Journal of Rajasthan Academy of Physical Sciences
ISSN: 0972-6306; URL: http://raops.org.in
International Conference on Mathematical and Statistical Computation (ICMSC-2022)
Swami Keshvanand Institute of Technology (SKIT), Jaipur, Rajasthan, (India) 3rd-5th March 2022
October, 2022, 77-88

COMPUTATIONAL APPROACH FOR TRANSIENT BEHAVIOUR OF FINITE SOURCE RETRIAL QUEUEING MODEL WITH MULTIPLE VACATIONS AND RENEGING

J. Indhumathi[a*], **A. Muthu Ganapathi Subramanian**[b] and **Gopal Sekar**[c]
[a,b,c]Mathematics, Kanchi Mamunivar Government Institute for Postgraduate Studies and Research, Puducherry, India
*E-mail: indhuspb10@gmail.com (Corresponding author)

Abstract: The main objective of this research paper is to analyze a transient behaviour of finite source retrial model with multiple vacations and reneging in which the primary arrival rate λ and reneging rate ξ which follows a Poisson distribution and the service time μ follows an exponential distribution. We have assumed that the finite calling population size M. The vacation time follows an exponential distribution with parameter α. All transitions are assembled by using an infinitesimal generator matrix. Fundamental Matrix method is used to obtain the time dependent and steady state solutions. Numerical studies have been done for time dependent probabilities of server free/busy/vacation and system performance measures for various values of $M, \lambda, \mu, \xi, \alpha, \sigma$ and t.

Keywords: Finite source; Retrial queue; Reneging; Fundamental Matrix; Multiple vacations.

1. Introduction

Our aim is to analyze the transient behaviour of finite source Retrial model with multiple vacations and reneging by new computational approach. If the arriving customers who find the server is busy then go for invisible queue (orbit) and retry for a service after some random time again and again from the orbit till to get the service which is known as **Retrial queues**. The practical application of this model is call center management, computer and communication systems, cloud computing and telephone exchange system. For example, in a call center, if a customer makes a phone call when all the agents are busy, the customer will try to make a phone call again after some random time.

Retrial queues and its studies have been found in Falin (1990) [2] and Artalejo (2009, 2010) [1]. Miller [5] (1983) discussed about Matrix-geometric solutions in stochastic Models. Considerable attention on finite source queues studies from various researchers such as Velika I. Dragieva and Tuan Phung-Duc (2020) [10] studied finite-source M/G/1 retrial queue with outgoing calls. Indhumathi, Muthu Ganapathi Subramanian, Gopal Sekar (2018) [3] have analysed Finite Source queues with catastrophe.

Transient analysis and reneging studies from Sherif Ammar (2015) [6-7] explained Transient Analysis of an M/M/1 Queue with Impatient Behaviour and Multiple Vacations. Also, in 2017 he analysed transient Solution of M/M/1 Vacation Queue with a Waiting Server and Impatient Customers. Indhumathi, Muthu Ganapathi Subramanian, Gopal Sekar (2021) [4] analysed Computational approach for transient behaviour of finite source retrial queueing model with exhaustive type single vacation, loss and feedback. Li, Yue, Qi (2008) [8] studied M/G/1 queueing system with balking, reneging and multiple vacations. Swathi, Vasanta Kumar (2018) [9] analysed M/M/1 Queuing System with Customer Reneging During Server Vacations Subject to Server Breakdown and Delayed Repair.

2. The Mathematical Model and its Solutions

Consider a finite source retrial queuing model with multiple vacations and reneging in which the primary arrival rate λ follows a Poisson distribution and the service time follows an exponential distribution with parameter μ. Further, we have assumed that the calling population is finite of size M. If the server is idle then the primary arrival will be served immediately and after completion of the service, it leaves the system. An effective algorithm is used to obtain transient probabilities and time dependent system performance measures.

If the server is not idle then the arriving customer goes to orbit becomes a group of repeated customers. This group of sources of repeated customers may be viewed as a sort of queue. Every such source produces a Poisson process of repeated customers with intensity σ. If incoming repeated customers find the server free, it is served and leaves the system after service, while the source which produced this repeated customer disappears.

The concept of multiple vacations is incorporated in this work in such a way that after completion of a service if the server finds no one in the orbit then he goes for a vacation. After completion of the vacation period if the server finds atleast one customer in the orbit then he returns to the system and waiting for service otherwise he will go for vacation once again.It follows an exponential distribution with parameter α. Reneging pertains to impatient customers. Few customers become impatient after being in orbit for some time and may leave the orbit. Reneging can occur during service as well as in the orbit with parameter ξ.

The retrial policy states that the probability of an customers from an orbit to try for service during the time interval $(t, t + \Delta t)$ given that there were n customers in orbit at time t is $n\sigma \Delta t + O(\Delta t)$. This regulation for access the server from the retrial group is called a classical retrial policy**.**

3. Representation of Random Process

Let C(t) be the random variable which represents the number of customers in the orbit at time t and S(t) be the random variable which represents the status of the server at time t. The random process is described as

$\{<C(t), S(t)> / C(t) = 1,2, 3\ldots, M; S(t) = 0\} \cup \{<C(t), S(t)> / C(t) = 0,1,2, 3\ldots, M-1; S(t) = 1\} \cup \{<C(t), S(t)> / C(t) = 0,1,2, 3\ldots, M; S(t) = 2\}$, whereS (t) = 0 if the server is free/idle at time t, S(t) = 1 if the server is busy at time t &S(t) = 2 if the server is in vacation at time t.

We define,

$P_{n0}(t)$: Probability that the server is idle when there are n customers in the orbit at t.

$P_{n1}(t)$: Probability that the server is busy when there are n customers in the orbit at t.

$P_{n2}(t)$: Probability that the server is in vacation when there are n customers in the orbit at t.

Kolmogorov balanced equations of this model are given below

The server is in idle:

$$\left.\begin{array}{l} P_{10}{}'(t) = -[(M-1)\lambda+\sigma]P_{10}(t)+(\mu+\xi)P_{11}(t)+\alpha P_{12}(t) \\ P_{20}{}'(t) = -[(M-2)\lambda+2\sigma]P_{20}(t)+(\mu+2\xi)P_{21}(t)+\alpha P_{22}(t) \\ \ldots \\ P_{M0}{}'(t) = -[M\sigma]P_{M0}(t)+\alpha P_{M2}(t) \end{array}\right\} \quad (1)$$

The server is in Busy:

$$\left.\begin{array}{l} P_{01}{}'(t) = -[(M-1)\lambda+\mu]P_{01}(t)+\sigma P_{10}(t) \\ P_{11}{}'(t) = -[(M-2)\lambda+\mu+\xi]P_{11}(t)+(M-1)\lambda P_{10}(t)+(M-1)\lambda P_{01}(t)+2\sigma P_{20}(t) \\ P_{21}{}'(t) = -[(M-3)\lambda+\mu+2\xi)]P_{21}(t)+(M-2)\lambda P_{20}(t)+(M-2)\lambda\theta P_{11}(t)+3\sigma P_{30}(t) \\ \ldots \end{array}\right\} \quad (2)$$

The server is in vacation

$$\left.\begin{array}{l} P_{02}{}'(t) = -[M\lambda]P_{02}(t)+\mu P_{01}(t) \\ P_{12}{}'(t) = -[(M-1)\lambda+\alpha]P_{12}(t)+M\lambda P_{02}(t) \\ P_{22}{}'(t) = -[(M-2)\lambda+\alpha]P_{22}(t)+(M-1)\lambda P_{12}(t) \\ \ldots. \\ P_{M2}{}'(t) = -\alpha P_{M2}(t)+\lambda P_{M-12}(t) \end{array}\right\} \quad (3)$$

In general,

$$\left.\begin{array}{l} P_{n0}{}'(t) = -[(M-n)\lambda+n\sigma]P_{n0}(t)+(\mu+n\xi)P_{n1}(t)+\alpha P_{n2}(t);\ for\ n=1,2,\ldots,M-1 \\ P_{n1}{}'(t) = -[(M-(n+1))\lambda+(\mu+n\xi)]P_{n1}(t)+(M-n)\lambda P_{n0}(t)+(M-n)\lambda P_{n-11}(t) \\ \qquad\qquad +(n+1)\sigma P_{n+10}(t);\ for\ n=1,2,\ldots,M-1 \\ P_{n2}{}'(t) = -[(M-n)\lambda+\alpha]P_{n2}(t)+(M-(n-1))\lambda P_{n-12}(t);\ for\ n=1,2,\ldots,M-1 \end{array}\right\} \quad (4)$$

The infinitesimal generator matrix L for this model is given below :

$$L = \begin{pmatrix} L_{00} & L_{01} & L_{02} & L_{03} & L_{04} & \ldots\ldots & L_{0M} \\ L_{10} & L_{11} & L_{12} & L_{13} & L_{14} & \ldots\ldots & L_{1M} \\ L_{20} & L_{21} & L_{22} & L_{23} & L_{24} & \ldots\ldots & L_{2M} \\ L_{30} & L_{31} & L_{32} & L_{33} & L_{34} & \ldots\ldots & L_{3M} \\ . & . & . & . & . & \ldots\ldots & \ldots\ldots \\ L_{M\text{-}10} & L_{M\text{-}11} & L_{M\text{-}12} & L_{M\text{-}13} & L_{M\text{-}14} & \ldots\ldots\ldots & L_{M\text{-}1M} \\ L_{M0} & L_{M1} & L_{M2} & L_{M2} & L_{M3} & \ldots\ldots\ldots & L_{MM} \end{pmatrix}$$

The matrices $L_{00,}L_{01},L_{10},L_{11},L_{21},L_{22},L_{32},L_{33},\ldots,L_{M\text{-}1M\text{-}1},L_{M0},L_{M2}$ are described in the L.

The infinitesimal transition rates of process ***X*** as follows

$$L_{00} = \begin{bmatrix} \text{-}[(M-1)\lambda+\mu] & 0 \\ \mu & \text{-}M\lambda \end{bmatrix}$$

$$L_{i\,i} = \begin{bmatrix} \text{-}((M\text{ - }i)\lambda+i\sigma) & \mu+\xi & \alpha \\ (M\text{ - }i)\lambda & -[(M\text{ - }(i+1))\lambda+\mu+i\xi] & 0 \\ 0 & 0 & -((M\text{ - }i)\lambda+\alpha) \end{bmatrix} for\ i=1,2,\ldots,M\text{ -}1$$

$$L_{i-1i} = \begin{bmatrix} (M-i)\lambda & 0 \\ 0 & (M-(i-1))\lambda \end{bmatrix}; for\ i=1,2,\ldots,M\text{ -}1$$

$$L_{i\,i+1} = \begin{bmatrix} i\sigma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} for\ i=1,2,\ldots,M-1$$

$$L_{M-1M} = \begin{bmatrix} M\sigma & 0 \\ 0 & 0 \end{bmatrix}$$

$$L_{M0} = \begin{bmatrix} -M\sigma & \alpha \end{bmatrix}; \ L_{M2} = \begin{bmatrix} \lambda & 0 & -\alpha \end{bmatrix}$$

Remaining all other entries are zero.

The equations (1), (2), (3) & (4) can be combined and expressed as

$$X'(t) = LX(t),\ where\ L = R^T$$
$$\&\ [X(t)]^T = \left[P_{01}(t),P_{02}(t),P_{10}(t),P_{11}(t),P_{12}(t),\ldots,P_{M\text{-}1\,0}(t),P_{M\text{-}1\,1}(t),P_{M\text{-}1\,2}(t),P_{M0}(t),P_{M2}(t)\right]$$

Solving the equations, we get, $X(t) = e^{tL}X_0$

When t = 0, $X_0 = X(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \end{bmatrix}^T$

4. Description of Computational Method

The following effective computational procedure is used to find the Time dependent probabilities of number of customers in the orbit at time *t*. The Time dependent Probabilities is denoted by

$$X(t) = \left[P_{01}(t), P_{02}(t), P_{10}(t), P_{11}(t), P_{12}(t), ..., P_{M\text{-}1\,0}(t)P_{M\text{-}1\,1}(t)P_{M\text{-}1\,2}(t), P_{M0}(t), P_{M2}(t)\right]^T$$

Step 1: Find the Eigen values this finite order matrix tL.

Step 2: Find the corresponding Eigen vectors x_i (t) = e^{λ_i} x_i

Step 3: The matrix $\phi(t) = (x_1$ (t), x_2 (t), … x_M (t)) is called fundamental matrix and it is non-singular.

Step 4: Find x (t) =ϕ (t)$\bar{c}$, where $\bar{c} = (c_1, c_2, ..., c_M)'$ is a constant column vector.

Step 5: using x (0) = $x_{0,}$ we obtain $\bar{c} = \phi^{-1}(0)$ $x_{0,}$

Step 6: put $\bar{c}$ in step 4, we get x (t) = ϕ (t)$\phi^{-1}(0)$ $x_{0.}$

Step 7: we know that X (t) = e^{At} x_0

Step 8: substitute X (t) in step 4 and comparing $x_{0,}$ we get $e^{At} = \phi(t)[\phi\ (0)]^{-1}$

Step 9: Extract the first column of this Exponential matrix tL & store in X (t).

Step 10: This probability vector **X (t)** provides time dependent probabilities of number of customers in the queue at time t.

5. System Performance Measures

The following system measures are used to bring out the Transient behaviour of the finite source retrial queueing model with multiple vacations and reneging under Computational study for various values of M,λ, μ, ξ, α, σ and t.

a. Probability that the server is idle at time t, $P_{idle}(t) = \sum_{n=1}^{M} P_{n0}(t)$

b. Probability that the server is busy at time t, $P_{busy}(t) = \sum_{n=0}^{M-1} P_{n1}(t)$

c. Probability that the server is in vacation at time t, $P_{vacation}(t) = \sum_{n=0}^{M} P_{n2}(t)$

Mean number of customers in the orbit at time t, $L_q(t) = \sum_{n=1}^{M} n\left(P_{n0}(t) + P_{n2}(t)\right) + \sum_{n=1}^{M-1} nP_{n1}(t)$

6. Numerical Computations

Transient Probabilities and System Performance Measures and Transient probabilities of this model have been done and expressed in the form of Tables for Various Values M,λ, μ, ξ, α, σ and t. All the numerical values are done by using SCILAB.

Table 1: Transient probability distribution of no. of customers in the orbit when the server is idle in the system for $\lambda = 2$, $\mu = 3$, $\alpha = 2$, $\sigma = 1$, $\xi = 1$, M = 3 and various values of *t*.

t	P10(t)	P20(t)	P30(t)
0.1	0.0479	0.0043	0.0000
0.2	0.1177	0.0222	0.0003
0.3	0.1677	0.0501	0.0015
0.4	0.1949	0.0813	0.0040
0.5	0.2060	0.1111	0.0078
0.6	0.2077	0.1375	0.0125
0.7	0.2049	0.1600	0.0175
0.8	0.2004	0.1787	0.0223
0.9	0.1957	0.1943	0.0265
1	0.1915	0.2074	0.0297

Table 2: Transient probability distribution of no. of customers in the orbit when the server is idle in the system for $\lambda = 4$, $\mu = 5$, $\alpha = 3$, $\sigma = 2$, $\xi = 2$, M = 6and various values of *t*.

t	P10(t)	P20(t)	P30(t)	P40(t)	P50(t)	P60(t)
0.1	0.0840	0.0645	0.0260	0.0054	0.0005	0.0000
0.2	0.0541	0.0966	0.0848	0.0373	0.0069	0.0001
0.3	0.0269	0.0828	0.1163	0.0786	0.0218	0.0009
0.4	0.0141	0.0643	0.1254	0.1123	0.0398	0.0024
0.5	0.0083	0.0503	0.1250	0.1362	0.0562	0.0039
0.6	0.0054	0.0410	0.1219	0.1525	0.0691	0.0049
0.7	0.0040	0.0351	0.1189	0.1637	0.0785	0.0052
0.8	0.0032	0.0314	0.1166	0.1715	0.0849	0.0049
0.9	0.0027	0.0291	0.1151	0.1769	0.0890	0.0043
1	0.0024	0.0276	0.1143	0.1808	0.0916	0.0036

Table 3: Transient probability distribution of no. of customers in the orbit when the server is idle in the system for $\lambda = 5$, $\mu = 6$, $\alpha = 4$, $\sigma = 3$, $\xi = 2$, M = 7and various values of *t*.

t	P10(t)	P20(t)	P30(t)	P40(t)	P50(t)	P60(t)	P70(t)
0.1	0.0563	0.0691	0.0475	0.0189	0.0041	0.0004	0.0000
0.2	0.0181	0.0546	0.0844	0.0726	0.0335	0.0068	0.0002
0.3	0.0057	0.0302	0.0763	0.1024	0.0708	0.0209	0.0010
0.4	0.0023	0.0175	0.0621	0.1116	0.0989	0.0358	0.0020
0.5	0.0011	0.0115	0.0517	0.1131	0.1171	0.0472	0.0026
0.6	0.0007	0.0085	0.0452	0.1127	0.1285	0.0547	0.0026
0.7	0.0005	0.0070	0.0415	0.1123	0.1354	0.0591	0.0022
0.8	0.0004	0.0063	0.0394	0.1120	0.1396	0.0615	0.0017
0.9	0.0004	0.0058	0.0383	0.1121	0.1421	0.0627	0.0012
1	0.0003	0.0056	0.0377	0.1122	0.1436	0.0633	0.0009

From **Table 1** to **Table 3**, we infer that as the value of t increases then the Transient state goes to steady state (i.e.) $P_{idle}(t) \to P_{idle}$.

Table 4: Transient probability distribution of no. of customers in the orbit when the server is busy in the system λ = 2, μ = 3, α = 2, σ = 1, ξ =1, M = 3 and various values of *t*.

t	P01(t)	P11(t)	P21(t)
0.1	0.4981	0.2155	0.0225
0.2	0.2537	0.2515	0.0538
0.3	0.1366	0.2427	0.0777
0.4	0.0811	0.2296	0.0953
0.5	0.0548	0.2211	0.1099
0.6	0.0421	0.2168	0.1233
0.7	0.0358	0.2152	0.1359
0.8	0.0323	0.2148	0.1478
0.9	0.0303	0.2150	0.1588
1	0.0289	0.2153	0.1686

Table 5: Transient probability distribution of no. of customers in the orbit when the server is busy in the system for λ = 4, μ = 5, α = 3, σ = 2, ξ = 2, M = 6and various values of *t*.

t	P01(t)	P11(t)	P21(t)	P31(t)	P41(t)	P51(t)
0.1	0.0871	0.2426	0.2125	0.0906	0.0192	0.0016
0.2	0.0119	0.1095	0.2012	0.1713	0.0713	0.0117
0.3	0.0035	0.0557	0.1581	0.1994	0.1189	0.0273
0.4	0.0016	0.0318	0.1230	0.2046	0.1548	0.0435
0.5	0.0008	0.0202	0.0987	0.2012	0.1798	0.0576
0.6	0.0005	0.0142	0.0829	0.1961	0.1967	0.0686
0.7	0.0004	0.0109	0.0728	0.1919	0.2080	0.0763
0.8	0.0003	0.0091	0.0665	0.1890	0.2158	0.0814
0.9	0.0002	0.0080	0.0626	0.1873	0.2211	0.0846
1	0.0002	0.0073	0.0602	0.1865	0.2249	0.0864

Table 6: Transient probability distribution of no. of customers in the orbit when the server is busy in the system for λ = 5, μ = 6, α = 4, σ = 3, ξ = 2, M = 7and various values of *t*.

t	P01(t)	P11(t)	P21(t)	P31(t)	P41(t)	P51(t)	P61(t)
0.1	0.0321	0.1595	0.2267	0.1658	0.0679	0.0148	0.0014
0.2	0.0029	0.0447	0.1351	0.2012	0.1645	0.0709	0.0126
0.3	0.0007	0.0158	0.0751	0.1707	0.2058	0.1267	0.0313
0.4	0.0003	0.0070	0.0457	0.1393	0.2170	0.1663	0.0495
0.5	0.0001	0.0038	0.0315	0.1180	0.2180	0.1910	0.0631
0.6	0.0001	0.0025	0.0244	0.1053	0.2170	0.2059	0.0717
0.7	0.0000	0.0019	0.0208	0.0980	0.2162	0.2148	0.0767
0.8	0.0000	0.0016	0.0188	0.0940	0.2160	0.2201	0.0792
0.9	0.0000	0.0015	0.0178	0.0919	0.2162	0.2233	0.0804
1	0.0000	0.0014	0.0173	0.0909	0.2166	0.2251	0.0810

From **Table 4** to **Table 6**, we infer that the value of t increases then the Transient state goes to steady state (i.e.) $P_{busy}(t) \to P_{busy}$.

Table 7: Transient probability distribution of no. of customers in the orbit when the server is vacation in the system for λ = 2, μ = 3, α = 2, σ = 1, ξ =1, M = 3 and various values of *t*

t	P02(t)	P12(t)	P22(t)	P32(t)
0.1	0.1568	0.0478	0.0068	0.0004
0.2	0.1648	0.1018	0.0306	0.0036
0.3	0.1316	0.1225	0.0586	0.0111
0.4	0.0954	0.1174	0.0793	0.0218
0.5	0.0670	0.1001	0.0890	0.0333
0.6	0.0474	0.0800	0.0891	0.0435
0.7	0.0347	0.0619	0.0828	0.0513
0.8	0.0266	0.0475	0.0733	0.0561
0.9	0.0216	0.0367	0.0628	0.0583
1	0.0185	0.0291	0.0528	0.0581

Table 8: Transient probability distribution of no. of customers in the orbit when the server is vacation in the system for λ = 4, μ = 5, α = 3, σ = 2, ξ = 2, M = 6 & various values of *t*.

t	P02(t)	P12(t)	P22(t)	P32(t)	P42(t)	P52(t)	P62(t)
0.1	0.0438	0.0548	0.0414	0.0195	0.0056	0.0009	0.0001
0.2	0.0086	0.0208	0.0353	0.0391	0.0270	0.0106	0.0018
0.3	0.0018	0.0050	0.0134	0.0259	0.0326	0.0237	0.0075
0.4	0.0005	0.0013	0.0039	0.0113	0.0230	0.0278	0.0148
0.5	0.0002	0.0004	0.0011	0.0041	0.0124	0.0235	0.0199
0.6	0.0001	0.0002	0.0004	0.0014	0.0058	0.0165	0.0216
0.7	0.0001	0.0001	0.0002	0.0005	0.0025	0.0104	0.0206
0.8	0.0001	0.0001	0.0001	0.0002	0.0011	0.0061	0.0180
0.9	0.0001	0.0001	0.0001	0.0001	0.0005	0.0034	0.0149
1	0.0000	0.0000	0.0001	0.0001	0.0002	0.0019	0.0119

Table 9: Transient probability distribution of no. of customers in the orbit when the server is vacation in the system λ = 5, μ = 6, α = 4, σ = 3, ξ = 2, M = 7 and various values of *t*.

t	P02(t)	P12(t)	P22(t)	P32(t)	P42(t)	P52(t)	P62(t)	P72(t)
0.1	0.0179	0.0322	0.0374	0.0284	0.0142	0.0045	0.0008	0.0001
0.2	0.0015	0.0045	0.0113	0.0209	0.0262	0.0213	0.0100	0.0021
0.3	0.0002	0.0005	0.0017	0.0053	0.0125	0.0199	0.0186	0.0077
0.4	0.0001	0.0001	0.0003	0.0010	0.0037	0.0101	0.0169	0.0128
0.5	0.0000	0.0000	0.0001	0.0002	0.0009	0.0038	0.0109	0.0142
0.6	0.0000	0.0000	0.0000	0.0000	0.0002	0.0013	0.0058	0.0128
0.7	0.0000	0.0000	0.0000	0.0000	0.0001	0.0004	0.0028	0.0103
0.8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0013	0.0077
0.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0006	0.0055
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0038

From **Table 7** to **Table** 9, we infer that the value of t increases then the Transient state goes to steady state (i.e.) $P_{vacation}(t) \to P_{vacation}$.

Table 10: Time dependent system measures for λ = 2, μ = 3, α = 2, σ = 1, ξ =1, M = 3 & various values of *t*

t	$P_{idle}(t)$	$P_{busy}(t)$	$P_{vacation}(t)$	$L_q(t)$
0.1	0.0522	0.7361	0.2117	0.3794
0.2	0.1403	0.5590	0.3008	0.6959
0.3	0.2193	0.4569	0.3238	0.9434
0.4	0.2801	0.4060	0.3139	1.1310
0.5	0.3248	0.3858	0.2894	1.2703
0.6	0.3577	0.3822	0.2601	1.3723
0.7	0.3824	0.3869	0.2307	1.4459
0.8	0.4015	0.3950	0.2036	1.4978
0.9	0.4165	0.4040	0.1795	1.5335
1	0.4286	0.4129	0.1586	1.5569

Table 11: Time dependent system measures for λ = 4, μ = 5, α = 3, σ = 2, ξ = 2, M = 6 & various values of *t*

t	$P_{idle}(t)$	$P_{busy}(t)$	$P_{vacation}(t)$	$L_q(t)$
0.1	0.1804	0.6537	0.1659	1.5626
0.2	0.2798	0.5770	0.1432	2.4362
0.3	0.3273	0.5629	0.1098	2.9555
0.4	0.3582	0.5592	0.0826	3.2719
0.5	0.3798	0.5584	0.0618	3.4633
0.6	0.3949	0.5590	0.0461	3.5768
0.7	0.4053	0.5603	0.0344	3.6421
0.8	0.4124	0.5620	0.0256	3.6779
0.9	0.4171	0.5638	0.0191	3.6960
1	0.4203	0.5655	0.0143	3.7037

Table 12: Time dependent system measures for λ = 5, μ = 6, α = 4, σ = 3, ξ = 2, M = 7& various values of *t*

t	$P_{idle}(t)$	$P_{busy}(t)$	$P_{vacation}(t)$	$L_q(t)$
0.1	0.1964	0.6683	0.1354	2.1765
0.2	0.2702	0.6320	0.0978	3.2634
0.3	0.3074	0.6262	0.0665	3.8493
0.4	0.3303	0.6249	0.0448	4.1656
0.5	0.3444	0.6255	0.0301	4.3334
0.6	0.3529	0.6269	0.0202	4.4198
0.7	0.3580	0.6284	0.0136	4.4625
0.8	0.3609	0.6299	0.0091	4.4821
0.9	0.3627	0.6312	0.0061	4.4899
1	0.3637	0.6322	0.0041	4.4920

From **Table 10** to **Table 12**, we infer that

1. $P_{busy}(t)$ increases as arrival rate λ increases for all values of t.
2. $P_{vacation}(t)$ decreases as arrival rate λ increases for all values of t.
3. As the value of t increases then the Transient state goes to steady state (i.e.) $P_{idle}(t) \to P_{idle}, P_{busy}(t) \to P_{busy}, P_{vacation}(t) \to P_{vacation}, L_q(t) \to L_q$

7. Special Cases

As $\sigma \to 0$ & $\xi \to 0$, this model coincides with transient behaviour of single server finite source queueing model with multiple vacations.

8. Conclusion

A new computational approach was used to evaluate the transient solution of finite source retrial queueing model with multiple vacations and reneging using Fundamental matrix method and infinitesimal generator matrix. Numerical studies have been done in an elaborate manner to determine the transient behaviour of probability distributions and time dependent System performance measures. Further we have to extend this work with multi sever queues and various other parameters.

Acknowledgement: The authors are thankful to the Referee for valuable comments and suggestions.

References

[1] Artalejo, J.R. (2010). Accessible bibliography on retrial queues: Progress in 2000-2009, Mathematical and Computer Modelling, **51**, 1071-1081.

[2] Falin, G. (1990). A survey of retrial queues, Queueing System, **7**, 127-167.

[3] Indhumathi, J., Muthu Ganapathi Subramanian, A. & Gopal Sekar (2018). Analysis of finite source queueing system with catastrophe, International Journal of Computer Sciences and Engineering, **7**(5) 5-9.

[4] Indhumathi, J., Muthu Ganapathi Subramanian, A. & Gopal Sekar (2021). Computational approach for transient behaviour of finite source retrial queueing model with exhaustive type single vacation, loss and feedback, Malaya Journal of Mathematik, S(1), 463-467.

[5] Li, L., Yue, D., & Qi, H. (2008). M/G/1 queueing system with balking, reneging and multiple vacations, Journal of Computational Information Systems, **4**(5) 2245-225.

[6] Miller, D.R., Matrix-geometric solutions in stochastic models, An algorithmic approach, by Marcel R. Neuts, The Johns Hopkins University Press, Baltimore, (1983).

[7] Sherif Ammar (2015). Transient Analysis of an M/M/1 Queue with Impatient Behaviour and Multiple Vacations, Applied Mathematics and Computations, **260**, 97-105.

[8] Sherif Ammar (2017). Transient Solution of M/M/1 Vacation Queue with a Waiting Server and Impatient Customers. Journal of Egyptian Mathematical Society, **25**, 337-342.

[9] Swathi, Ch., Vasanta Kumar, V. (2018). Analysis of M/M/1 Queuing System with Customer Reneging During Server Vacations Subject to Server Breakdown and Delayed Repair, International Journal of Engineering & Technology, **7**, 4.10, 552-557.

[10] Velika I. Dragieva & Tuan Phung-Duc (2020). A finite-source M/G/1 retrial queue with outgoing calls, Annals of operation Research, **293**, 101-121.