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MULTI OBJECTIVE LINEAR FRACTIONAL PROGRAMMING PROBLEM: A FUZZY GOAL PROGRAMMING APPROACH

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Abstract : This paper presents fuzzy goal programming approach for the solution of multiobjective linear fractional programming problem (MOLFPP). In the FGP model formulation, firstly MOLFPP is transformed into another equivalent multiobjective programming problem and equivalence between them is established. Secondly the objectives are transformed into fuzzy goals (membership functions) by means of assigning an aspiration level to each of them and suitable membership function is defined for each objectives. Then achievement of the highest membership value of each of fuzzy goals is formulated by minimizing the negative deviational variables. The proposed methodology is illustrated with numerical example in order to support the proposed methodology.

Key Words: Multiobjective linear fractional programming problem, Fuzzy goal programming, Membership function, compromise optimal solution.

2010 Mathematics Subject Classification : 90C29, 90C32

1. Introduction

Decision-making problems such as production planning, water resource management etc. involve multiple conflicting objectives with constraints and can be described by multiple objective programming models. Wallenius [19] , Zimmermann [23, 24] , Yager [20] , Hanan [6], Narasimhan [14], Rubin and Narasimhan [17], Ying-Yung [21], Chanas [2], Rommelfanger [18], Gupta and Chakraborty [3,5] and many researchers used and modified the concept of multiobjective decision making problems and also discussed different approaches to tackle the multiobjective programming problem. Balbas and Galperin *et al.* [1] gave a sensitivity analysis in multi objective optimization. Yan and Wei *et al*. [23] constructed an efficient solution structure of multi objective linear programming.

Jain and Lachhwani [8] considered multiobjective programming problem with fuzzy relational equations. Afterwards, Jain and Lachhwani [9] obtained the solution of multi objective linear fractional programming problem by converting it into fuzzy programming problem.

Numerous methods for multi objective optimization problems have been suggested in the literature. Each method appears to have some advantages as well as disadvantages. In the context of each application, some of the methods seem more appropriate than others. However, the issue of choosing a proper method in a given context is still a subject of active research. A number of researchers have worked for fuzzy mathematical programming problem using goal programming approach like Pal and Moitra *et al.* [15] suggested a goal programming procedure for fuzzy multiobjective linear fractional programming problem. Chao-Fang *et al.* [4] proposed a generalized varying domain optimization method using fuzzy goal programming for multi objective optimization problem with priorities. Pramanik and Roy [16] gave a procedure for solving multi level programming problem in a large hierarchical decentralized organization through linear fuzzy goal programming approach. Ibrahim [7] presented fuzzy goal programming (FGP) algorithm for solving decentralized bi-level multi objective (DBL-MOP) problem with a single decision maker at the upper level and multiple decision makers at the lower level. Li and Hu [10] proposed a satisfying optimization method based on goal programming for fuzzy multi objective optimization problem with the aim of achieving the higher desirable satisfying degree. Lachhwani and Poonia [12] used fuzzy goal programming approach to solve multi level linear fractional programming problem (MLFPP). Recently, Lachhwani [11] suggested fuzzy goal programming approach to solve multi objective quadratic programming problem. Regarding the presently available procedures, a FGP approach seems to be most appropriate for multi objective programming problem.

A multi objective linear fractional programming (MOLFP) problem seeks to optimize more than one objective function in the form of a ratio in which denominator and numerator both contains linear and quadratic forms. We assume that the set of feasible solutions is a convex polyhedral with a finite number of extreme points and the denominator of the objective functions is non-zero in the constraint set. The aim of this paper is to extend FGP approach introduced by Mohamed [13] to solve MOLFP problem. The paper is organized as follows: In section 2, we discuss formulation of MOLFPP and its equivalence with another multiobjective programming problem in context of compromise optimal solution. In next section, we discuss proposed FGP approach to tackle MOLFPP and formulate mathematical models related to it. To illustrate the proposed methodology, numerical example is considered in section 4. Concluding remarks are given in the last section.

2. Problem Formulation

The general format of classical multi objective linear fractional programming (MOLFP) problem can be stated as:

$$Max.\{Z_1(X), Z_2(X),...,Z_k(X)\} \tag{1}$$

where

$$Z_i(X) = \frac{(C_i X + \alpha_i)}{(D_i X + \beta_i)} \quad \forall\ i = 1,...,k$$

subject to, $X \in S = \left\{ X \in R^n \middle| AX \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, X \geq 0, b \in R^m, \forall\ i = 1,...,k \right\}$

Here C_i and D_i $(i = 1,...,k)$ are row vectors with n-components, α_i, β_i are scalars, X and b are column vectors with n and m components respectively. It is assumed that $D_i X + \beta_i > 0$ $(i = 1,...,k)$ for all $X \in S$. Now we consider another equivalent multiobjective non linear programming problem as follows:

$$Max.\{Z_1(X), Z_2(X),...,Z_k(X)\} \tag{2}$$

where $Z_i(X) = (C_i X + \alpha_i)Y_i \quad \forall\ i = 1,...,k$

subject to, $X \in S = \left\{ X \in R^n \middle| AX \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, X \geq 0, b \in R^m, \right\}$

$(D_i X + \beta_i)Y_i = 1$,

and $\alpha_i, \beta_i \geq 0, Y_i > 0 \quad \forall\ i = 1,...,k$

Where problem (2) is obtained from (1) by the transformation $Y_i = 1/(D_i X + \beta_i)$ with the equality constraints $(D_i X + \beta_i)Y_i = 1$. Now we prove the equivalence between MOLFPP (1) and multiobjective programming problem (2) in continuation of following related definitions as:

***Definition 1*:** An ideal solution (ideal point) X^* is the finite optimal solution to the single objective programming problem *i.e.*

Maximize Z

$$X \in S = \left\{ X \in R^n \middle| AX \begin{pmatrix} \le \\ = \\ \ge \end{pmatrix} b, X \ge 0, b \in R^m, \right\},$$

Definition 2: $X^0 \in S$ is an efficient solution to problem (1) – (2) if and only if there exists no other $X \in S$ such that $Z_i \ge Z_i^0$ for all $i=1,2,\ldots,k$ and $Z_i > Z_i^0$ for at least one *i*.

For our purpose, we define ideal solution (ideal point) of single objective and compromise efficient solution for multi objective programming problem.

Definition 3. For problem (1), a compromise optimal solution is an efficient solution selected by the decision maker (DM) as being the best solution where the selection is based on the DM's explicit or implicit criteria.

Zeleny [25] as well as most authors describes the act of finding a compromise optimal solution to problem as "……an effort or emulate the ideal solution as closely as possible".

Our FGP model for determining compromise optimal (efficient) solution based on the finding of the totality or subset of efficient solutions with the DM, then choosing one best solution on some explicit or implicit algorithm.

Theorem 1. If (1) reaches at a compromise optimal solution $\overline{X} = \overline{X}^*$. Then (2) also reaches at same compromise optimal solution $(\overline{X}^*, \overline{Y_i}^*) \in S$ and the values of objective functions at these points are equal.

Proof: Let $\overline{X}^*$ be a compromise optimal solution of problem (1). It follows that corresponding values of $\overline{Y_i}^* = 1/(D_i\overline{X}^* + \beta_i)$ can be obtained using values of $\overline{X}^*$ in new introduced constraints. This implies that (2) also reaches at some compromise optimal solution $(\overline{X}^*, \overline{Y_i}^*) \in S$ and the values of the objective functions at problem (1) and (2) are equal as:

$$Z_i(\overline{X}^*) = \frac{(C_i\overline{X}^* + \alpha_i)}{(D_i\overline{X}^* + \beta_i)} = (C_i\overline{X}^* + \alpha_i)\overline{Y_i}^*$$

Now, in the field of fuzzy programming, the fuzzy goals are characterized by their associated membership functions. The linear membership function (as shown in figure 1) for the *i* th fuzzy goal can be defined as:

$$\mu_i\left(Z_i\left(X\right)\right)=\begin{cases} 0 & if\ Z_i\left(X\right)\leq \underline{Z_i} \\ \dfrac{Z_i\left(X\right)-\underline{Z_i}}{\overline{Z_i}-\underline{Z_i}} & if\ \underline{Z_i}\leq Z_i\left(X\right)\leq \overline{Z_i} \\ 1 & if\ Z_i\left(X\right)\geq \overline{Z_i} \end{cases} \tag{3}$$

Where $\overline{Z_i}$ and $\underline{Z_i}$ are the values of maximum and minimum value of each individual objective function.

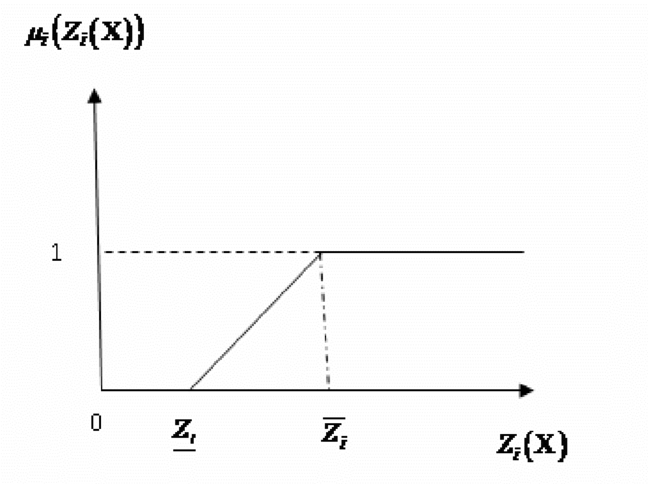

Figure 1. Membership function $\mu_i\left(Z_i\left(\mathrm{X}\right)\right)$

Also in the fuzzy decision making environment, the achievement of the objective goals to their aspired levels to the extent possible is actually represented by the possible achievement of their respective membership values to the highest degree.

3. Fuzzy Goal Programming Formulation

In fuzzy goal programming approaches, the highest degree of membership functions is 1. So, as in Mohamed [13] for the defined membership function in (3), the flexible membership goals with the aspired level 1 can be expressed as:

$$\mu_i\left(Z_i\left(X\right)\right)+d_i^- - d_i^+ = 1$$

$$i.e. \quad -\overline{Z_i}+Z_i\left(X\right)+(\overline{Z_i}-\underline{Z_i})d_i^- -(\overline{Z_i}-\underline{Z_i})d_i^+ = 0, \forall i=1,...,k \tag{4}$$

where $d_i^-(\geq 0)$ and $d_i^+(\geq 0)$ with $d_i^- d_i^+ = 0$ represent the under and over deviational variables respectively from the aspired levels. It can be easily realized that the membership goals in expression (4) are inherently non linear equation and this may reduce computational difficulties in the solution process. The i th membership goal with aspired level 1 can also be presented as:

$$-\overline{Z}_i + (C_i X + \alpha_i)Y_i + (\overline{Z}_i - \underline{Z}_i)d_i^- - (\overline{Z}_i - \underline{Z}_i)d_i^+ = 0 \tag{5}$$

In conventional GP, the under and/or over deviational variables are included in the achievement function for minimizing them and that depends upon the type of the objective functions to be optimized. In this approach, only the under deviational variables d_i^- is required to be minimized to achieve the aspired levels of the fuzzy goals. It may be noted that any over deviation from a fuzzy goal indicate the full achievement of the membership value. However, for model simplification the expression (5) can be considered as a general form of goal expression of the above stated membership goals. It may be noted that when a membership goal is fully achieved, $d_i^- = 0$, and when its achievement is zero, $d_i^+ = 1$ are found in the solution. Now, if the most widely used and simplest version of GP (*i.e.* minsum GP) is introduced to formulate the model of the problem (1) under consideration, then FGP model formulation becomes:

<u>Model I</u> Find X so as to Minimize $\chi = \sum_{i=1}^{k} w_i d_i^-$ (6)

Subject to, $-\overline{Z}_i + (C_i X + \alpha_i)Y_i + (\overline{Z}_i - \underline{Z}_i)d_i^- - (\overline{Z}_i - \underline{Z}_i)d_i^+ = 0$

$$X \in S = \left\{ X \in R^n \,\middle|\, AX \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, X \geq 0, Y > 0, b \in R^m \right\},$$

$$(D_i X + \beta_i)Y_i = 1$$

and $\alpha_i,\ \beta_i \geq 0,\ d_i^-, d_i^+ \geq 0, \quad \forall i = 1,\dots,k$

where χ represents the fuzzy achievement function consisting of the weighted under deviational variables where the numerical weights $w_i \geq 0, (\forall i = 1,\dots,k)$ represent the relative importance of achieving the aspired level of the respective fuzzy goals subject to the constraints in the decision making situation. To assess the relative importance of the fuzzy goals properly, the weighted scheme suggested by Mohamed [13] can be used to assign the values to $w_i\ (\geq 0), i = 1,...,k$. In the present formulation w_i can be determined as:

$$w_i = \frac{1}{\overline{Z_i} - \underline{Z_i}}$$

The above model can also be rewritten as:

<u>Model II</u> Find X so as to Minimize $\chi = \sum_{i=1}^{k} d_i^-$ (7)

Subject to, $-\overline{Z_i} + (C_i X + \alpha_i) Y_i + (\overline{Z_i} - \underline{Z_i}) d_i^- - (\overline{Z_i} - \underline{Z_i}) d_i^+ = 0$

$$X \in S = \left\{ X \in R^n \middle| AX \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, X \geq 0, Y > 0, b \in R^m \right\},$$

$$(D_i X + \beta_i) Y_i = 1$$

and $\alpha_i,\ \beta_i \geq 0,\ d_i^-, d_i^+ \geq 0, \quad \forall i = 1,...,k$

In model II the numerical weights are taken as unity.

<u>Model III</u> Find X so as to Minimize $\chi = \sum_{i=1}^{k} d_i^-$ (8)

Subject to, $-\overline{Z_i} + (C_i X + \alpha_i) Y_i + (\overline{Z_i} - \underline{Z_i}) d_i^- \geq 0$

$$X \in S = \left\{ X \in R^n \middle| AX \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, X \geq 0, Y > 0, b \in R^m \right\},$$

$$(D_i X + \beta_i) Y_i = 1$$

and $\alpha_i,\ \beta_i \geq 0,\ d_i^-, d_i^+ \geq 0, \quad \forall i = 1,...,k$.

However, model I, II and III can be easily solved using non linear techniques.

4. Numerical Example

The following example is considered to illustrate the above approach:

Example 1 Maximize $\{Z_1(X), Z_2(X)\}$

where $$Z_1(X) = \frac{(2x_1 + 20x_2 + 12)}{(-2x_1 - 5x_2 + 15)}$$

$$Z_2(X) = \frac{(3x_1 + 30x_2 + 51)}{(-4x_1 - 10x_2 + 30)}$$

subject to, $x_1 + 15x_2 \le 2$

$3x_1 + 20x_2 \le 4$, and $x_1, x_2 \ge 0$

Using the proposed methodology, the FGP model I is obtained as:

Model I Find X (x_1, x_2, y_1, y_2) so as to Minimize $\chi = (2.56944d_1^- + 1.88775d_2^-)$

Subject to, $(2x_1 + 20x_2 + 12)y_1 + 0.389189d_1^- - 0.389189d_1^+ - 1.189189 = 0$

$(3x_1 + 30x_2 + 51)y_2 + 0.52973d_2^- - 0.52973d_2^+ - 2.22973 = 0$

$(-2x_1 - 5x_2 + 15)y_1 = 1$

$(-4x_1 - 10x_2 + 30)y_2 = 1$

$x_1 + 15x_2 \le 2$

$3x_1 + 20x_2 \le 4$

and $x_1, x_2, d_1^-, d_2^-, d_1^+, d_2^+ \ge 0$, $y_1, y_2 > 0$

Solving the above problem using non linear techniques or software package (as shown in figure 2(a) and figure 2(b)), the compromise optimal solution obtained as:

$$x_1 = 1.33333,\ x_2 = 0\ \ y_1 = 0.08108,\ y_2 = 0.04054,$$

$$d_1^- = 0.24170 \times 10^{-5}, d_2^- = 0.45198 \times 10^{-5},\ d_1^+ = 0,\quad d_2^+ = 0.$$

Also achieved values of membership functions are: $\mu_1(Z_1(X)) = 0.99999$, $\mu_2(Z_2(X)) = 0.99994$ which are the highest membership values of each of the fuzzy goals with the FGP model I.

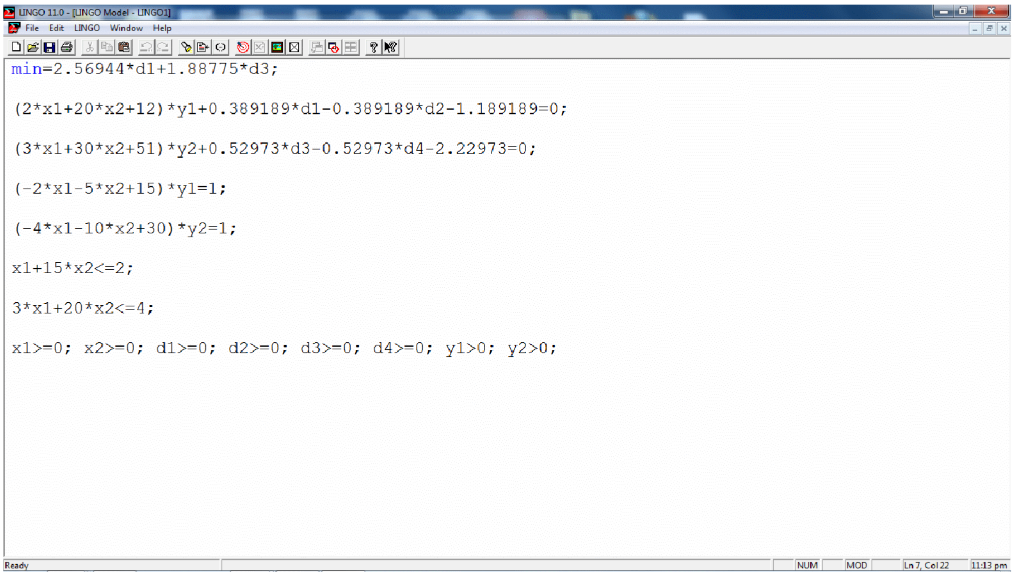

Figure 2(a). Description of model II in LINGO 11. 0 (trial version)

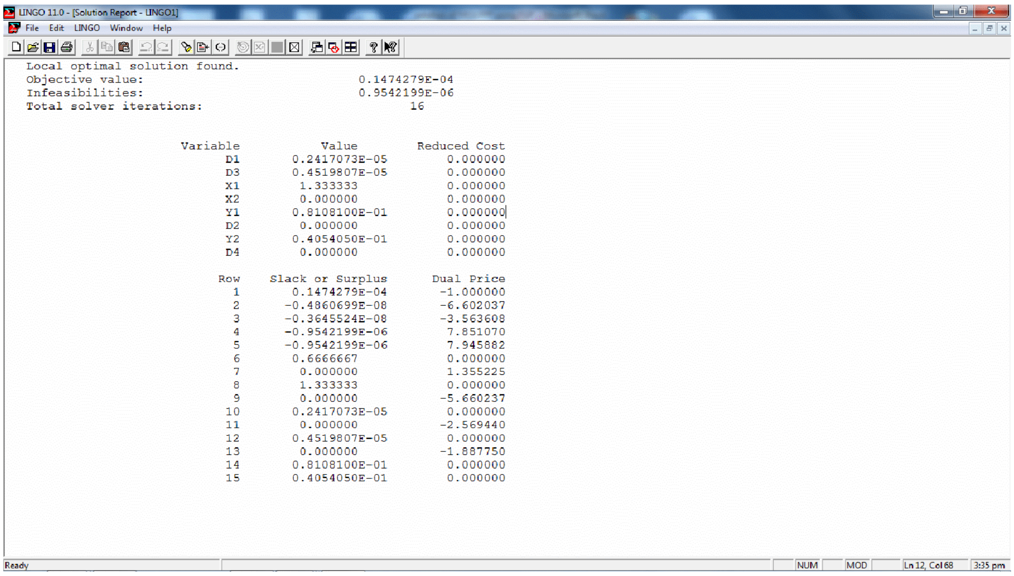

Figure 2(b). Solution of model II in LINGO 11. 0 (trial version)

5. Conclusion

An effort has been made to solve MOLFP problem using fuzzy goal programming approach. The proposed methodology yields a compromise optimal solution of MOLFP problem with a higher degree of satisfaction. The proposed technique is efficient and requires less computational work as it finally converts MOLFP problem into non linear programming problem which can be easily solved using non linear techniques or software packages like LINGO, CPLEX etc. Certainly there are many other points for future research in the area MOLFP problem based on fuzzy goal programming approach and should be studied. Some of these points are:

(i) An efficient algorithm should be carried out to solve multiobjective linear fractional programming problem with homogeneous constraints using fuzzy goal programming approach.

(ii) A solution methodology is required to treat multiobjective integer programming problem with fuzzy goal programming approach.

References

[1] Balbas, A., Galperin, E. and Jimenez Guerra, P. (2005). Sensitivity in multi objective optimization: The generalized evolvent theorem, *Non Linear Analysis* **63**, 725-733.

[2] Chanas, S. (1989). Fuzzy programming in multi objective linear programming: A parametric approach, *Fuzzy Sets and System* **29**, 303 – 313.

[3] Chakraborty, M. and Gupta, S. (2005). Fuzzy mathematical programming for multi objective linear fractional programming problem, *Fuzzy Sets and Systems* **125**, 335-342.

[4] Chao-Fang, Hu., Chang-Jun, T. and Shao-Yuan, Li. (2007). A fuzzy gaol programming approach to multi-objective optimization problem with priorities, *European Journal of Operational Research* **176**, 1319-1333.

[5] Gupta, S. and Chakraborty, M. (1997). Multi objective linear programming: A fuzzy programming approach, *International Journal of Management and System* **13**(2), 207- 214.

[6] Hanan, E. L. (1979). On the efficiency of the product operator in fuzzy programming with multiple objective, *Fuzzy Sets and Systems* **2**, 259-263.

[7] Ibrahim, A. B. (2009). Fuzzy goal programming algorithm for solving decentralized bi- level multiobjective programming problem, *Fuzzy Sets and Systems* **160**, 2701-2713.

[8] Jain, S. and Lachhwani, K. (2009). Multiobjective programming problem with fuzzy relational equations, *International Journal of Operations Research* **6**(2), 55-63.

[9] Jain, S. and Lachhwani, K. (2009). Solution of a multi objective fractional programming problem using fuzzy programming approach, *Proc. Nat. Acad. Sci. India Sect. A.* **79**, 267-272.

[10] Li, S. and Hu, C. (2009). Satisfying optimization method based on goal programming for fuzzy multiple objective optimization problem, *European Journal of Operational Research* **193**, 329-341.

[11] Lachhwani, K. (2012). Fuzzy goal programming approach to multi objective quadratic programming problem, *Proc. Nat. Acad. of Sci. India*, In Press: **82**(4), doi:10.1007/s40010-012-0040-x

[12] Lachhwani, K. and Poonia, M.P. (2012). Mathematical solution of multi level fractional programming problem with fuzzy goal programming approach. *Journal of Industrial Engineering International*, Springer Open Access **8**(12) doi:10.1186/2251-712x-8-16

[13] Mohamed, R. H. (1997). The relationship between goal programming and fuzzy programming, *Fuzzy Sets and Systems* **89**, 215-222.

[14] Narasimhan, R. (1980). Goal programming in fuzzy environment, *Decision Science* **11**, 325-336.

[15] Pal, B.B., Moitra, B. N. and Maulik, U. (2003). A goal programming procedure for fuzzy multiobjective linear fractional programming problem, *Fuzzy Sets and Systems* **139**, 395-405.

[16] Pramanik, S. and Roy, T. K. (2007). Fuzzy goal programming approach to multi level programming problem, *European Journal of Operational Research* **176**, 1151-1166.

[17] Rubin, P. A. and Narasimhan, R. (1984). Fuzzy goal programming with nested priorities, *Fuzzy Sets and Systems* **14**, 115-129.

[18] Rommelfanger, H. (1989). Interactive decision making in fuzzy Linear Optimization problems, *European Journal of Operational Research* **41**, 210 -217.

[19] Wallenius, J. (1975). Comparative evaluation of some interactive approaches to multicriteria optimization, *Management Science* **21**, 1387-1396.

[20] Yager, R. R. (1979). Mathematical programming with fuzzy constraints and preference on the objective, *Kybernates* **8**, 258-291.

[21] Ying-Yung, F. (1983). A method using fuzzy mathematical programming to solve vector maximum problem, *Fuzzy Sets and Systems* **9**, 129-136.

[22] Yan, H., Wei, Q. and Wang, J. (2005). Constructing efficient solution structure of multi objective linear programming, *Journal of Mathematical Analysis and Application* **307**, 504-523.

[23] Zimmermann, H. J. (1978). Fuzzy programming and linear programming with several objectives function, *Fuzzy Sets and Systems* **1**, 46-55.

[24] Zimmermann, H. J. (1981). Fuzzy mathematical programming, *Computers & Operations Research* **4**, 291-298.

[25] Zeleny, M. (1982), *Multiple Criteria Decision Making*, McGraw-Hill book company, New York.