The result is sharp with the extremal function given by (4).

**Proof.** The proof of the above theorem is similar to that of Theorem 2.9. Therefore we omit the details involved.

**Theorem 2.11.** Let the function  $f(z)$  defined by (1) be in the class  $T(n, m, j, \alpha, \lambda)$  and let  $c$  be a real number such that  $c > -1$ . Then the function  $F(z)$  defined by

$$F(z) = z - \int_0^z t^{c-1} f(t) dt, \quad (c > -1) \quad (19)$$

also belongs to the class  $T(n, m, j, \alpha, \lambda)$ .

**Proof.** From the representation (19) of  $f(z)$  it follows that

$$F(z) = z - \sum_{k=j+1}^{\infty} b_k z^k, \quad \text{where } b_k = \left( \frac{c+1}{c+k} \right) \alpha_k.$$

Therefore, we have

$$\begin{aligned} & \sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^n [(\alpha+1) \{1 + \lambda(k-1)\}^m - \alpha] b_k \\ & \leq \sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^n [(\alpha+1) \{1 + \lambda(k-1)\}^m - \alpha] \alpha_k \leq 1 \end{aligned}$$

Since  $f(z) \in T(n, m, j, \alpha, \lambda)$ .

Hence by Theorem 2.1,  $F(z) \in T(n, m, j, \alpha, \lambda)$ .

**Theorem 2.12.** Let the function  $f(z)$  defined by (1) be in the class  $T(n, m, j, \alpha, \lambda)$  and let  $c$  be a real number such that  $c > -1$ . Then the function  $F(z)$  given by (19) is univalent in  $|z| < R^*$  where

$$R^* = \inf_k \left[ \frac{(c+1)\{1+(k-1)\lambda\}^n \left[ (\alpha+1)\{1+(k-1)\lambda\}^m - \alpha \right]}{(c+k)} \right]^{\frac{1}{k-1}}, \quad k \geq j+1. \tag{20}$$

The result is sharp.

**Proof .** From (19), we have

$$f(z) = \frac{z^{(1-c)}[z^c F(z)]'}{c+1} = z - \sum_{k=j+1}^{\infty} \frac{(c+k)}{(c+1)} a_k z^k.$$

In order to obtain the required result, it suffices to show that

$|F'(z) - 1| < 1$ , whenever  $|z| < R^*$ , where  $R^*$  is given by (20).

Now  $|F'(z) - 1| \leq \sum_{k=j+1}^{\infty} \frac{k(c+k)}{(c+1)} a_k |z|^{k-1}.$

Thus  $|F'(z) - 1| < 1$  if  $\sum_{k=j+1}^{\infty} \frac{k(c+k)}{(c+1)} a_k |z|^{k-1} \leq 1.$  (21)

But, Theorem 2.1 confirms that

$$\sum_{k=j+1}^{\infty} \{1+\lambda(k-1)\}^n \left[ (\alpha+1)\{1+\lambda(k-1)\}^m - \alpha \right] \leq 1. \tag{22}$$

Thus

$$\frac{k(c+k)}{(c+1)} |z|^{k-1} < [1+\lambda(k-1)]^n [(\alpha+1)\{1+\lambda(k-1)\}^m - \alpha]$$

That is, if

$$|z| < \left[ \frac{(c+1)}{k(c+k)} \{1+(k-1)\lambda\}^n \left[ (\alpha+1)\{1+(k-1)\lambda\}^m - \alpha \right] \right]^{\frac{1}{k-1}}, \quad k \geq j+1 \tag{23}$$

Therefore, the function given by (19) is univalent in  $|z| < R^*$ .

Let the function  $f_\nu(z)$ , ( $\nu = 1, 2$ ) be defined by (5). The modified Hadamard product  $f_1(z)$  and  $f_2(z)$  is defined by

$$(f_1 * f_2)(z) = z - \sum_{k=j+1}^{\infty} a_{k1} a_{k2} z^k. \quad (24)$$

**Theorem 2.13.** Let each of the function  $f_\nu(z)$ , ( $\nu = 1, 2$ ) defined by (5) be in the class

$T(n, m, j, \alpha, \lambda)$ . Then  $f_1 * f_2(z) \in T(n, m, j, \alpha, \lambda)$  where

$$\beta = \frac{(1+\lambda)^n [(\alpha+1)(1+\lambda j)^m - \alpha]^2 - (1+\lambda j)^m}{(1+\lambda j)^m - 1} \quad (25)$$

The result is sharp.

**Proof.** Employing the technique used by Schild and Silverman [9], we need to find largest

$\beta = \beta(n, m, j, \alpha, \lambda)$  such that

$$\sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^n [(\beta+1)\{1 + \lambda(k-1)\}^m - \beta] \alpha_{k,1} \alpha_{k,2} \leq 1.$$

Since  $\sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^n [(\alpha+1)\{1 + \lambda(k-1)\}^m - \alpha] \alpha_{k,1} \leq 1$

and  $\sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^n [(\alpha+1)\{1 + \lambda(k-1)\}^m - \alpha] \alpha_{k,2} \leq 1.$

By the Cauchy-Schwarz inequality, we have

$$\sum_{k=j+1}^{\infty} [1 + \lambda(k-1)]^n [(\alpha+1)\{1 + \lambda(k-1)\}^m - \alpha] \sqrt{\alpha_{k,1} \alpha_{k,2}} \leq 1.$$

and thus it is sufficient to show that

$$\begin{aligned} [1 + \lambda(k-1)]^n [(\beta+1)\{1 + \lambda(k-1)\}^m - \beta] \alpha_{k,1} \alpha_{k,2} \\ \leq [1 + \lambda(k-1)]^n [(\alpha+1)\{1 + \lambda(k-1)\}^m - \alpha] \sqrt{\alpha_{k,1} \alpha_{k,2}} \end{aligned}$$

That is

$$\sqrt{\alpha_{k,1} \alpha_{k,2}} \leq \frac{[(\alpha + 1)\{1 + \lambda(k - 1)\}^m - \alpha]}{[(\beta + 1)\{1 + \lambda(k - 1)\}^m - \beta]}, \quad (k \geq j + 1).$$

Note that  $\sqrt{\alpha_{k,1} \alpha_{k,2}} \leq \frac{1}{[1 + \lambda(k - 1)]^n [(\alpha + 1)\{1 + \lambda(k - 1)\}^m - \alpha]}.$

Consequently, we need only to prove that

$$\frac{1}{[1 + \lambda(k - 1)]^n [(\alpha + 1)\{1 + \lambda(k - 1)\}^m - \alpha]} \leq \frac{[(\alpha + 1)\{1 + \lambda(k - 1)\}^m - \alpha]}{[(\beta + 1)\{1 + \lambda(k - 1)\}^m - \beta]}$$

Or, equivalently that

$$\begin{aligned} \beta \{ [1 + \lambda(k - 1)]^m - 1 \} + [1 + \lambda(k - 1)]^m &\leq [1 + \lambda(k - 1)]^n [(\alpha + 1)\{1 + \lambda(k - 1)\}^m - \alpha]^2 \\ \beta &\leq \frac{[1 + \lambda(k - 1)]^n [(\alpha + 1)\{1 + \lambda(k - 1)\}^m - \alpha]^2 [1 + \lambda(k - 1)]^m}{[1 + \lambda(k - 1)]^m - 1} \end{aligned} \tag{26}$$

Since right hand side of (26) is an increasing function of  $k$ , letting  $k = j + 1$  in equation (26), we have

$$\beta \leq \frac{(1 + \lambda j)^n [(\alpha + 1)(1 + \lambda j)^m - \alpha]^2 - (1 + \lambda j)^m}{(1 + \lambda j)^m - 1}$$

which proves the main assertion of Theorem 2.13. Finally, by taking the function

$$f_v(z) = z - \frac{1}{(1 + \lambda j)^n [(\alpha + 1)(1 + \lambda j)^m - \alpha]} z^{j+1}. \tag{27}$$

We can see the result is sharp.

**Theorem 2.14.** Let  $f_1(z) \in T(n, m, j, \alpha, \lambda)$  and  $f_2(z) \in T(n, m, j, \gamma, \lambda)$  Then

$f_1 * f_2(z) \in T(n, m, j, \xi, \lambda)$  where

$$\begin{aligned} \xi &= \xi(n, m, j, \gamma, \lambda) \\ &= \frac{(1 + \lambda j)^n [(\alpha + 1)(1 + \lambda j)^m - \alpha][(\gamma + 1)(1 + \lambda j)^m - \gamma] - (1 + \lambda j)^m}{(1 + \lambda j)^m - 1} \end{aligned}$$

The result is best possible for the function

$$f_1(z) = z - \frac{1}{(1 + \lambda j)^n [(\alpha + 1)\{1 + \lambda j\}^m - \alpha]} z^{j+1}$$

and 
$$f_2(z) = z - \frac{1}{(1 + \lambda j)^n [(\gamma + 1)\{1 + \lambda j\}^m - \gamma]} z^{j+1}$$

**Proof.** Proceeding as in the proof of Theorem 2.13, we obtain

$$\xi \leq \frac{[1 + \lambda(k-1)]^n [(\alpha + 1)\{1 + \lambda(k-1)\}^m - \alpha][(\gamma + 1)\{1 + \lambda(k-1)\}^m - \gamma] - \{1 + \gamma(k-1)\}^m}{[1 + \lambda(k-1)]^m - 1},$$

( $k \geq j+1$ )

Since the right hand side of (29) is an increasing function of  $k$ , setting  $k=2$  in (29), we obtain (28).

This completes the proof of Theorem 2.14

**Corollary 2.4.** Let the function  $f_\nu(z)$  defined by

$$f_\nu(z) = z - \sum_{k=j+1}^{\infty} a_{k,\nu} z^k, \quad (a_{k,\nu} \geq 0, \nu = 1, 2, 3) \quad (30)$$

be in the class  $T(n, m, j, \alpha, \lambda)$ .

$$(f_1 * f_2 * f_3)(z) \in T(n, m, j, \delta, \lambda),$$

where,

$$\delta = \frac{(1 + \lambda j)^{2n} [(\alpha + 1)(1 + \lambda j)^m - \alpha]^2 - (1 + \lambda j)^m}{(1 + \lambda j)^m - 1}. \quad (31)$$

The result is best possible for the functions

$$f_\nu(z) = z - \frac{1}{(1 + \lambda j)^n [(\alpha + 1)(1 + \lambda j)^m - \alpha]^2} z^{j+1}. \quad (32)$$

**Proof.** From Theorem 2.13, we have ( $\nu = 1, 2, 3$ ),

$$(f_1 * f_2)(z) \in T(n, m, j, \beta, \lambda) \text{ where } \beta \text{ is given by (25).}$$

Now, using Theorem 2.14, we get

$$(f_1 * f_2 * f_3)(z) \in T(n, m, j, \delta, \lambda) \text{ where } \delta \text{ is given by (31).}$$

This completes the proof of corollary.

**Theorem 2.15.** Let the functions  $f_v(z)$  ( $v=1,2$ ) defined by (5) be in the class  $T(n,m,j,\alpha,\delta)$ , then the function

$$h(z) = z - \sum_{k=j+1}^{\infty} (\alpha_{k,1}^2 + \alpha_{k,2}^2)z^k \tag{33}$$

belongs to the class  $T(n,m,j,\eta,\lambda)$ , where

$$\eta = \eta(n,m,j,\alpha,\lambda) = \frac{(1+\lambda j)^n [(\alpha+1)(1+\lambda j)^m - \alpha]^2 - 2(1+\lambda j)^m}{2\{(1+\lambda j)^m - 1\}}. \tag{34}$$

The result is sharp for the function defined by (27).

**Proof.** By virtue of Theorem 2.1, we have

$$\begin{aligned} & \sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^n [(\alpha+1)\{1+\lambda(k-1)\}^m - \alpha] \alpha_{k,1}^2 \\ & \leq \left[ \sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^n [(\alpha+1)\{1+\lambda(k-1)\}^m - \alpha] \alpha_{k,1} \right]^2 \\ & \leq 1, \end{aligned} \tag{35}$$

and

$$\begin{aligned} & \sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^n [(\alpha+1)\{1+\lambda(k-1)\}^m - \alpha] \alpha_{k,2}^2 \\ & \leq \left[ \sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^n [(\alpha+1)\{1+\lambda(k-1)\}^m - \alpha] \alpha_{k,2} \right]^2 \\ & \leq 1. \end{aligned} \tag{36}$$

It follows from (35) and (36) that

$$\sum_{k=j+1}^{\infty} \frac{1}{2} \left[ \{1+\lambda(k-1)\}^n \{(\alpha+1)\{1+\lambda(k-1)\}^m - \alpha\} \right]^2 (\alpha_{k,1}^2 + \alpha_{k,2}^2) \leq 1. \tag{37}$$

Therefore, we need to find the largest  $\eta$  such that

$$\begin{aligned} & \left[ \{1+\lambda(k-1)\}^n (\eta+1) \{1+\lambda(k-1)\}^m - \eta \right] \\ & \leq \frac{1}{2} \left[ \{1+\lambda(k-1)\}^n (\alpha+1) \{1+\lambda(k-1)\}^m - \alpha \right]^2 \end{aligned}$$

That is

$$n \leq \frac{\{1+\lambda(k-1)\}^n \{(\alpha+1)\{1+\lambda(k-1)\}^m - \alpha\}^2 - 2\{1+\lambda(k-1)\}^m}{2 \left[ \{1+\lambda(k-1)\}^m - 1 \right]}. \tag{38}$$

Since right hand side of (38) is an increasing function of  $k$ , we readily have

$$n \leq \frac{\{1 + \lambda j\}^n [\{(\alpha + 1)\{1 + \lambda j\}^m - \alpha\}]^2 - 2\{1 + \lambda j\}^m}{2[1 + \lambda(k - 1)^m - 1]}$$

and Theorem 2.15 follows at once.

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