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P-GEODETIC VERTEX CRITICAL GRAPHS

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Abstract: A vertex subset S of a graph G = (V, E) is said to be a geodetic set if every vertex in G is in $u - v$ geodesic for some vertices u and v in S. The minimum cardinality of such a set is the geodetic number and is denoted by g(G). A graph G is said to be p-geodetic vertex critical if for every subset $X \subseteq V(G)$ with cardinality p satisfies $g(H) \leq g(G)$, where H is the subgraph induced by $V - X$ without any isolated vertex. The decrease in geodetic number cannot be more than one and p should be minimum. In this paper the authors discussed on the p-geodetic vertex critical graphs and the structural properties of such graphs.

Keywords: geodetic, geodetic number, p-geodetic vertex critical.

AMS Subject Classification: 05C69, 05C76

1. Introduction

The graphs G discussed in this paper are simple, connected and non-directional. Degree of a vertex v in a graph G is the number of vertices which are adjacent to v. A vertex with degree one is called a pendant vertex and the vertex adjacent to the pendant vertex is called a support vertex. In a graph G, δ or $\delta(G)$ and Δ or $\Delta(G)$ represents the minimum and maximum degrees of a graph G. The distance between the vertex $v \in V(G)$ and the farthest vertex from v is the eccentricity of the vertex v and is denoted as e(v). The maximum eccentricity of the vertices in a graph G represents the diameter of G (diam(G)) and the minimum is the radius of G (rad(G)). The connectivity of a graph G is the minimum number of vertices whose removal gives a disconnected graph which is denoted as $\kappa(G)$. For a graph G = (V, E), $S \subseteq V(G)$ is said to be an independent set if no two vertices in S are adjacent. The maximum cardinality of such a set is the independence number $\alpha(G)$. A vertex subset $A \subseteq V(G)$ is said to be a covering of G if every $e \in E(G)$ is incident with at least one vertex in A and the minimum cardinality of such a set is the covering number of G which is denoted as $\beta(G)$. The maximal connected subgraph of a graph G without any cut-vertex is the block.

Let G and H be connected graphs of order m and n respectively. Corona product of G and H is denoted as $G \circ H$ and is obtained by taking one copy of G and m copies of H such that the i^{th} vertex of G is connected with every vertex in the i^{th} copy of H where $1 \leq i \leq m$. Join of G and H is denoted as G + H and is obtained by taking G U H together with all edges joining V (G) and V (H). Cartesian product of G and H is denoted as $G \square H$ with vertex set V (G) × V (H) such that two vertices (p, q) and (a, b) are adjacent if p = a and qb ∈ E(H) or q = b and pa ∈ E(G). Line graph of a graph G is denoted as L(G) and is obtained by taking V (L(G)) = E(G) such that two vertices in L(G) are adjacent iff the corresponding edges are adjacent in G. The shortest path between the vertices u and v is the $u - v$ geodesic. I[u, v] represents the set of all elements in some $u - v$ geodesic including u and v. For a graph G = (V, E), S ⊆ V(G) is said to be a geodetic set [12, 3] if I[S] = V (G), where I[S] is defined as the union of I[u, v] for any u, v ∈ S. The minimum cardinality of geodetic set is the geodetic number [12, 3] and is represented as g(G). The vertex subset S ⊆ V (G) is said to be a dominating set [13] if all vertices in S^c are adjacent to atleast one vertex in S and the minimum cardinality of such a set is the domination number (γ(G)) [13]. If the geodetic set S is also a dominating set, then S is called geodetic dominating set [11] and the minimum cardinality of such set is called the geodetic domination number [11] which is represented as $\gamma_g(G)$.

A vertex v is said to be an extreme vertex or link-complete vertex if the subgraph induced by the adjacent vertices of v is complete. The number of extreme vertices in a graph G is its extreme order and is denoted as ex(G). If the geodetic number of a graph G is equal to the extreme order of G, then the graph is called an extreme geodesic graph [3]. A graph G is said to be geodetic [16] if for any x, y ∈ V (G) is connected by a unique shortest path. Let H be the graph obtained by adding (removing) an edge e from graph G. If H is not geodetic then G is called upper (lower) geodetic critical [16]. If the geodetic (geodomination) number of a graph G reduces by the removal of any vertex, then G is called geodomination critical [17]. G is called geodomination bicritical [17] if its geodomination number reduces by removing any pair of vertices in G. A graph G is called domination vertex critical [9] or γ-critical if γ(G) reduces by removing any vertex from G. If γ(G) not decreases on removing any vertices, then G is called domination vertex stable or γ-stable [18].

Definition 1.1. A Pineapple graph K^m_n is obtained by adding n pendant vertices to a vertex in K_m provided $m \geq 3$ and $n \in N$.

Definition 1.2. An n (n > 2) crown graph is constructed by removing horizontal edges from $K_{n,n}$.

Definition 1.3. A Friendship graph Fn is of order 2n+1 and size 3n is constructed by using n copies of K_3 with a common vertex.

Definition 1.4. Glue graph of a graph G is denoted as Gg and is obtained from G by making adjacency between the vertices with same eccentricity in G.

Definition 1.5. Center of a graph G is the set of all vertices in G whose eccentricity equals to the radius of the graph and it is denoted as C(G).

Definition 1.6. A graph G is said to be self-centered if its diameter and radius are same. In other words, if C(G) = V (G), then G is called a self-centered graph. If the diameter of self-centered graph is k, then G is called k-self-centered graph.

Definition 1.7. The n-Sun graph is a graph with order 2n and which consists of K_n as its central part and an outer ring on n vertices such that each of these vertices is adjacent to both endpoints of the closest outer edge of K_n. For example, consider 4-Sun graph as shown in figure 1, it consists K_4 as its central part and an outer ring of 4 vertices such that each of these vertices is adjacent to the both end points of the closest outer edge of K_4.

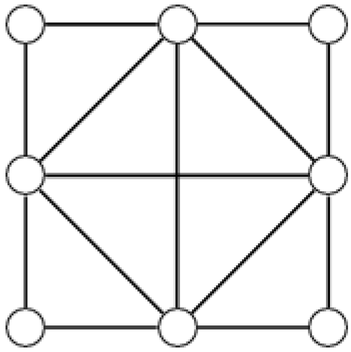

Figure 1: 4-Sun

Definition 1.8. The density (D) of a graph G = (V, E) is defined as $D = \frac{2|E|}{|V|(|V-1|)}$.

Definition 1.9. [15] A graph G is said to be k-total domination edge critical or k – γ_t–edge critical if $\gamma_t(G) = k$ and $\forall$ e $\in$ E(G), $\gamma_t(G + e) < k$. If $\gamma_t(G + e) = k – 2$, $\forall$ e $\in$ E(G), then G is called supercritical graph.

Theorem 1.1. [3, 8, 10, 11] Let G be a non-trivial connected graph with order n and diameter d, then

(i) $g(G) \leq n - d + 1$.

(ii) $\Delta \leq n - d + 1$

(iii) $2 \leq \max\{g(G), \gamma(G)\} \leq \gamma_g(G) \leq n$

(iv) $\gamma_g(G) \leq n - \lfloor \frac{2d}{3} \rfloor$.

Theorem 1.2. [7, 13] Let G be a connected graph with order n, diameter d and minimum degree δ where δ > 1, then

(i) $\left\lceil \frac{d+1}{3} \right\rceil \leq \gamma(G)$.

(ii) $d \leq \frac{3n}{\delta+1} - 1$.

Theorem 1.3. [3, 4] Set of extreme vertices in a graph G is a subset of every geodetic set in G.

Theorem 1.4. [1] Let G be a graph with order n and have k connected components. Then $|E(G)| \leq \frac{(n-k+1)(n-k)}{2}$.

2. p-Geodetic vertex critical graphs

In this section, we introduced and studied the new concept of critical graph called p-geodetic vertex critical graphs. The graphs G and the induced subgraphs of G discussed here are without any isolated vertices.

Definition 2.1. A graph G is said to be p-geodetic vertex critical if for all $X \subseteq V(G)$ with cardinality p satisfies $g(H) \leq g(G)$, where H is the subgraph induced by $V - X$ without any isolated vertex. The decrease in geodetic number cannot be more than one and p should be minimum. For example, consider the graph C_5 which is given in Figure2. The set S = {a, c, d} forms the minimum geodetic set and g(G) = 3. G′ be the graph obtained by the removal of any one of the vertex from G. Then G′ is isomorphic to P_4. But $g(P_4) = 2 < g(G)$. Hence C_5 is 1- geodetic vertex critical.

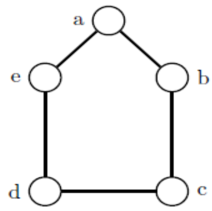

Figure 2: C_5

Theorem 2.1. Complete graph K_n (n > 2) is 1-geodetic vertex critical.

Proof. Clearly $g(K_n) = n$ [12]. Let $v \in V(K_n)$, then the subgraph induced by $V(K_n) - \{v\}$ is isomorphic to K_{n-1}. But $g(K_{n-1}) = n-1$. Hence K_n is 1-geodetic vertex critical.

Corollary 2.1.1. Glue graph of a self-centered graph is 1-geodetic vertex critical.

Proof. Let G be a self-centered graph with order n. Then its glue graph Gg ≅ Kn. Hence Gg is 1-geodetic vertex critical.

Corollary 2.1.2. The line graph of a Star graph (S_n) is 1-geodetic vertex critical.

Proof. Let $L(S_n)$ be the line graph of S_n. But $L(S_n) \cong K_{n-1}$ [20]. Hence L(Sn) is 1-geodetic vertex critical.

Theorem 2.2. Cycle C_n is 1-geodetic vertex critical.

Proof. For the cycle graph, $g(C_n) = \begin{cases} 2; n \text{ is even} \\ 3; n \text{ is odd} \end{cases}$ [12]. Let $u \in V(C_n)$, the subgraph induced by $V(C_n) - \{u\}$ is isomorphic to P_{n-1}. But $g(P_{n-1}) = 2$ [12]. Therefore, C_n is 1-geodetic vertex critical.

Proposition 2.1. Star graph S_n is 1-geodetic vertex critical only if the vertex which is removing from Sn is any of the pendant vertex.

Proof. Since S_n has n − 1 pendant vertices, $g(S_n) = n - 1$ [12]. Let v be any of the pendant vertices in S_n. Then the subgraph induced by $V(S_n) - \{v\}$ is isomorphic to S_{n-1} which has n − 2 pendant vertices and hence geodetic number is n − 2. Hence S_n is 1-geodetic vertex critical. Suppose v is the central vertex in S_n. Then the subgraph induced by $V(S_n) - \{v\}$ is a totally disconnected graph. But we are considering only the graph without any isolated vertex.

Remark 2.1. Bistar graph $S_{m,n}$ is 1-geodetic vertex critical only if the vertex which is removing from $V(S_{m,n})$ is not any one of the central vertices.

Theorem 2.3. The complete bipartite graph $K_{m,n}$ with m, n > 2 is 1-geodetic vertex critical.

Proof. Let G be the complete bipartite graph with bipartition X and Y, where |X|=m and |Y |=n. Also, $g(K_{m,n})=\min\{m, n,4\}$[12]. Let $V(X)= \{v_1, v_2, v_3, \ldots, v_m\}$ and $V(Y) = \{u_1, u_2, u_3, \ldots, u_n\}$. Without loss of generality take m<n, then $g(K_{m,n}) = \min\{m, 4\}$. Let H_1 and H_2 be the subgraphs induced by $K_{m,n} - \{v_i\}$ and $K_{m,n} - \{u_j\}$ respectively. Then $g(H_1) = \min\{m - 1, 4\}$ and $g(H_2) = \min\{m, 4\}$. Hence $K_{m,n}$ is 1-geodetic vertex critical provided m, n > 2.

Theorem 2.4. The graph $P_n\Box K_2$ is 1-geodetic vertex critical.

Proof. Let $V(P_n\Box K_2) = \{w_1, w_2, w_3, \ldots, w_{2n}\}$. Clearly $g(P_n\Box K_2) = 2$. Let G be the subgraph induced by $V(P_n\Box K_2) - \{w_j\}$ for some $w_j \in V(P_n\Box K_2)$. Choose two distinct vertices $w_i, w_k \in V(G)$ such that $d(w_i, w_k) = diam(G)$. Then U $I[w_i, w_k] = V(G)$. Therefore, the minimum geodetic set of G is $S = \{w_i, w_k\}$ and hence g(G) = 2.

Theorem 2.5. The n-crown graph (n ≥ 3) is 1-geodetic vertex critical.

Proof. Let A and B be the partitions of n-crown graph G with $V(A) = \{a_1, a_2, \ldots, a_n\}$ and $V(B) = \{b_1, b_2, \ldots, bn\}$. Clearly g(G) = 2. Let G' is obtained by the removal of a vertex from G. Choose $a_i, b_j \in V(G')$ such that $d(a_i, b_j) = diam(G')$. Then U $[a_i, b_j] = V(G')$ and the set $\{a_i, b_j\}$ forms the minimum geodetic set. Therefore g(G) = 2.

Theorem 2.6. For each m ∈ N and n > 4, there exists a graph G with order n and g(G) = n − m such that G is both 1-geodetic vertex critical and extreme geodesic.

Proof. Consider the graph $G = K_{p1} \cup K_{p2} \cup \ldots \cup K_{pt} + K_m$ as shown in Figure3, where $n = p_1+p_2+\ldots+p_t+m$. The vertices in each Kpi ($1 \le i \le t$) are extreme and $I[V(K_{p1} \cup K_{p2} \cup \ldots \cup K_{pt})] = V(G)$. Therefore g(G) = ex(G) = n − m and hence G is an extreme geodesic graph [4]. Let H be the subgraph induced by $V(G) - \{v\}$ for any $v \in V(G)$. If $v \in V(G) - V(K_m)$, then H has n − m − 1 extreme vertices which forms the minimum geodetic set of H. Therefore g(H) = n − m − 1. If $v \in V(K_m)$, then g(H) = n − m. Hence G is 1-geodetic vertex critical.

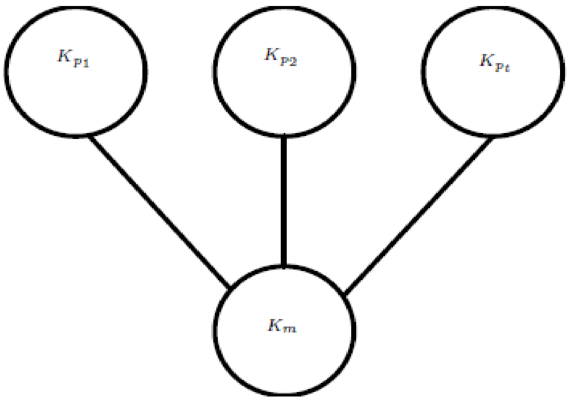

Figure 3: $K_{p1} \cup K_{p2} \cup \ldots \cup K_{pt} + K_m$

Corollary 2.6.1. The Friendship graph F_n is 1-geodetic vertex critical.

Proof. From the above theorem, Fn = $\bigcup_{i=1}^{n} K_{2i}$ +K_1. Fn consists of 2n extreme vertices and one cut vertex. Clearly $g(F_n) = 2n$. Let H is obtained by the removal of the cut vertex. Then H contain n P_2' s and g(H) = 2n. If H is obtained by the removal of any of the extreme vertex in F_n, then H is a connected graph with 2n − 1 extreme vertices. These set of extreme vertices forms the minimum geodetic set of H. Hence g(H) = 2n − 1 and therefore F_n is 1-geodetic vertex critical.

Corollary 2.6.2. Let G be a non-trivial connected graph with order n and g(G) = n − 1. Then G is 1-geodetic vertex critical.

Proof. Since g(G) = n − 1, G ≅ $\bigcup_{i=1}^{t} K_{ni}$ + K_1 [19] where $n_1 + n_2 + \ldots + nt + 1 = n$. Then from theorem 2.6 G is 1-geodetic vertex critical.

Remark 2.2. Line graph of a 1-geodetic vertex critical graph need not be 1-geodetic vertex critical.

For example, consider the butterfly graph F_2 which is 1-geodetc vertex critical. $L(F_2)$ represents the line graph of the butterfly graph as shown in Figure 4. The vertices a and d are extreme and the set S = {a, d} forms the minimum geodetic set of $L(F_2)$. Therefore $g(L(F_2)) = 2$. Construct the new graph L by removal of the vertex d from V $(L(F_2))$. Then the vertices a, c and e are the extreme vertices in L and the minimum geodetic set is S' = {a, c, e}. Hence g(L) = 3. Similarly, the geodetic number of the induced subgraph formed by the removal of the vertex a from V $(L(F_2))$ is 3. Therefore $L(F_2)$ is not 1-geodetic vertex critical. The geodetic number of the induced subgraph formed by the removal of any vertex other than a and d is 2.

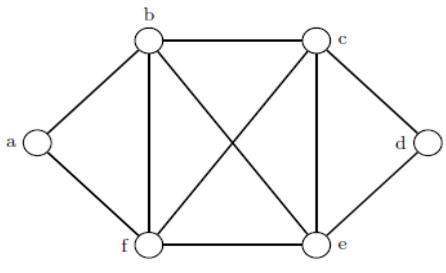

Figure 4: $L(F_2)$

Observation 2.1. The Wheel graph W_n (n > 7) is not 1-geodetic vertex critical.

Theorem 2.7. Let G be a 2-self-centered graph with order n ≥ 5 and having minimum edges. Then G is 1-geodetic vertex critical only if G is the Petersen graph.

Proof. The 2-self-centered graph G with order n ≥ 5 and having minimum number of edges can be any one of the following graphs [2].

(i) G is constructed by adding an additional vertex v to $S_{m,n}$ and joining v to each of the pendant vertices in $S_{m,n}$ as shown in Figure5. The set S = {a, b, v} forms the minimum geodetic set of G and hence g(G) = 3. The subgraph induced by V(G) − {v} is isomorphic to $S_{m,n}$ and its geodetic number is m + n. Hence G is not 1-geodetic vertex critical.

(ii) G is formed from K_3(p, q, r) by adding a new vertex w and joining w to each of the pendant vertices in K_3(p, q, r) as shown in Figure 6. Clearly g(G) = 4 and the set S = {v_1, v_2, v_3, w} forms the minimum geodetic set of G. Let H be the graph constructed by removing the vertex w from G. Then g(H) = p + q + r > 4. Hence G is not 1-geodetic vertex critical.

(iii) G is Petersen graph: Geodetic number of Petersen graph is 4 and it can be easily verified that the geodetic number of subgraphs induced by V (G) − {v} is 3 for any v ∈ V (G).

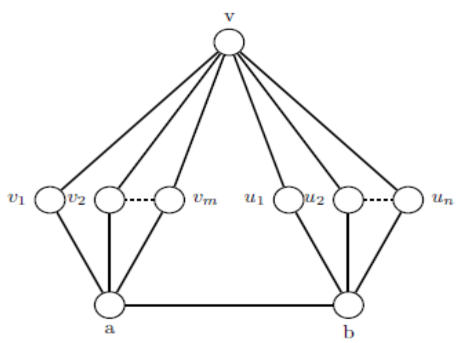

Figure 5: Graph mentioned in (i) of proof of the theorem 2.7

Theorem 2.8. A nested triangle graph is 1-geodetic vertex critical.

Proof. A nested triangle graph G consist of 3n vertices and n triangles. Also diam(G) = n. Choose the vertices u, v and w such that d(u, v) = d(u, w) = n, where v and w are in the outer most triangle (A) and u is in the inner most triangle (B) as shown in Figure7.

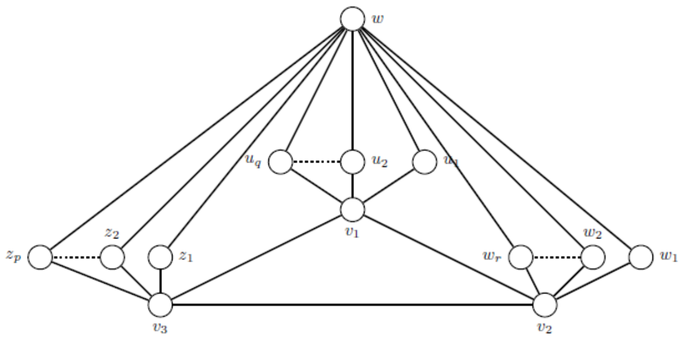

Figure 6: Graph mentioned in (ii) of proof of the theorem 2.7

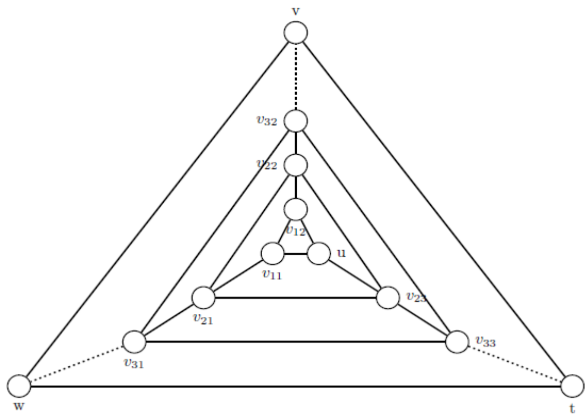

Figure 7: Nested triangle graph

Then U I[u, v] U U I[u, w] = V (G). Hence, the set, S = {u, v, w} forms the minimum geodetic set of G and hence g(G) =3. Let H be the subgraph induced by V (G) − {x} for any x ∈ V (G).

Claim: g(H) = 2 or 3.

Case 1: x ∈ B

Choose the vertices y and z from B and w from A such that d(y, w) = d(z, w) = diam(H) = n. Also, U I[y, w] U U I[z, w] = V (H). Then the set S_1 = {y, z, w} forms the minimum geodetic set of H and hence g(H) = 3.

Case 2: x ∈ A

Choose the vertices a and b from A and c from B such that d(a, c) = d(b, c) = diam(H) = n. As similar to the previous case, the set S_2 = {a, b, c} forms the minimum geodetic set of H and hence g(H) = 3.

Case 3: $x \notin A$ and $x \notin B$

Choose the vertices u from B and v from A such that d(u, v) = n + 1. Then U I[u, v] = V (H). Hence the set $S_3 = \{u, v\}$ forms the minimum geodetic set of H and hence g(H) = 2.

In all the three cases, g(H) is either 2 or 3. Hence G is 1-geodetic vertex critical.

Theorem 2.9. Let H be any non-trivial connected graph of order p, then $H \circ S_m$ $(m > 2)$ is 1-geodetic vertex critical.

Proof. For the graph $H \circ S_m$, $g(H \circ S_m) = p(m - 1)$. The pendant vertices in each copy of S_m are extreme in $H \circ S_m$. Let H_1 be the subgraph induced by $V(H \circ S_m) - \{v\}$ for any $v \in V(H \circ S_m)$. Let v be any of the extreme vertex in $H \circ S_m$. Then $g(H_1) = (p - 1)(m - 1) + m - 2 = p(m - 1) - 1$. Let $v \in V(H)$, then H_1 has $m - 1$ pendant vertices and $(p - 1)(m - 1)$ extreme vertices which together forms the minimum geodetic set of H. Therefore $g(H_1) = (p - 1)(m - 1) + m - 1 = p(m - 1)$. Suppose v is a central vertex in some i^{th} copy of S_m then also we can verify that $g(H_1) = p(m - 1)$. Hence $H \circ S_m$ $(m > 2)$ is 1-geodetic vertex critical.

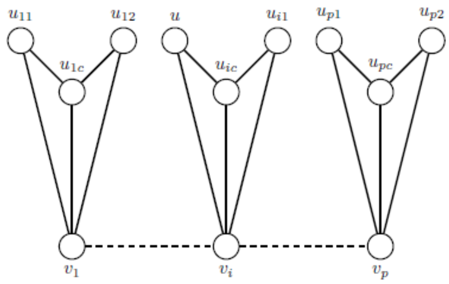

Figure 8: $G \circ S_3$

Theorem 2.10. The Pineapple graph K^p_q $(p > 2)$ is 1-geodetic vertex critical.

Proof. For a Pineapple graph, $g(K^p_q) = p-1+q$. Let $V(K^p_q) = \{u_1, u_2, \ldots u_p, w_1, w_2, \ldots w_q\}$, where $w_i (1 \leq i \leq q)$ represents the pendant vertices in the graph and these pendant vertices are linked to the vertex u_j. Let H be the subgraph induced by $V(K^p_q) - \{u\}$ for any $u \in V(K^p_q)$. If u is any of the pendant vertices, then $g(H) = p+q - 2$. Also, $u \neq u_j$, otherwise H contains isolated vertices. Let u be any one of the vertexes in K_p other than u_j then H contains q pendant vertices and p -2 extreme vertices which together forms the minimum geodetic set of H and hence Pineapple graph is 1-geodetic vertex critical.

Theorem 2.11. Let G be any non-trivial connected graph with order n. If $\delta(G) > 1$, then $G \circ K_m$ is 1-geodetic vertex critical.

Proof. In $G \circ K_m$, vertices in each copy of K_m are extreme. Let S be the set of these extreme vertices. Since $I[S] = V(G \circ K_m)$, S is a minimum geodetic set of $G \circ K_m$ and $g(G \circ K_m) = nm$. For any $v \in V(G \circ K_m)$, H be the subgraph induced by $V(G \circ K_m) - \{v\}$. If $v \in S$, then H contains $nm - 1$ extreme vertices which forms the minimum geodetic set of H. Therefore $g(H) = nm - 1$. Let $\delta(G) = 1$ and u be a pendant vertex in G with the support vertex w. Construct H by removing w from $G \circ K_m$. Then H contains two components such that u is also extreme in H. Then H contains nm + 1 extreme vertices and therefore $G \circ K_m$ is not 1-geodetic vertex critical when $\delta(G) = 1$. For $\delta(G) > 1$ and $v \in S^c$, we can easily verify that H contains nm extreme vertices and g(H) = nm. Therefore, if $\delta(G) > 1$, $G \circ K_m$ is 1-geodetic vertex critical.

Theorem 2.12. The n-Sun graph is 1-geodetic vertex critical.

Proof. In n-Sun graph G, vertices which are not in K_n are extreme. Let S denote the set of these extreme vertices. But I[S] = V(G). Hence g(G) = n. Let H be the subgraph induced by V (G) − {v} for some v ∈ V (G). Let $v \in S^c$ and u, w ∈ S such that v is adjacent to u and w in G. Clearly in H, u and w are pendant vertices. These two pendant vertices together with the n − 2 extreme vertices in H forms the minimum geodetic set of H and hence g(H) = n. Now consider v ∈ S.

Case 1: n = 3

H consists of two extreme vertices and hence g(H) ≥ 2. We can easily verify that these two vertices do not form the geodetic set. Adding one more vertex from S^c to these extreme vertices, we get a minimum geodetic set. Hence g(H) = 3.

Case 2: n > 3

Let $S_1 = S - \{v\}$. Clearly S_1 is the set of extreme vertices in H and $I[S_1] = V(H)$. Therefore g(H) = n − 1. Hence n-Sun graph is 1-geodetic vertex critical.

Theorem 2.13. Union of 1-geodetic vertex critical graphs is 1-geodetic vertex critical.

Proof. Let the graph G be the union of 1-geodetic vertex critical graphs H_i where $1 \leq i \leq n$. Then, for any $v \in V(H_i)$, $g(H_i - v) = g(H_i)$ or $g(H_i - v) = g(H_i) - 1$. Clearly $g(G) = g(H_1) + g(H_2) + \ldots + g(H_n)$ and v ∈ V (G) is a vertex in some H_i. Then g(G − v) =g(G) or g(G − v) = g(G) − 1. Hence G is 1-geodetic vertex critical

Corollary 2.13.1. Every supercritical graph is 1-geodetic vertex critical.

Proof. Since G is supercritical, $G \cong K_{p1} \cup K_{p2} \cup \ldots \cup K_{pn}$ where each $p_i > 1$ $(1 \leq i \leq n)$ [14]. But from theorem 2.1, each K_{pi} is 1-geodetic vertex critical. Then from the above theorem G is 1-geodetic vertex critical.

Remark 2.3. The above theorem does not hold for p-geodetic vertex critical graphs with p > 1. For example, take $G_1 = G_2 = K_4 - \{e\}$ where $e \in E(K_4)$. Clearly $K_4 - \{e\}$ is 2-geodetic vertex critical but not $G_1 \cup G_2$.

The following theorem gives an example for a general p-geodetic (p > 1) vertex critical graph.

Theorem 2.14. Let G be a connected graph obtained by removing an edge from K_p (p > 3), then G is p − 2 geodetic vertex critical.

Proof. Let u_i and u_j be the two non-adjacent vertices in G such that $d(u_i, u_j) = 2$ and V (G) = {u1, u2, . . . , up}. Since $\cup I[u_i, u_j] = V(G)$, g(G) = 2. Let H be the induced subgraph formed by the removal of vertices in G. If H is formed by the removal of u_i, or u_j, then $H \cong K_{p-1}$ and g(H) = p−1 > 2. Hence G is not 1-geodetic vertex critical. Clearly for the graph H, which is constructed by the removal of $p - k$ ($1 \leq k \leq p - 3$) vertices including u_i, or u_j or both has the geodetic number greater than 2. Hence G is not geodetic vertex critical. Now construct H by the removal of $p - 2$ vertices, then $H \cong K_2$. Hence $g(H) = 2$ and therefore G is a $p - 2$ geodetic vertex critical graph. H is constructed by the removal of $p - 2$ vertices other than ui and uj is a totally disconnected graph. Hence that case is neglected.

3. Structural properties of 1-geodetic vertex critical graphs

Proposition 3.1. If G is a connected 1-geodetic vertex critical graph of order n ($n > 4$) and $g(G) \leq 3$. Then G is a block.

Proof. Let H be the subgraph induced by $V(G) - \{x\}$ for any $x \in V(G)$. Since G is 1-geodetic vertex critical, either $g(H) = 2$ or $g(H) = 3$. Assume G has atleast one cut-vertex say v. Then the subgraph induced by $V(G) - \{v\}$ has atleast two components and hence $g(H) \geq 4$, which is a contradiction. Therefore, G has no cut vertex and hence G is a block.

Remark 3.1. The connectivity of a 1-geodetic vertex critical graph G with $g(G) \leq 3$ cannot be one.

Proposition 3.2. Let G be a triangle free graph with order n and $\delta > 1$. If G is 1-geodetic vertex critical, then for all $v \in V(G)$

(i) $g(G - v) \leq \beta(G)$.

(ii) $g(G - v) \leq 2|M|$ where M is a maximal matching of G.

Proof. For a triangle free graph with $\delta > 1$ satisfies the inequalities $\gamma_g(G) \leq n - \alpha(G)$ and $\gamma_g(G) \leq 2|M|$ where M is a maximal matching of G [8]. But $g(G) \leq \gamma_g(G)$ and $\alpha(G) + \beta(G) = n$. Since G is 1-geodetic vertex critical, $g(G - v) = g(G)$ or $g(G - v) = g(G) - 1$, $\forall v \in V(G)$. Then $g(G - v) \leq \alpha(G) + \beta(G) - \alpha(G) = \beta(G)$ and $g(G - v) \leq 2|M|$ $\forall v \in V(G)$.

Proposition 3.3. Let G be a 1-geodetic vertex critical graph with order n and have at least one universal vertex, then G satisfies $g(G - e) \leq g(G - v) + 3$, $\forall e \in E(G)$ and $\forall v \in V(G)$.

Proof. In a graph G, $\forall e \in E(G)$, $\gamma_g(G - e) \leq \gamma_g(G) + 2$ [8]. Since $\gamma(G) = 1$, $\gamma_g(G) = g(G)$ [8]. Then we have $g(G - e) \leq g(G) + 2$. But $g(G) = g(G - v)$ or $g(G) = g(G - v) + 1$ $\forall v \in V(G)$. Hence $g(G - e) \leq g(G - v) + 3$.

Proposition 3.4. Let G be a 1-geodetic vertex critical graph with order n and diameter d, then $\frac{1}{2}(g(G)-1+\lfloor \frac{5d}{3} \rfloor) \leq n$.

Proof. For any $v \in V(G)$, $g(G - v) = g(G)$ or $g(G - v) = g(G) - 1$. But, $\gamma_g(G - v) \leq \Delta + \gamma_g(G) - 1$, for any $v \in V(G)$ [8]. Then from theorem 1.1, $g(G - v) \leq n - d + 1 + n - \lfloor \frac{2d}{3} \rfloor - 1$. On simplifying we get the result.

Theorem 3.1. Let G be a 1-geodetic vertex critical graph with $\delta > 1$. Then for any $v \in V(G)$, the graph $G - v$ has atmost $\frac{g(G)}{2}$ components.

Proof. We have g(G − v) = g(G) or g(G − v) = g(G)-1. Assume G − v has more than $\frac{g(G}{2}$ components. But the geodetic number of each of these components is atleast two. Then g(G − v) > g(G) which is a contradiction. Hence G − v has atmost $\frac{g(G)}{2}$ components.

Theorem 3.2. Let G be a 1-geodetic vertex critical graph with order n and diameter d. Then

$$D_H \leq \frac{2(1-d) + (2n - g(G))^2}{4(n-1)(n-2)}$$

where D_H is the density of the graph H which is formed by removal of any vertex v from G.

Proof. Since G is 1-geodetic vertex critical, from theorem 3.1 H has atmost $\frac{g(G)}{2}$ components. Then from theorem 1.4

$$|E(H)| \leq \frac{\left(n - \frac{g(G)}{2}\right)\left(n - 1 - \frac{g(G)}{2}\right)}{2}$$

$$= \frac{(2n - g(G))(2n - 2 - g(G))}{8}$$

$$= \frac{2(g(G) - 2n) + (2n - g(G))^2}{8}$$

From theorem 1.1 we have g(G) ≤ 2n − d + 1. Therefore g(G) − 2n ≤ 1 − d. Hence

$$|E(H)| \leq \frac{2(1-d) + (2n - g(G)^2}{8}$$

The density of H is given by

$$D_H = \frac{2|E(H)|}{(n-1)(n-2)}$$

$$\leq \frac{2(1-d) + (2n - g(G))^2}{4(n-1)(n-2)}$$

Observation 3.1. Let G be a 1-geodetic vertex critical graph. If ex(G) /= 0, then g(G) ≥ γ(G).

4. Conclusion

In this paper, the authors introduced and studied the p-geodetic vertex critical graphs and their structural properties. In this study, the authors have focused mainly on the 1-geodetic vertex critical graphs. Further studies can be done for p-geodetic vertex critical graphs for p > 1 and also the criticality concerning the removal of edges.

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