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A new subclass of close-to-convex functions with fekete-szego problem

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Abstract : In this paper, a new subclass K_s (t, λ, *A*, *B*) of close-to-convex functions, defined by means of subordination is investigated. Some results such as inclusion relationships, coefficient estimates, covering theorem, distortion property and Fekete-Szego problem for this class are derived. The results obtained here is extension of earlier known work.

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic and univalent in the open unit disk $U := \{z : z \in C \quad and \quad |z| < 1\}$.

Let *S*, S^* and *K* denote the usual subclasses of *A*, whose members are univalent, starlike and close-to-convex in **U**, respectively. Also let $S^*(\alpha)$ denote the class of starlike functions of order α, $0 \le \alpha < 1$.

For any two analytic functions *f* and *g* in **U**, we say that *f* is subordinate to *g* in **U**, written as $f(z) \prec g(z)$. if there exist a Schwarz function *w (z)* such that f (z) = g (w(z)), for $z \in$**U**. In particular, if *g* is univalent in **U**, then *f* is subordinate to *g* iff $f(0) = g(0)$ and $f(U) \subset g(U)$.

Gao and Zhou [2] discussed a class K_s of analytic functions related to the starlike functions, that is the subclass of $f(z) \in S$ satisfying the inequality

$$\Re\left(\frac{z^2 f'(z)}{g(z)g(-z)}\right) < 0 \qquad (z \in U),$$

where $g \in S^*\left(\frac{1}{2}\right)$.

Recently various interesting generalization of the class K_s have been studied from number of different viewpoints. We choose to recall here the investigations by Wang *et al.* ([13],[14]), Kowalczyk and Less-Bomba [6], Xu *et al.* [15], Seker [10], Cho *et al.* [1] and Goswami *et al.* [3] that introduced the generalization of class K_s and provided some properties for analytic functions in each classes.

In class K_s the assumption that g (z) is starlike of order $\frac{1}{2}$ makes the function $\frac{-g(z)g(-z)}{z}$ starlike. So instead of $\frac{-g(z)g(-z)}{z}$, we can consider $\frac{g(z)g(tz)}{tz}$ with $0 < |t| \leq 1$, because if $g(z) \in S^*\left(\frac{1}{2}\right)$ then $\frac{g(z)g(tz)}{tz}$ is also a starlike function which motivates us to define a new subclass $K_s(t, \lambda, A, B)$ of close-to-convex functions as follows:

1.1 Definition : Let the function f (z) be analytic in **U** and normalized by (1.1), then $f \in K_s(t, \lambda, A, B)$ if there exists a function $g(z) \in S^*\left(\frac{1}{2}\right)$ such that

$$\frac{zf'(z) + \lambda z^2 f''(z)}{\frac{g(z)g(tz)}{tz}} \prec \frac{1 + Az}{1 + Bz} \tag{1.2}$$

$(0 \leq \lambda \leq 1, -1 \leq B < A \leq 1, z \in \mathbf{U})$.

For t = -1, we obtain the class $K_s(\lambda, A, B)$, recently studied by Wang and Chen [14]. The class $K_s(t, \lambda, A, B)$ contains the several known classes of analytic functions for different suitable selections of parameters. We also note that the class $K_s(t, \lambda, A, B)$ is closely related to the class $MK(A, B)$ for meromorphic functions [11].

The transformation $\frac{1 + Az}{1 + Bz}$ involve in the class $K_s(t, \lambda, A, B)$ is analytic and convex univalent in **U**. Moreover

$$0 \le \frac{1-\mathrm{A}}{1-\mathrm{B}} < \Re\left(\frac{1+\mathrm{Az}}{1+\mathrm{Bz}}\right) < \frac{1+\mathrm{A}}{1+\mathrm{B}} \qquad (-1 \le B < A \le 1, \quad z \in \mathbf{U}).$$

In the present work, by using the principle of subordination we obtain inclusion relation, coefficient estimates, covering theorem, distortion property and Fekete-Szego problem for functions in the functional class $K_s(t, \lambda, A, B)$. Result obtained here unify and extend the known results due to corresponding results by Wang and Chen [14], Gao and Zhou [2], Kowalczyk and Les-Bomba [5], Cho *et al.* [1] etc.

Throughout our present discussion, we assume that $-1 \le B < A \le 1$, $0 < |t| \le 1$, $0 \le \lambda \le 1$.

2. Properties of starlike functions

In beginning we prove the following result of starlike functions:

Theorem 2.1 *Let* $0 \le \alpha_1 + \alpha_2 + ... + \alpha_n - n + 1 < 1$, $0 < t_i \le 1$ and $\phi_i \in S^*(\alpha_i)$; $i = 1, 2, ..., n$. Then

$$\frac{1}{z^{n-1}} \prod_{i=1}^{n} \frac{1}{t_i} \phi_i(t_i z) \in S^*(\alpha_1 + \alpha_2 + ... + \alpha_n - n + 1).$$

Proof: Let $_i \in S^*(\alpha_i)$, by definition we know that

$$\Re\left(\frac{z\phi_i(z)}{\phi_i(z)}\right) > \alpha_i; \qquad i = 1, 2, ..., n.$$

Next, we suppose

$$h(z) = \frac{1}{z^{n-1}} \prod_{i=1}^{n} \frac{1}{t_i} \phi_i(t_i z).$$

Then, we easily find that

$$\frac{zh'(z)}{h(z)} = -(n-1) + \sum_{i=1}^{n} \frac{t_i z \phi_i'(t_i z)}{\phi_i(t_i z)}$$

it follows

$$\Re\left(\frac{zh'(z)}{h(z)}\right) = -(n-1) + \sum_{i=1}^{n} \Re\left(\frac{t_i z \phi_i'(t_i z)}{\phi_i(t_i z)}\right)$$

$$> (\alpha_1 + \alpha_2 + \ldots + \alpha_n - n + 1)$$

which concludes the result .

For n = 2, $\alpha_1 = \alpha_2 = \frac{1}{2}$ and $t_1 = 1$, Theorem 2.1 gives the following corollary:

Corollary 2.2 *Let* $g(z) \in S^*\left(\frac{1}{2}\right)$ and $0 < |t| \leq 1$, then $\frac{g(z)g(tz)}{tz} \in S^*$.

Theorem 2.3 : Let

$$g(z) = z + \sum_{n=1}^{\infty} b_n z^n \in S^*\left(\frac{1}{2}\right). \tag{2.1}$$

Then

$$\left|b_n + b_2 b_{n-1} t + b_3 b_{n-2} t^2 + \ldots + b_{n-1} b_2 t^{n-2} + b_n t^{n-1}\right| \leq n. \qquad (0 < |t| \leq 1).$$

Proof : Let

$$G(z) = \frac{g(z)g(tz)}{tz}. \tag{2.2}$$

According to the corollary (2.2), G(z) is a member of the function class S^*. If let

$$G(z) = z + \sum_{n=2}^{\infty} c_n z^n, \tag{2.3}$$

then it is well known that

$$|c_n| \leq n. \tag{2.4}$$

Substituting the series expansions of g (z) and G (z) in (2.2) and compare the coefficients of both the sides of this equation, we get

$$c_n = b_n + b_2 b_{n-1} t + b_3 b_{n-2} t^2 + \ldots + b_{n-1} b_2 t^{n-2} + b_n t^{n-1}. \tag{2.5}$$

Now (2.4) and (2.5) give the desired result.

3. Inclusion relationships

We first prove the following inclusion relationship for the class $K_s(t, \lambda, A, B)$, which tells us that $K_s(t, \lambda, A, B)$ is a subclass of close-to-convex functions.

Lemma 3.1 : Let $\gamma \geq 0$ and $f \in K$, then

$$\frac{1+\gamma}{z^{\gamma}}\int_0^z f(t)t^{\gamma-1}dt \in K.$$

This lemma is a special case of Theorem 4, obtained by Wu [12].

Theorem 3.2 : Let $0 \leq \lambda \leq 1$ and $-1 \leq B < A \leq 1$. Then

$$K_s(t, \lambda, A, B) \subset K\ S.$$

Proof : Suppose that

$$F(z) = (1-\lambda)f(z) + \lambda z f'(z) \text{ and } G(z) = \frac{g(z)g(tz)}{tz}$$

with $f \in K_s(t, \lambda, A, B)$. Then the condition (1.2) can be written as

$$\frac{zF'(z)}{G(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in U).$$

By corollary 2.2, we know that $G \in S^*$, thus we have

$$F(z) = (1-\lambda)f(z) + \lambda z f'(z) \in K.$$

We now split it into two cases to prove

(1) when $\lambda = 0$. It is obvious that $f = F \in K$.

(2) when $0 < \lambda \leq 1$. By noting that $F = (1-\lambda)f + \lambda z f'$, we find that

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z F(t) t^{\frac{1}{\lambda}-2} dt.$$

Since

$$\gamma = \frac{1}{\lambda} - 1 \geq 0,$$

by lemma 3.1, we know that $f \in K$. Therefore

$$K_s(t, \lambda, A, B) \subset K\ S.$$

The proof of Theorem 3.2, is evidently completed.

Lemma 3.3 : *(See[8]) Let* $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$. Then

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{1+A_2 z}{1+B_2 z}.$$

Theorem 3.4 : *Let* $0 \leq \lambda \leq 1$ and $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$. Then

$$K_s(t, \lambda, A_1, B_1) \subset K_s(t, \lambda, A_2, B_2).$$

Proof : Suppose that $f \in K_s(t, \lambda, A_1, B_1)$. Then

$$\frac{tz\left(zf'(z)+\lambda z^2 f''(z)\right)}{g(z)g(tz)} \prec \frac{1+A_1 z}{1+B_1 z}.$$

Since $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$. By lemma 3.3, we have

$$\frac{tz\left(zf'(z)+\lambda z^2 f''(z)\right)}{g(z)g(tz)} \prec \frac{1+A_1 z}{1+B_1 z} \prec \frac{1+A_2 z}{1+B_2 z},$$

it follows that $f(z) \in K_s(t, \lambda, A_2, B_2)$, which implies the inclusion result.

4. Coefficient estimates

In this section, we obtain the coefficient estimates of functions belonging to the class $K_s(t, \lambda, A, B)$.

Lemma 4.1 : *(See [9]) Let*

$$h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$$

be analytic in **U** and

$$k(z) = 1 + \sum_{n=1}^{\infty} k_n z^n$$

be analytic and convex in ***U***, *if* $h(z) \prec k(z)$, then

$$|h_n| \leq |k_1| \qquad (n \in N).$$

Theorem 4.2 : Let $0 \leq \lambda \leq 1$ and $-1 \leq B < A \leq 1$. If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K_s(t, \lambda, A, B),$$

then

$$|a_n| \leq \frac{1}{1+\lambda(n-1)}\left(1+\frac{(A-B)(n-1)}{2}\right) \qquad (n \in N). \tag{4.1}$$

Proof : From the definition of $K_s(t, \lambda, A, B)$, we know that there exists a function with positive real part

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \qquad (z \in U)$$

such that

$$p(z) = \frac{zf'(z) + \lambda z^2 f''(z)}{\frac{g(z)g(tz)}{tz}} \prec \frac{1+Az}{1+Bz} \tag{4.2}$$

By lemma 4.1,we have

$$|p_n| \le (A-B) \qquad (n \in N). \tag{4.3}$$

At the same time, Theorem 2.3 gives if

$$G(z) = \frac{g(z)g(tz)}{tz} = z + \sum_{n=2}^{\infty} c_n z^n.$$

then $|c_n| \le n.$ (4.4)

Upon substituting the series expression of function $f(z)$, $G(z)$, $p(z)$ into equality (4.2) and comparing the coefficient of two sides of this equation, we get

$$na_n[1 + \lambda(n-1)] = c_n + p_1 c_{n-1} + p_2 c_{n-2} + \dots + p_{n-2} c_2 + p_{n-1} \qquad (n \in N). \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we have

$$|a_n| \le \frac{1}{1+\lambda(n-1)} \left(1 + \frac{(A-B)(n-1)}{2} \right) .$$

This evidently completes the proof of Theorem 4.2.

Theorem 4.3 : Let $g \in S^*\left(\frac{1}{2}\right)$ be a function given by (2.1) and $-1 \le B < A \le 1$, $0 \le \lambda \le 1$. If an analytic function f in **U** defined by (1.1) satisfies the inequality

$$(1+|B|)\sum_{n=2}^{\infty} \{1+\lambda(n-1)\} n |a_n| + (1+|A|) \sum_{n=2}^{\infty} |c_n| \le (A-B), \tag{4.6}$$

where for n=2,3... the coefficients c_n is given in Theorem 2.3, then $f \in K_s$(t, λ, *A*, *B*) and it is generated by g. In particular, if

$$(1+|B|)\sum_{n=2}^{\infty} \{1+\lambda(n-1)\}n|a_n| \leq (A-B),$$

then $f \in K_s$(t, λ, *A*, *B*) and is generated by g(z) = z.

Proof : We set for f given by (1.1) and g defined by (2.1)

$$M = \left|zF'(z) - \frac{g(z)g(tz)}{tz}\right| - \left|-BzF'(z) + \frac{Ag(z)g(tz)}{tz}\right|,$$

where zF' (z) = zf' (z) + λ $z^2 f$'' (z)

$$M = \left|\sum_{n=2}^{\infty} \{1+\lambda(n-1)\}na_n z^n - \sum_{n=2}^{\infty} c_n z^n\right| - \left|z(A-B) - \sum_{n=2}^{\infty} \{1+\lambda(n-1)\}na_n z^n B + A\sum_{n=2}^{\infty} c_n z^n\right| \tag{4.7}$$

Hence, for $z \in \mathbf{U}$, using (4.6) we have the inequalities

$$M \leq \sum_{n=2}^{\infty} \{1+\lambda(n-1)\}n|a_n||z|^n + \sum_{n=2}^{\infty} |c_n||z|^n$$

$$-\left((A-B)|z| + \sum_{n=2}^{\infty} \{1+\lambda(n-1)\}n|a_n||z|^n|B| - |A|\sum_{n=2}^{\infty} |c_n||z|^n\right)$$

$$= -(A-B)|z| + (1+|B|)\sum_{n=2}^{\infty} \{1+\lambda(n-1)\}n|a_n||z|^n + (1+|A|)\sum_{n=2}^{\infty} |c_n||z|^n$$

$$< \left((A-B) + (1+|B|)\sum_{n=2}^{\infty} \{1+\lambda(n-1)\}n|a_n| + (1+|A|)\sum_{n=2}^{\infty} |c_n|\right)|z|$$

$$\leq 0.$$

From the above calculation we obtain that M < 0. Thus by (4.7), we have

$$\left|zF'(z) - \frac{g(z)g(tz)}{tz}\right| < \left|-BzF'(z) + \frac{Ag(z)g(tz)}{tz}\right|, \qquad (z \in U)$$

which is equivalent to the inequality (1.2). Thus $f \in K_s$(t, λ, *A*, *B*) and it completes the proof.

5. Covering theorem

In this section, we give the covering theorem for the class K_s(t, λ, *A*, *B*)

Theorem 5.1 *Let f* ∈ K_s(t, λ, *A*, *B*). Then the unit disk **U** is mapped by *f* on a domain that contain the disk $|\omega| < r_1$, where $r_1 = \frac{2(1+\lambda)}{[2+(A-B)]+4(1+\lambda)}$

Proof : Suppose that *f* ∈ K_s(t, λ, *A*, *B*), and let ω_0 be any complex number such that $f(z) \neq \omega_0$ for $z \in$ **U**. Then $\omega_0 \neq 0$ and

$$\frac{\omega_0 f(z)}{\omega_0 - f(z)} = z + \left(a_2 + \frac{1}{\omega_0}\right) z^2 + \ldots$$

is univalent in **U** by Theorem 3.2. This leads to

$$\left| a_2 + \frac{1}{\omega_0} \right| \leq 2. \tag{5.1}$$

On the other hand, from Theorem 4.2, we know that

$$|a_2| \leq \frac{1}{(1+\lambda)}\left[1 + \frac{(A-B)}{2}\right]. \tag{5.2}$$

Combining (5.1) and (5.2), we deduce that

$$|\omega_0| \geq \frac{1}{|a_2|+2} \geq \frac{2(1+\lambda)}{[2+(A-B)]+4(1+\lambda)} = r_1,$$

we thus complete the proof of Theorem 5.1.

6. Distortion theorem

Now, we prove the distortion theorem for the class K_s(t, λ, *A*, *B*).

Theorem 6.1 : Let $0 \leq \lambda \leq 1$, -1 B < A ≤ 1 and *f* ∈ K_s(t, λ, *A*, *B*).

(i) If λ = 0, then for |z|= r < 1, we have

$$\int_0^r \frac{1-Al}{(1-Bl)(1+l^2)} dl \leq |f(z)| \leq \int_0^r \frac{1+Al}{(1+Bl)(1-l^2)} dl \qquad ....(6.1)$$

(ii) If 0 < λ ≤ 1, then for |z|= r < 1, we have

$$\frac{1}{\lambda} r^{1-\frac{1}{\lambda}} \int_0^r \int_0^s \frac{1-Al}{(1-Bl)(1+l^2)} dl \; s^{\frac{1}{\lambda}-2} ds \leq |f(z)| \leq \frac{1}{\lambda} r^{1-\frac{1}{\lambda}}$$

$$\int_0^r \int_0^s \frac{1+Al}{(1+Bl)(1-l^2)} dl \; s^{\frac{1}{\lambda}-2} ds. \tag{6.2}$$

Proof : Suppose that $f \in K_s(t, \lambda, A, B)$. Then from the definition of subordination between analytic functions, we deduce that

$$\frac{1-Ar}{1-Br} \le \frac{1-A|\omega(z)|}{1-B|\omega(z)|} \le \left| \frac{t\{z^2 f'(z) + \lambda z^3 f''(z)\}}{g(z)g(tz)} \right|$$

$$= \left| \frac{zf'(z) + \lambda z^2 f''(z)}{\frac{g(z)g(tz)}{tz}} \right| = \left| \frac{1+A\omega(z)}{1+B\omega(z)} \right|$$

$$\le \frac{1+A|\omega(z)|}{1+B|\omega(z)|} \le \frac{1+Ar}{1+Br} \qquad (|z| = r < 1) \tag{6.3}$$

where ω is a schwarz function with $\omega(0) = 0$ and $|\omega(z)| < 1 \quad (z \in \mathbf{U})$.

Since $G(z) = \frac{g(z)g(tz)}{tz}$ $(z \in \mathbf{U})$ is an starlike function. It is well known that

$$\frac{r}{1+r^2} \le |G(z)| \le \frac{r}{1-r^2} \qquad (|z| = r < 1) \tag{6.4}$$

It now follows from (6.3) and (6.4) that

$$\frac{1-Ar}{(1-Br)(1+r^2)} \le |zf'(z) + \lambda z^2 f''(z)| \le \frac{1+Ar}{(1+Br)(1-r^2)} \qquad (|z| = r < 1). \tag{6.5}$$

Upon integrating (6.5) from 0 to r, we have

$$\int_0^r \frac{1-Al}{(1-Bl)(1+l^2)} dl \le |(1-\lambda)f(z) + \lambda z f'(z)| \le \int_0^r \frac{1+Al}{(1+Bl)(1-l^2)} dl \tag{6.6}$$

To complete the proof we consider the following two cases:

(i) when $\lambda = 0$. From (6.6) we easily get (6.1).

(ii) when $0 < \lambda \le 1$. From the proof of Theorem 3.2 together with (6.6), we readily arrive at (6.2).

7. Fekete - Szego Problem

In this section we require the following lemmas:

Lemma 7.1 : *(See[5],[7])*. If $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ ($z \in \mathbf{U}$) is a function with positive real part, then for any complex number μ

$$|c_2 - \mu c_1^2| \le 2\max\{1, |2\mu - 1|\}$$

and the result is sharp for the functions given by $p(z) = \dfrac{1+z^2}{1-z^2}$ and $p(z) = \dfrac{1+z}{1-z}$ ($z \in \mathbf{U}$).

Lemma 7.2 *(See[4])*. Let $G(z) = z + \sum_{n=2}^{\infty} c_n z^n + \ldots \in S^*$.

Then $|c_3 - \lambda c_2^2| \le \max\{1, |3 - 4\lambda|\}$

which is sharp for the Koebe function k if $\left|\lambda - \frac{3}{4}\right| \ge \frac{1}{4}$ and for $\left(k(z^2)\right)^{\frac{1}{2}} = \dfrac{z}{1-z^2}$ if $\left|\lambda - \frac{3}{4}\right| \le \frac{1}{4}$.

Theorem 7.3 : For a function $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ belonging to the class $K_s(t, \lambda, A, B)$ and $\mu \in C$, the following sharp estimate holds

$$|a_3 - \mu a_2^2| \le \frac{(A-B)}{3(1+2\lambda)}\max\{1, |2\gamma_1 - 1|\} + \max\frac{1}{3(1+2\lambda)}\{1, |3 - 4\mu_1|\}$$

$$+2(A-B)\left|\frac{1}{3(1+2\lambda)} - \frac{\mu}{2(1+\lambda)^2}\right| \qquad (7.1)$$

where

$$\gamma_1 = \frac{(1+2\lambda)}{(A-B)}\left\{\frac{(A-B)(1+B)}{2(1+2\lambda)} + \frac{3\mu(A-B)^2}{8(1+\lambda)^2}\right\}$$

$$\mu_1 = \frac{3}{4}\frac{(1+2\lambda)\mu}{(1+\lambda)^2}$$

Proof : Let $f(z) \in K_s(\text{t}, \lambda, A, B)$, then

$$\frac{zf'(z)+\lambda z^2 f''(z)}{G(z)} = \frac{1+A\omega(z)}{1+B\omega(z)} \qquad (z \in U), \tag{7.2}$$

where G(z) is given by (2.2) and ω(z) is schwarz function which is analytic in **U** with ω(0) = 0 and |ω (z)| ≤ 1.

Let

$$h(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + d_1 z + d_2 z^2 + \dots \tag{7.3}$$

then $\Re e\, h(z) > 0$ and h(0) = 1. From (7.2) and (7.3), we get

$$\frac{zf'(z)+\lambda z^2 f''(z)}{G(z)} = \frac{1-A+h(z)(1+A)}{1-B+h(z)(1+B)}. \tag{7.4}$$

Using the series expansions in (7.4), we have

$$1 + (2a_2(1+\lambda) - c_2)\, z + (3a_3(1+2\lambda) - c_3 - 2c_2 a_2(1+\lambda) + c_2^2)\, z^2 + \dots$$

$$= 1 + \frac{d_1(A-B)z}{2} + \frac{(A-B)}{2}\left\{d_2 - d_1^2\left(\frac{1+B}{2}\right)\right\} z^2 + \dots \tag{7.5}$$

Equating of coefficients in (7.5) gives us

$$a_2 = \frac{2c_2 + d_1(A-B)}{4(1+\lambda)} \tag{7.6}$$

$$a_3 = \frac{1}{3(1+2\lambda)}\left(c_3 + \frac{(A-B)}{2}\left(d_1 c_2 + d_2 - \frac{d_1^2(1+B)}{2}\right)\right) \tag{7.7}$$

Therefore, we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{(A-B)}{6(1+2\lambda)}\left|d_2 - \gamma_1 d_1^2\right| + \frac{1}{3(1+2\lambda)}\left|c_3 - \mu_1 c_2^2\right|$$

$$+ \frac{(A-B)}{2}\left|c_2\right|\left[\frac{1}{3(1+2\lambda)} - \frac{\mu}{2(1+\lambda)^2}\right]\left|d_1\right|$$

Now, the desired result follows upon using lemma 7.1 and 7.2.

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