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A BACKLOGGED REPLENISHMENT POLICY WITH LINEAR INCLINING DEMAND AND VARIABLE DETERIORATION

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Abstract: In this paper, an EOQ (Economic Order Quantity) model is presented for deteriorating items with generalized deterioration and linear inclining demand over a finite time horizon. Shortages in this model are allowed and backlogged. The concept of variable deterioration is also taken into consideration. In addition, the shortages are neither completely lost nor completely backlogged assuming the backlogging rate which is reciprocal to the waiting time for the next replenishment. The present model is formulated by assuming the ascending order of the shortages period and the cycle length of the system. Approximate expressions are derived for the determination of the shortages period, scheduling period, maximum inventory level and total average cost. The main objective of the model is to compute the optimal cost by minimizing both the shortage time and the ordering cycle time of the system. The results of the system are illustrated by a couple of numerical examples and sensitivity analysis on some parameters is carried out.

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1. Introduction

In real market situations, the harvest of food grains like wheat, sugarcane, paddy etc. is annual. Since there are a large number of people living in the rural area of India, there will

be a constant demand of these food grains throughout the year. Due to poverty, many of them are bound to sell a part of their food grains after the harvest. Therefore, that section of people cannot store enough food grains to last the whole production cycle. As a result, the demand of food grains increases with the time during a cycle. In last few years, inventory problems involving time variable demand patterns received the attention of several researchers. In reality, many physical goods deteriorate due to damage, spoilage, dryness, evaporation, etc. overtime during their normal storage period. The deterioration also depends on the preserving facility and inventory levels. The nature of deterioration cannot be ignored while determining the optimal inventory levels. Determination of the optimal models has been the subject of study for many inventory researchers. Owing to the natural process “deterioration”, Whitin [24] presented the model where the deterioration of items starts at the end of prescribed period of storage. The concept of deterioration is not new in nowadays. Items like volatile liquids, alcohol, gasoline, human blood, vegetable, radioactive substances, etc. having no self-life at all are called the decaying items. Many researchers studied inventory problems and their solutions from time to time. Ghare and Schrader [7] first pointed out the effect of the process “decay” in inventory analysis. However, they studied a simple economic order quantity (EOQ) model with exponential decay. Aggarwal [1] also reconsidered Ghare and Schrader’s [7] model with constant rate of deterioration. Later, researchers studied the inventory models with parameters depending upon the time in one form or the other. Dave and Patel [6] considered such models with variable demand. However, the deterioration rate in all these models also was assumed to be constant. Models with time-dependent deterioration of inventories were developed by Covert and Philip [5] with instantaneous replenishment. Gupta and Agarwal [11] studied an EOQ inventory model without shortage by considering both time-dependent deterioration and linearly increasing demand. Nahamias [16], Raafat [19] and Goyal and Giri [8], Li et al. [14] and Bakker et al. [2] presented a complete and up-to-date survey of perished inventory literature for deteriorating inventory models.

In several existing models, it is observed that items have long life, but there are items like food grains, fruits, photographic films, electronics items, etc do not have a long life. Besides, certain types of inventories deteriorate with time. Therefore, the decay of items plays a vital role in practical situations. The first attempt to describe the optimal policies for deteriorating items was made by Hollier and Mak [12]. The standard EOQ (Economic Order Quantity) model assumes a constant and known demand rate over infinite planning horizon. However, most of the items experience a suitable demand pattern during the saturation phase of their life cycle and for a finite period of time. Furthermore, the demand rate is deterministic and varies with time: modification of the EOQ model is clearly required. Many studies have extended the EOQ model to accommodate time-varying demand patterns. Wee [23] and Jalan et al. [13] presented their model by considering rapid changes in demand pattern. Bhunia and Maiti [3] presented a deterministic inventory replenishment problem for deteriorating items with time-dependent linear decreasing demand and shortages over the finite time horizon. Ghosh and Chaudhuri [9] presented the model with the generalized demand as a quadratic function of time and constant deterioration.

As partial backlogging is a function of waiting time, the willingness of a customer to wait during this period is reciprocal to the length of the waiting time. It is applicable to food grains, fruits, fashionable items, and high-tech products like supercomputers, laptops, android mobiles etc. In this context, Goswami and Chaudhuri [10] assumed the deterministic demand pattern as a linear function of time in their models with shortages. Chang and Dye [4] developed the deteriorating model with time-varying demand and partial backlogging. Ouyang et al. [17] studied the backlogging inventory model with exponential declining demand in which the backlogging rate is reciprocal to the waiting time for the next replenishment. Pradhan and Tripathy [18] formulated the three-parameter Weibull distributed inventory model assuming partial backlogging and shortages. Mashud [15] described the inventory model with different types of demand and fully backlogged shortages including demand as a linear function of time. The three-parameter Weibull distributed inventory model with a quadratic demand pattern is presented by Singh et al. [20]. Some models developed with the consideration of quadratic holding cost. In this regard, Swain et al. [22] formulated the model with both generalized demand pattern and deterioration. Singh et al. [21] developed an optimal policy for deteriorating items with trapezoidal-type demand, generalized variable deterioration and shortages.

In this paper, an EOQ model is developed in which (1) shortages in the system are permitted and partially backlogged, (2) variable deterioration rate is assumed, and (3) the demand is a linear increase in time. The main objective of the present model is to determine the shortage period, the length of the cycle, the ordering quantity and the average total cost. The obtained solutions are verified by a couple of numerical examples. The sensitivity analysis of one of the numerical examples concerning various parameters of the system is also discussed. Finally, a summary and suggestions for future research of the model are provided.

2. Assumptions and Notations

The following assumptions of the inventory system are stated.

- (i) The system needs a single type of item.
- (ii) Replenishment occurs at an infinite rate.
- (iii) Demand rate is a linear inclining function of time.
- (iv) Deterioration rate is a variable function of time.
- (v) Neither replacement nor repairing of deteriorated units during the time period under consideration.
- (vi) A small part of demand for the shortages is considered as partially backlogged. The backlogging rate acts as a variable and is dependent on the length of the waiting time for the next replenishment.

The following notations are used.

$$\theta_0(t) = \alpha t, \quad 0 < \alpha \ll 1: \text{variable deterioration rate.}$$

$$D(t) = \begin{cases} a + bt, & I(t) > 0 \\ D_0, & I(t) \leq 0 \end{cases} : \text{represents linear inclining demand at any time } t, \text{ where}$$

a & b ($a \geq 0$ & $b \neq 0$) and D_0 are positive constants. Here the constants a and b represent the initial and self-increased demand rate.

$B_0(t) = \frac{1}{1 + \delta_0(T-t)}$: backlogging rate for the negative inventory. Here $\delta_0 > 0$ denotes the backlogging parameter.

c_1, c_2, c_3, c_4 & c_5 : cost of holding the inventory \$ per unit per unit time, cost of deterioration of the item \$ per unit time, cost of replenishment of the item \$ per order, cost of shortage \$ per unit time and cost of opportunity due to lost sales \$ per unit, respectively.

t_s & T : starting time of the shortage and span of each cycle, respectively.

W_{\max}, S_{\max} & I_0 : the highest level for each ordering cycle, a quantity of backlogged demand, and EOQ for each ordering cycle, respectively.

$I(t)$: Inventory level at any instant t .

$AT(t_s, T)$: Minimum total average cost.

3. The Mathematical Formulation

At the start of the model, the maximum level of inventory is W_{\max} units of items at time $t = 0$. During the period $[0, t_s]$, the inventory level reduces to demand and deterioration. At that time t_s , the inventory level achieves zero, after that partially backlogged shortages are allowed during the time interval $[0, t_s]$. (See Figure 1)

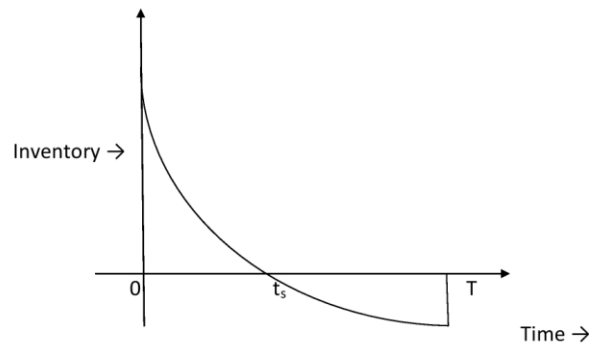


Fig. 1: Inventory-Time relationship.

The differential equations governing the inventory status $0 \leq t \leq T$ can be written as

$$\frac{dI(t)}{dt} + \theta_0(t)I(t) + D(t) = 0, \quad 0 \leq t \leq t_s \quad (1)$$

where $\theta_0(t) = \alpha t$ (variable deterioration) and $D(t) = a + bt$ (linear inclining demand rate);

and

$$\frac{dI(t)}{dt} + D_0 B_0(t) = 0, \quad t_s \leq t \leq T \quad (2)$$

where $B_0(t) = \frac{1}{1 + \delta_0(T-t)}$, $\delta_0 > 0$.

From Eq. (1), the integrating factor (I.F.) is

$$I.F. = e^{\int \alpha t dt} = e^{\frac{\alpha t^2}{2}} \quad (3)$$

and the solution is given by

$$I(t) \cdot e^{\frac{\alpha t^2}{2}} = -\int (a + bt) e^{\frac{\alpha t^2}{2}} dt + c, \text{ where } c \text{ is the constant of integration.}$$

Ignoring the terms containing the higher power of α ($0 < \alpha \ll 1$) and using the boundary condition $I(t_s) = 0$, the solution of Eq. (1) is given by

$$I(t) = \left[\begin{array}{l} \left\{ a \left(t_s + \frac{\alpha t_s^3}{6} \right) + b \left(\frac{t_s^2}{2} + \frac{\alpha t_s^4}{8} \right) \right\} - at \\ -\frac{1}{2} \left\{ b + \alpha \left(at_s + \frac{bt_s^2}{2} \right) \right\} t^2 + \frac{\alpha at^3}{3} + \frac{\alpha bt^4}{8} \end{array} \right], \quad 0 \leq t \leq t_s \quad (4)$$

The maximum inventory is obtained by substituting the condition $I(0) = W_{\max}$ and is given by

$$I(0) = W_{\max} = I(t) = a \left(t_s + \frac{\alpha t_s^3}{6} \right) + b \left(\frac{t_s^2}{2} + \frac{\alpha t_s^4}{8} \right). \quad (5)$$

The solution of Eq. (2) with $I(t_s) = 0$ is given by

$$I(t) = \frac{D_0}{\delta_0} \left[\ln \{1 + \delta_0 (T - t)\} - \ln \{1 + \delta_0 (T - t_s)\} \right], \quad t_s \leq t \leq T \quad (6)$$

Maximum amount of backlogged demand per cycle is computed from Eq. (6) by setting $S_{\max} = -I(T)$, i.e.,

$$S_{\max} = -I(T) = \frac{D_0}{\delta_0} \left[\ln \{1 + \delta_0 (T - t)\} \right]. \quad (7)$$

Hence, the EOQ per cycle is given by

$$I_0 = W_{\max} + S_{\max} = a \left(t_s + \frac{\alpha t_s^3}{6} \right) + b \left(\frac{t_s^2}{2} + \frac{\alpha t_s^4}{8} \right) + \frac{D_0}{\delta_0} \left[\ln \{1 + \delta_0 (T - t_s)\} \right]. \quad (8)$$

The total minimum average cost equals the ratio of the sum of the cost holding the inventory, cost of deterioration, cost of replenishment, cost of shortages and the cost of opportunity due to lost sales with the cycle length of the system.

(i) Cost of holding the inventory (CI):

$$\begin{aligned} CI &= c_1 \int_0^{t_s} I(t) dt \\ &= c_1 \int_0^{t_s} \left[\left\{ a \left(t_s + \frac{\alpha t_s^3}{6} \right) + b \left(\frac{t_s^2}{2} + \frac{\alpha t_s^4}{8} \right) \right\} - at \right. \\ &\quad \left. - \frac{1}{2} \left\{ b + \alpha \left(at_s + \frac{bt_s^2}{2} \right) \right\} t^2 + \frac{\alpha at^3}{3} + \frac{\alpha bt^4}{8} \right] dt \\ &= c_1 \left[\frac{a}{2} \left(t_s^2 + \frac{\alpha t_s^4}{6} \right) + \frac{b}{3} \left(t_s^3 + \frac{\alpha t_s^5}{5} \right) \right], \end{aligned} \quad (9)$$

(by ignoring terms containing $\alpha^2, \alpha^3, \dots$; as $0 < \alpha \ll 1$).

(ii) Cost of deterioration (CW):

$$\begin{aligned} CW &= c_2 \left[W - \int_0^{t_s} D(t) dt \right] = c_2 \left[W - \int_0^{t_s} (a + bt) dt \right] \\ &= \alpha c_2 \left(\frac{at_s^3}{6} + \frac{bt_s^4}{8} \right). \end{aligned} \quad (10)$$

(iii) Cost of replenishment (CR):

$$CR = c_3. \quad (11)$$

During the shortage period, two cases need to be determined. The first one is to derive the shortage cost for the backlogged items, and the other is to determine the opportunity cost due to lost sales.

(iv) Cost of shortages (CB):

$$\begin{aligned} CB &= c_4 \left[-\int_{t_s}^T I(t) dt \right] \\ &= \frac{c_4 D_0}{\delta_0} \left[T - t_s - \frac{1}{\delta_0} \ln [1 + \delta_0 (T - t_s)] \right]. \end{aligned} \quad (12)$$

(v) Cost of opportunity due to lost sales (CS):

$$\begin{aligned} CS &= c_5 D_0 \int_{t_s}^T \left[1 - \frac{1}{1 + \delta_0 (T - t)} \right] dt \\ &= c_5 D_0 \left[T - t_s - \frac{1}{\delta_0} \ln [1 + \delta_0 (T - t_s)] \right]. \end{aligned} \quad (13)$$

Therefore, $AT = AT(t_s, T)$, the average minimum total cost per unit time per cycle is

$$\begin{aligned} AT &= AT(t_s, T) = \frac{1}{T} [CI + CW + CR + CB + CS] \\ &= \frac{1}{T} \left[c_1 \left\{ \frac{a}{2} \left(t_s^2 + \frac{\alpha t_s^4}{6} \right) + \frac{b}{3} \left(t_s^3 + \frac{\alpha t_s^5}{5} \right) \right\} + \alpha c_2 \left(\frac{a t_s^3}{6} + \frac{b t_s^4}{8} \right) \right] \\ &\quad + \frac{1}{T} \left[c_3 + \frac{D_0 (c_4 + \delta_0 c_5)}{\delta_0} \left\{ T - t_s - \frac{1}{\delta_0} \ln [1 + \delta_0 (T - t_s)] \right\} \right]. \end{aligned} \quad (14)$$

Thus, the optimal values of shortages, the cycle and the minimum average cost are obtained from the system of equations

$$\frac{\partial AT(t_s, T)}{\partial t_s} = 0 \quad (15)$$

and

provided that

$$\frac{\partial^2 AT(t_s, T)}{\partial t_s^2} > 0, \quad \frac{\partial^2 AT(t_s, T)}{\partial T^2} > 0 \quad \& \quad \frac{\partial^2 AT(t_s, T)}{\partial t_s^2} \times \frac{\partial^2 AT(t_s, T)}{\partial T^2} - \left(\frac{\partial^2 AT(t_s, T)}{\partial t_s \partial T^2} \right)^2 > 0$$

(Given in the Appendix).

From Eq. (15), we have

$$\frac{\partial AT(t_s, T)}{\partial t_s} = \frac{1}{T} \left[c_1 \left\{ a \left(t_s + \frac{\alpha t_s^3}{3} \right) + b \left(t_s^2 + \frac{\alpha t_s^4}{3} \right) \right\} + \frac{\alpha c_2}{2} (a t_s^2 + b t_s^3) \right]$$

$$\frac{\partial AT(t_s, T)}{\partial T} = 0. \tag{16}$$

$$-\frac{1}{T} \left[\frac{D_0 (c_4 + \delta_0 c_5) (T - t_s)}{1 + \delta_0 (T - t_s)} \right] = 0, \tag{17}$$

$$-\frac{c_3}{T^2} - \frac{D_0 (c_4 + \delta_0 c_5)}{\delta_0 T^2} \left[T - t_s - \frac{1}{\delta_0} \ln \{ 1 + \delta_0 (T - t_s) \} \right] = 0. \tag{18}$$

and from Eq. (16), we have

$$\frac{\partial AT(t_s, T)}{\partial T} = \frac{1}{T} \left[\frac{D_0 (c_4 + \delta_0 c_5) (T - t_s)}{1 + \delta_0 (T - t_s)} \right]$$

$$-\frac{1}{T^2} \left[c_1 \left\{ \frac{a}{2} \left(t_s^2 + \frac{\alpha t_s^4}{6} \right) + \frac{b}{3} \left(t_s^3 + \frac{\alpha t_s^5}{5} \right) \right\} + \alpha c_2 \left(\frac{a t_s^3}{6} + \frac{b t_s^4}{8} \right) \right]$$

Firstly, t_s^* & T^* , the optimal times are computed from both the Eqs. (17) and (18).

Thereafter, the optimal order quantity (I_0^*) and optimal minimum average total cost per unit time ($AT^*(t_s, T)$) can be calculated by substitution of the values of t_s^* & T^* in the Eqs. (8) and (14), respectively.

4. Numerical Illustrations

The proposed inventory model is illustrated with a couple of examples with the following data sets:

Example 1:

Let

$\alpha = 0.002$; $a = 4.4$; $b = 3.2$; $D_0 = 12$; $\delta = 2$; $c_1 = 0.4$; $c_2 = 1.2$; $c_3 = 8$; $c_4 = 2.4$; $c_5 = 2$ are the parameters with their proper units.

Solving Eqs. (17) and (18), we get $t_s^* = 1.77487$ unit time and $T^* = 1.89044$ unit time. Then, the optimal order quantity is $I_0^* = 14.1135$ units and the minimum average total cost is

$$AT^*(t_s, T) = 7.20912 \text{ per unit time, provided that the conditions } \frac{\partial^2 AT(t_s, T)}{\partial t_s^2} = 30.1952 \geq 0$$

$$, \frac{\partial^2 AT(t_s, T)}{\partial T^2} = 26.8028 \geq 0 \text{ and } \frac{\partial^2 AT(t_s, T)}{\partial t_s^2} \times \frac{\partial^2 AT(t_s, T)}{\partial T^2} - \left(\frac{\partial^2 AT(t_s, T)}{\partial t_s \partial T} \right)^2 = 90.9205 \geq 0 .$$

Example 2:

Let $\alpha = 0.01$; $a = 4$; $b = 3$; $D_0 = 8$; $\delta = 2$; $c_1 = 0.5$; $c_2 = 1.5$; $c_3 = 10$; $c_4 = 2.5$; $c_5 = 2$ are the parameters with their proper units.

Solving Eqs. (17) and (18), we get $t_s^* = 1.77123$ unit time and $T^* = 2.01638$ unit time.

Then, the optimal order quantity is $I_0^* = 13.4607$ units and the minimum average total cost is $AT^*(t_s, T) = 8.55375$ per unit time, provided that the conditions $\frac{\partial^2 AT(t_s, T)}{\partial t_s^2} = 15.549 \geq 0$,

$$\frac{\partial^2 AT(t_s, T)}{\partial T^2} = 11.6113 \geq 0 \text{ and } \frac{\partial^2 AT(t_s, T)}{\partial t_s^2} \times \frac{\partial^2 AT(t_s, T)}{\partial T^2} - \left(\frac{\partial^2 AT(t_s, T)}{\partial t_s \partial T} \right)^2 = 45.7218 \geq 0 .$$

5. Sensitivity Analysis

The sensitivity analysis of the proposed system is carried out by changing the value of parameters one at a time by +25%, +5%, -5% and -25% and without changing the remaining parameters illustrated Example-1. The notable points are discussed in Table-1.

- (i) t_s^* & T^* increase while $AT^*(t_s, T)$ decreases with decreasing value of α & c_2 .
Here t_s^* , T^* & $AT^*(t_s, T)$ all are insensitive to change in α & c_2 .
- (ii) t_s^* & T^* increase while $AT^*(t_s, T)$ increases with decreasing value of a , b & c_1 .
Here t_s^* , T^* & $AT^*(t_s, T)$ are moderately sensitive to change in a , b & c_1 .
- (iii) t_s^* & $AT^*(t_s, T)$ decrease while T^* increases with decreasing value of D_0 , δ_0 , c_4 & c_5 . Here t_s^* , T^* & $AT^*(t_s, T)$ are moderately sensitive to change in D_0 , δ_0 , c_4 & c_5 .
- (iv) t_s^* , T^* & $AT^*(t_s, T)$ decrease with decreasing value of c_3 . Here t_s^* , T^* & $AT^*(t_s, T)$ are highly sensitive to change in c_3 .

Table 1: Sensitivity Analysis

Parameters	% Change in parameters	t_s^*	T^*	$AT^*(t_s, T)$	% Change in $AT^*(t_s, T)$
α	+25 %	1.77332	1.88895	7.21246	+0.04633
	+5 %	1.77456	1.89014	7.20979	+0.00929
	-5 %	1.77518	1.89073	7.20845	-0.00929
	-25 %	1.77643	1.89193	7.20577	-0.04647
a	+25 %	1.71175	1.83451	7.56964	+5.00089
	+5 %	1.76187	1.87888	7.28246	+1.01732
	-5 %	1.78807	1.90218	7.13514	-1.02620
	-25 %	1.84290	1.95112	6.83268	-5.22172
b	+25 %	1.67766	1.79917	7.50748	+4.13865
	+5 %	1.75355	1.87035	7.27173	+0.86848
	-5 %	1.79731	1.91161	7.14484	-0.89165
	-25 %	1.90072	2.00964	6.86874	-4.72152
D_0	+25 %	1.78211	1.87114	7.25543	+0.64238
	+5 %	1.77662	1.88569	7.22029	+0.15494
	-5 %	1.77292	1.89580	7.19666	-0.17284
	-25 %	1.76218	1.92664	7.12825	-1.12177
δ	+25 %	1.77889	1.88120	7.23482	+0.35649
	+5 %	1.77576	1.88841	7.21480	+0.07879
	-5 %	1.77393	1.89257	7.20312	-0.08323
	-25 %	1.76958	1.90237	7.17535	-0.46843
c_1	+25 %	1.62504	1.75341	7.84447	+8.81314
	+5 %	1.74122	1.85945	7.34361	+1.86555
	-5 %	1.81080	1.92364	7.07035	-1.92492
	-25 %	1.98389	2.08510	6.46465	-10.3268
c_2	+25 %	1.77378	1.88940	7.21168	+0.03551
	+5 %	1.77465	1.89023	7.20963	+0.00707
	-5 %	1.77509	1.89064	7.20861	-0.00707
	-25 %	1.77596	1.89148	7.20656	-0.03551
c_3	+25 %	1.92823	2.06442	8.22019	+14.0249
	+5 %	1.80745	1.92718	7.41867	+2.90673
	-5 %	1.74118	1.85256	6.99539	-2.96472
	-25 %	1.59313	1.68740	6.09179	-15.4988
c_4	+25 %	1.77801	1.88196	7.22916	+0.27791
	+5 %	1.77555	1.88859	7.21345	+0.06006
	-5 %	1.77417	1.89237	7.20461	-0.06256
	-25 %	1.77102	1.90110	7.18452	-0.34123
c_5	+25 %	1.77979	1.87722	7.24058	+0.43639
	+5 %	1.77599	1.88740	7.21624	+0.09876
	-5 %	1.77368	1.89371	7.20150	-0.10569
	-25 %	1.76792	1.90988	7.16478	-0.61505

6. Conclusions

In this paper, an ordering inventory system for variable deteriorated items with time-dependent linear demand pattern and partial backlogging is considered. The inspiration for taking on present model is the thought of a linear inclining demand rate. A numerical study has been discussed to observe the sensitivity of the effect of changes in parameters.

The proposed EOQ model can be extended in a few ways. Initially, we might extend the linear inclining demand to a generalized demand patterns with varying times. We could extend the proposed generalized deterioration to Weibull distributions like two-parameter and three-parameter and the Gamma distributed models. Besides, we could extend this deterministic model into stochastic and quantity discount inventory models.

References

- [1] Aggarwal, S.P. (1978). A note on an order-level inventory model for a system with constant rate of deterioration, *Opsearch*. **15**, 184-187.
- [2] Bakker, M., Riezebos, J. and Teunter, R.H. (2012). Review of inventory system with deterioration since 2001, *European Journal of Operational Research*. **221**, 275-284.
- [3] Bhunia, A.K. and Maiti, M. (1997). A deterministic inventory replenishment problem for deteriorating items with time-dependent demand and shortages for the finite time horizon, *Opsearch*. **34**(1), 51-61.
- [4] Chang, H.J. and Dye, C.Y. (1999). An EOQ model for deteriorating items with time varying demand and partial backlogging, *Journal of the Operational Research Society*. **50**, 1176-1182.
- [5] Covert, R.P. and Philip, G.C. (1973). An EOQ model for items with Weibull distribution deterioration, *AIIE Transaction*. **5**, 323-326.
- [6] Dave, U. and Patel, L.K. (1981). (T, Si) Policy inventory model for deteriorating items with time proportional demand, *Journal of the Operational Research Society*. **32**, 137-142.
- [7] Ghare, P.M. and Schrader, G.P. (1963). A model for exponentially decaying inventories, *Journal of Industrial Engineering*. **14**, 238-243.
- [8] Goyal, S.K. and Giri, B.C. (2001). Recent trends in modeling of deteriorating inventory, *European Journal of Operational Research*. **134**, 1-16.
- [9] Ghosh, S.K. and Chaudhuri, K.S. (2006). An EOQ model with a quadratic demand, time proportional deterioration and shortages in all cycles, *International Journal of Systems Sciences*. **37**, 663-672.
- [10] Goswami, A. and Chaudhuri, K.S. (1991). EOQ model for an inventory with a linear trend in demand and finite rate of replenishment considering shortages, *International Journal of Systems Sciences*. **22**, 181-187.
- [11] Gupta, P.N. and Agarwal, R.N. (2000). An order level inventory model with time dependent deterioration, *Opsearch*. **37**(4), 351-359.

- [12] Hollier, R.H. and Mak, K.L. (1983). Inventory replenishment policies for deteriorating items in a declining market, *International Journal of Production Research*. **21**, 813-826.
- [13] Jalan, A., Giri. R. and Chaudhury, K.S. (1996). EOQ model for items with Weibull distribution deterioration shortages and trended demand, *International Journal of Production Economics*. **113**, 852-861.
- [14] Li, R., Lan, H. and Mawhinney, J.R. (2010). A review on inventory study, *Journal of Service Science and Management*. **3**, 117-129.
- [15] Mashud, A.H.M. (2020). An EOQ deteriorating inventory model with different types of demand and fully backlogged shortages, *International Journal of Logistics Systems and Management*. **36**(1), 16-45.
- [16] Nahmias, S. (1982). Perishable inventory theory: A review, *Operations Research*. **30**, 680-708.
- [17] Ouyang, L.Y., Wu, K.S. and Cheng, M.C. (2005). An inventory model for deteriorating items with exponential declining demand and partial backlogging, *Yugoslav Journal of Operations Research*. **15**(2), 277-288.
- [18] Pradhan, L.M. and Tripathy, C.K. (2013). An EOQ model for three-parameter weibull deteriorating item with partial backlogging, *LogForum*. **9**(1), 35-42.
- [19] Raafat, F. (1991). Survey of literature on continuously deteriorating inventory model, *Journal of the Operational Research Society*. **42**, 27-37.
- [20] Singh, T., Pattanayak, H., Nayak, A.K. and Sethy, N.N. (2018). An optimal policy with three-parameter Weibull distribution deterioration, quadratic demand, and salvage value under partial backlogging, *International Journal of Rough Sets and Data Analysis*. **5**(1), 79-98.
- [21] Singh, T., Sethy, N.N., Nayak, A.K. and Pattnaik, H. (2021). An optimal policy for deteriorating items with generalized deterioration, trapezoidal-type demand, and shortages, *International Journal of Information Systems and Supply Chain Management*. **14**(1), 23-54.
- [22] Swain, P., Mallick, C., Singh, T., Mishra, P.J. and Pattanayak, H. (2021). Formulation of an optimal ordering policy with generalised time-dependent demand, quadratic holding cost and partial backlogging, *Journal of Information and Optimization Sciences*. **42**(5), 1163-1179.
- [23] Wee, H.M. (1995). A deterministic lot-size inventory model for deteriorating items with shortages and a declining market, *Computer & Operations Research*. **22**(3), 345-356.
- [24] Whitin, T.M. (1957). *Theory of inventory management*, Princeton University Press, Princeton, NJ. 62-72.

Appendix:

$$\frac{\partial^2 AT(t_s, T)}{\partial t_s^2} = \frac{1}{T} \left[c_1 \left\{ a(1 + \alpha t_s^2) + b \left(2t_s + \frac{4\alpha t_s^3}{3} \right) \right\} + \frac{\alpha c_2}{2} (2at_s + bt_s^2) + \frac{D_0(c_4 + \delta_0 c_5)}{(1 + \delta_0(T - t_s))^2} \right]$$

$$\frac{\partial^2 AT(t_s, T)}{\partial T^2} = \frac{1}{T} \left[\frac{D_0(c_4 + \delta_0 c_5)}{(1 + \delta_0(T - t_s))^2} \right] - \frac{2}{T^2} \left[\frac{D_0(c_4 + \delta_0 c_5)(T - t_s)}{1 + \delta_0(T - t_s)} \right]$$

$$+ \frac{2}{T^3} \left[c_1 \left\{ \frac{a}{2} \left(t_s^2 + \frac{\alpha t_s^4}{6} \right) + \frac{b}{3} \left(t_s^3 + \frac{\alpha t_s^5}{5} \right) \right\} + \alpha c_2 \left\{ \frac{at_s^3}{6} + \frac{bt_s^4}{8} \right\} \right] + \frac{2c_3}{T^3}$$

$$+ \frac{2D_0(c_4 + \delta_0 c_5)}{T^3 \delta_0} \left[T - t_s - \frac{1}{\delta_0} \ln[1 + \delta_0(T - t_s)] \right]$$

and

$$\frac{\partial^2 AT(t_s, T)}{\partial t_s \partial T} = -\frac{1}{T} \left[\frac{D_0(c_4 + \delta_0 c_5)}{(1 + \delta_0(T - t_s))^2} \right] - \frac{1}{T^2} \left[c_1 \left\{ a \left(t_s + \frac{\alpha t_s^3}{3} \right) + b \left(t_s^2 + \frac{\alpha t_s^4}{3} \right) \right\} + \frac{\alpha c_2}{2} (at_s^2 + bt_s^3) \right]$$

$$+ \frac{1}{T^2} \left[\frac{D_0(c_4 + \delta_0 c_5)(T - t_s)}{1 + \delta_0(T - t_s)} \right].$$

